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ABELIAN QUASIGROUPS ARE *T*-QUASIGROUPS

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Abstract

By means of known results with respect to algebras of a congruence modular variety it is proved that abelian (in the sense of McKenzie) quasigroups, i.e. quasigroups coinciding with their centre, are T-quasigroups and conversely.

In the literature on quasigroups it was accepted medial quasigroups to call abelian quasigroups. However, the latest investigations of the centre of quasigroups show that the class of abelian (in the sense of McKenzie [1]) quasigroups is the class of T-quasigroups introduced and thoroughly studied in [2,3] and including medial quasigroups.

In this note we shall prove this fact by means of known results with respect to algebras of modular varieties.

Let A be an universal algebra. According to the definition from [1,4] the *centre of A* is the set $Z(A)$ of all pairs $(a,b)\in A^2$ such that for each term operation $t(x,y_1,\dots,y_n)$ of A, each $\bar{u},\bar{v}\in A^n$:

$$t(a,\bar{u})=t(a,\bar{v})\Leftrightarrow t(b,\bar{u})=t(b,\bar{v}).$$

The centre $Z(A)$ is a congruence on A.

An algebra A is called *abelian* if $Z(A) = A^2$.

In [5,6] the concept of the h-centre Z_h, $h\in Q$, of a quasigroup $Q(\cdot)$ was introduced and studied. Z_h is a normal subset of $Q(\cdot)$ and defines the normal congruence $\theta_Z(\cdot)$. It was proved that if Z_h forms a subquasigroup of $Q(\cdot)$, then this subquasigroup is a T-quasigroup [5,6], and each T-quasigroup $Q(\cdot)$ coincides with its h-centre ($Z_h = Q$) for each $h\in Q$.

In [7] it was proved that $\theta_Z = Z(Q)$ where $Z(Q)$ is the centre of the corresponding primitive quasigroup $Q(\cdot,\backslash,/)$. These results mean that the definition of Z_h is an inner characterization of the centre $Z(Q)$. It also implies that in the variety of all primitive quasigroups the abelian quasigroups are

T-quasigroups, and conversely. It is a long way to prove this. The aim of this note is to prove directly that a primitive quasigroup $Q(\cdot,\backslash,/)$ is abelian if and only if it is a T-quasigroup by means of known results on universal algebras of a congruence modular (shortly, modular) variety. To prove this we need a number of necessary concepts and results with respect to algebras of modular varieties.

First we remind that if an algebra A lies in a modular variety, then its centre can be characterized by means of a commutator of congruences. For the first time the theory of commutators and the centre was given for universal algebras of permutable (or Mal'cev) varieties and thoroughly studied by J.D.H.Smith in [8]. Later this theory was developed by many authors in algebras of modular varieties. In [9] J.D.H.Smith called a quasigroup $Q(\cdot,\backslash,/)$ coinciding with its centre (i.e.an abelian primitive quasigroup in the sense of McKenzie) a $\mathfrak{Z}$-quasigroup.

Let α,β be congruences on an algebra A of a modular variety ($\alpha,\beta\in$ConA). According to [10] define the congruence Δ_α^β on α by

$$\Delta_\alpha^\beta = <((a,a),(b,b))|(a,b)\in\beta>_\alpha,$$

i.e. Δ_α^β is the congruence generated in α (viewed as a subalgebra of $A\times A$). In other words, Δ_α^β is the smallest congruence relation on α containing the set $\{((a,a),(b,b))|(a,b)\in\beta\}$.

The *commutator* $[\alpha,\beta]$ of two congruences α,β on A is defined as follows:

$$[\alpha,\beta]=\{(x,y)|(x,x)\Delta_\alpha^\beta(x,y)\}.$$

A congruence θ on an algebra A of a congruence modular variety is called *central* if $[\theta,A^2]=0_A$, where

$$0_A=\{(a,a)|a\in A\}.$$

In this case the centre $Z(A)$ is exactly the largest central congruence on A ([1], **Lemma 5.2**). Hence, an algebra A of a modular variety is abelian iff $[A^2,A^2]=0_A$.

All abelian algebras in a modular variety form a subvariety. A variety is called abelian if every algebra of this variety is abelian.

According to **Definition 5.3** [1] a congruence α on A is called *abelian* if $[\alpha,\alpha]=0_A$. It is evident that each central congruence is abelian since

$$[\alpha,\alpha]\le[\alpha,Q^2]=0_A$$

implies

$$[\alpha,\alpha]=0_A.$$

Let f be a n-ary term operation of an algebra A,

$$\bar{x} = x_1^n, \bar{y} = y_1^n, \bar{z} = z_1^n \in A^n,$$

and

$$f(\bar{x} - \bar{y} + \bar{z}) = f(x_1 - y_1 + z_1, x_2 - y_2 + z_2, \ldots, x_n - y_n + z_n).$$

Definition 1[1]. An algebra A of a modular variety is called *affine* if there is an abelian group $\bar{A} = \langle A, +, - \rangle$ having the same universe as A, and a 3-ary term operation $t(x,y,z)$ on A, such that:

1) $t(x,y,z) = x - y + z$ for all $x, y, z \in A$;

2) $f(\bar{x} - \bar{y} + \bar{z}) = f(\bar{x}) - f(\bar{y}) + f(\bar{z})$

for each n-ary term operation f and $\bar{x}, \bar{y}, \bar{z} \in A^n$.

The condition **2)** is equivalent to the following statement:

each operation (and each term operation) of the algebra A is affine with respect to the group $\bar{A}$, i.e. for any given n-ary operation F there are endomorphisms $\alpha_1, \alpha_2, \ldots, \alpha_n$ of $\bar{A}$ and an element $a \in A$ such that

$$F(x_1^n) = \sum_{i=1}^{n} \alpha_i x_i + a$$

(see [1], p.46).

According to **Corollary 5.9** [1] in a modular variety every abelian algebra is affine, and conversely.

We also need the following result due to C.Herrmann [11] (see also **Theorem 3.4** [12]).

Theorem 1 [11]. *If $\mathfrak{R}$ is a modular variety, then there exists a term $p(x,y,z)$ such that for each algebra $A \in \mathfrak{R}$, each abelian $\alpha \in ConA$, each $a \in A$ on the α-class d/α containing the element d, operations "+,-" of an abelian group are defined such that for all $a, b, c \in d/\alpha$:*

$$p(a,b,c) = a - b + c,$$

and for each signature n-ary operation for all $\bar{a}, \bar{b}, \bar{c} \in A^n$ such that for $i \le n$ a_i, b_i, c_i are α-equivalent, the equality

$$f(\bar{a} + \bar{b} - \bar{c}) = f(\bar{a}) + f(\bar{b}) - f(\bar{c})$$

holds.

From **Theorem 1, Definition 1** and **Corollary 5.9** [1] it follows

Corollary 1. *Let* α *be an abelian congruence of an algebra* A *of a modular variety and the* α*-class* d/α *forms a subalgebra of* A *for some* $d \in A$. *Then this subalgebra is affine and so an abelian algebra.*

At last, we remind that the primitive quasigroup is an algebra $Q(\cdot,\backslash,/)$ with three binary operations which satisfy the laws:

$$x(x\backslash y)=y,\ (x/y)y=x,\ x\backslash(xy)=y,\ (xy)/y=x.$$

It is known that the class of all primitive quasigroups forms a permutable (and thus a modular) variety.

A quasigroup $Q(\cdot)$ is called a *T-quasigroup* [2,3], if there exist an abelian group $Q(+)$, its automorphosms α,β and an element $a \in Q$ such that

$$xy=\alpha x+\beta y+a$$

for all $x,y \in Q$.

Medial quasigroups are a special case of T-quasigroups when the automorphisms α and β commute.

Now we can easy prove for quasigroups the following

Lemma. *Let* α *be an abelian congruence on a quasigroup* $Q(\cdot,\backslash,/)$ *and a* α*-class* H *be a subquasigroup of* $Q(\cdot,\backslash,/)$. *Then* H *is a T-quasigroup.*

Proof. By **Corollary 1** $H(\cdot,\backslash,/)$ is an affine quasigroup so

$$xy=\alpha x+\beta y+a,$$

where α,β are endomorphisms of the abelian group $(H,+,-)$. Put $x=0$ (0 is the zero of $Q(+)$), then

$$0y=\beta y+a.$$

Hence β is a permutation on Q since $Q(\cdot)$ is a quasigroup. Analogously, α is a permutation on Q. Hence, $H(\cdot)$ is a T-quasigroup.

Note that according to **Lemma 33** [3], if $Q(\cdot)$ is a T-quasigroup and

$$xy=\alpha x+\beta y+a$$

where $\alpha,\beta \in AutQ(+)$, $Q(+)$ is an abelian group, then

$$x\backslash y=-\beta^{-1}\alpha x+\beta^{-1}y-\beta^{-1}a,$$

$$x/y=\alpha^{-1}x-\alpha^{-1}\beta y-\alpha^{-1}a,$$

i.e. $Q(\backslash)$ and $Q(/)$ are also T-quasigroups.

Theorem 2. *A quasigroup* $Q(\cdot,\backslash,/)$ *is abelian if and only if it is a T-quasigroup.*

Proof. If $Q(\cdot,\backslash,/)$ is abelian, then $[Q^2,Q^2]=0$ and by **Lemma** $Q(\cdot,\backslash,/)$ is a T-quasigroup.

Conversely, let $Q(\cdot,\backslash,/)$ be a T-quasigroup:

$$xy=\alpha x+\beta y+a \tag{1}$$

Prove that $Q(\cdot,\backslash,/)$ is affine. For this consider the Mal'cev term

$$t(x,b,y)=(x/(b\backslash b))\cdot(b\backslash y)$$

for $Q(\cdot,\backslash,/)$, i.e.

$$t(x,b,y)=R_{e_b}^{-1}x\cdot L_b^{-1}y, \tag{2}$$

(here

$$be_b=b,\quad R_{e_b}x=xe_b,\quad L_bx=bx),$$

since

$$b\backslash b=e_b,\quad x/e_b=R_{e_b}^{-1}x,\quad b\backslash y=L_b^{-1}y.$$

From (2) it follows that $t(x,b,y)$ is a loop with the identity element $be_b=b$. But (1) means that the abelian group $Q(+)$ is principally isotopic to the quasigroup $Q(\cdot)$ so according to **Albert's Theorem** the loop $t(x,b,y)$ is an abelian group isomorphic to $Q(+)$. Moreover, from the proof of **Albert's Theorem** (see [13, p. 17]) it follows that

$$t(x,b,y)=x-b+y$$

since the loop $t(x,b,y)$ is principally isotopic to $Q(+)$. Therefore the condition **1)** from the definition of an affine algebra holds.

The condition **2)** is also true since it is equivalent to the fact that each of the operations $(\cdot)$, $(/)$ and $(\backslash)$ has the form

$$f(x,y)=\alpha x+\beta y+a,$$

where α,β are some endomorphisms of the same abelian group $Q(+)$. This completes the proof.

Corollary 2. *If $Q(\cdot,\backslash,/)$ is a T-quasigroup, then $[\alpha,\beta]=0$ for all congruences α, β of $Q(\cdot,\backslash,/)$.*

Indeed, $[\alpha,\beta]\leq[Q^2,Q^2]=0$.

Now we can note that primitive T-quasigroups as abelian algebras of a modular variety have all properties established with respect to these algebras. Many of such properties can be found in [2] in addition to the properties of the T-quasigroups from [2,3,5,6].

In conclusion remark that if we take some variety of primitive quasigroups, then the abelian quasigroups can be a special case of T-quasigroups. For example, according to **Theorem 7** [6] the abelian quasigroups in the variety of all idempotent quasigroups are medial distributive quasigroups, and in the variety of all commutative quasigroups the abelian quasigroups are medial (commutative) quasigroups (**Corollary 6** [6]).

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