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ONE-SIDED *T*-QUASIGROUPS AND IRREDUCIBLE BALANCED IDENTITIES

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Abstract

Left and right T-quasigroups are considered. It is proved that all primitive left (right) T-quasigroups form the variety which can be characterized by two identities. Some varieties of primitive left (right) T-quasigroups and T-quasigroups characterized by irreducible balanced identities are picked out.

Introduction.

It is known that all primitive quasigroups isotopic to groups form the variety characterized by one identity [1].

The class of linear quasigroups plays the important role in this variety. As V.D.Belousov has shown in [1] these quasigroups are closely connected with irreducible balanced identities in quasigroups.

A quasigroup $Q(\cdot)$ is called linear (over a group) if a group $Q(+)$, its automorphisms φ,ψ and an element $c\in Q$ exists such that

$$xy=\varphi x+c+\psi y \qquad (1)$$

for all $x,y\in Q$.

The automorphisms φ,ψ are called *determining automorphisms* for the quasigroup $Q(\cdot)$.

In [2] the concept of linear quasigroup was generalized as follows.

A quasigroup $Q(\cdot)$ is called a *left (right) linear quasigroup* if there exist group $Q(+)$, its automorphism φ (ψ) and an one-to-one mapping β (α) of Q onto Q such that

$$xy=\varphi x+\beta y \quad (xy=\alpha x+\psi y)$$

for all $x,y\in Q$.

As it was shown in [2], left (right) linear quasigroups are closely connected with the left (right) nucleus in quasigroups. They also arised in [1] in the investigation of irreducible balanced identities in quasigroups.

All primitive left linear quasigroups form the variety characterized by the following identity:

$$[x(u\backslash y)]z=[x(u\backslash u)]\cdot(u\backslash yz). \tag{2}$$

Analogously, all primitive right linear quasigroups are characterized by the identity

$$x[(y/u)z]=(xy/u)\cdot[(u/u)z] \tag{3}$$

(**Corollary 2** [2]).

All primitive linear quasigroups also form the variety which can be characterized by the identities (2) and (3) (**Corollary 3** [2]) or the unique identity

$$xy\cdot uv=xu\cdot(\alpha_u y\cdot v) \tag{4}$$

where α_u is a mapping of Q in Q depending on u (**Theorem 1** [3]). It is easy to see that α_u is an one-to-one mapping of Q onto Q:

$$\alpha_u y=[u\backslash(u/u)y\cdot u]/(u\backslash u).$$

The T-quasigroups, i.e. the quasigroups linear over abelian groups, are the special case of linear quasigroups. These quasigroups were introduced and studied in detail in [4,5]. The well known medial quasigroups are a special case of T-quasigroups.

In [6] it was proved that the T-quasigroups play a role in the theory of quasigroups comparable to that of abelian groups among groups. Namely, a quasigroup coincides with its centre iff it is a T-quasigroup (see **Theorem 6** [6]).

In [6] the variety of all primitive T-quasigroups is characterized by two identities: (4) and the identity

$$xy\cdot uv=(\beta_x v\cdot y)\cdot ux, \tag{5}$$

where

$$\beta_x v=[(x((x/x)v))/x]/(x\backslash x).$$

In this article we consider the one-sided T-quasigroups (left and right T-quasigroups) and prove that all primitive left (right) T-quasigroups form the variety, which can be characterized by two identities. We also pick out a number of varieties of primitive left (right) T-quasigroups and T-quasigroups characterized by irreducible balanced identities.

1. Left (right) T-quasigroups and their characterization.

The following case of a left linear quasigroup $Q(\cdot)$ arised in [1] due to V.D.Belousov when he studied quasigroups with irreducible balanced identities:

$$xy = \varphi x + \beta y,$$

where $Q(+)$ is an abelian group, φ is its automorphism, β is an one-to-one mapping of Q onto Q. Using this we say that a quasigroup $Q(\cdot)$ is a *left (right) T-quasigroup*, briefly, a *LT-quasigroup* (*RT-quasigroup*) if $Q(\cdot)$ is a left (right) linear quasigroup over an abelian group.

First, we recall that the primitive quasigroup $Q(\cdot,\backslash,/)$ corresponds to each quasigroup $Q(\cdot)$, where

$$xy = z \Leftrightarrow x \backslash z = y \Leftrightarrow z / y = x.$$

We also note that according to **Lemma 1** [2] a left linear quasigroup, which is simultaneously a right linear quasigroup, is a linear quasigroup. From this **Lemma** it immediately follows that if a LT-quasigroup is a RT-quasigroup, then it is a T-quasigroup.

Theorem 1. *All primitive LT-quasigroup form the variety characterized by the following two identities*

$$[x(u \backslash y)]z = [x(u \backslash u)] \cdot (u \backslash yz), \tag{6}$$

$$(x / u)(u \backslash y) = (y / u)(u \backslash x). \tag{7}$$

All primitive RT-quasigroups are characterized by the identity (7) *and the following identity*

$$x[(y / u)z] = (xy / u)[(u / u)z]. \tag{8}$$

Proof. According to **Corollary 2** [2] the identity (6) means that $Q(\cdot)$ is left linear over a group $Q(+)$. But (7) implies that $Q(+)$ is an abelian group. Really, write (7) as follows

$$R_u^{-1}x \cdot L_u^{-1}y = R_u^{-1}y \cdot L_u^{-1}x, \tag{9}$$

where R_u, L_u are the translations of $Q(\cdot)$ with respect to an element $u \in Q$:

$$R_u x = xu, \qquad L_u x = ux.$$

Fixing in (9) the element u, we obtain that

$$x \circ y = y \circ x,$$

where $Q(\circ)$ is a loop principally isotopic to $Q(\cdot)$. Hence, the loop $Q(\circ)$ is commutative. By the **Albert's theorem** (see, for example, **Theorem 1.4** [7]) the loop $Q(\circ)$ is an abelian group. Thus, $Q(\cdot)$ is a LT-quasigroup.

Conversely, if $Q(\cdot)$ is a LT-quasigroup, then it is left linear over an abelian group $Q(+)$ and by **Corollary 2** [2] $Q(\cdot)$ satisfies the identity (6). Next, since the group $Q(+)$ is abelian, then by the **Albert's theorem** each loop, isotopic to $Q(+)$, is commutative. Hence, the equality (9) is satisfied for all $x,y,u \in Q$, i.e. the identity (7) holds. This completes the proof for the LT-quasigroups.

The proof for the RT-quasigroups is similar if we take into account that the identity (8) characterizes the variety of all right linear quasigroups (see **Corollary 2** [2]).

In the introduction it was noted that the variety of all primitive T-quasigroups is characterized by two identities (4) and (5). From **Theorem 1** an another characterization of T-quasigroups follows.

Corollary 1. *The variety of all primitive T-quasigroups can be characterized by three identities* (6),(7) *and* (8).

Indeed, it follows from above that if a LT-quasigroup $Q(\cdot)$ is also a RT-quasigroup, then $Q(\cdot)$ is a T-quasigroup. The converse follows from **Theorem 1.**

2. LT-quasigroups, RT-quasigroups, T-quasigroups and balanced identities.

Now we recall that an identity

$$w_1 = w_2$$

defined on a quasigroup $Q(\cdot)$ is called *balanced* if each variable x, which occurs on one side w_1 of the identity, occurs on the another side w_2 too and if no variable occurs in w_1 or w_2 more than once. This definition is due to A.Sade (see [8]). All balanced identities can be separated on two kinds. An identity $w_1 = w_2$ is kind 1 if the elements in w_1 and w_2 are equally ordered and is kind 2 otherwise.

An identity $w_1 = w_2$ is called *reducible* [1] if either

(i) each of w_1 and w_2 contains a "free element" x so that w_1 is of the form u_1x or xv_1 and w_2 likewise is the form u_2x or xv_2 (where u_i or v_i represents a subword of the word w_i for $i=1,2$); or

(ii) w_1 has the product xy of two free elements x and y as a subword and w_2 has one of the product xy or yx as a subword, or the dual of this statement.

An identity which is not reducible is called *irreducible*.

V.D.Belousov has proved the following remarkable theorem (**Theorem 3** [1]): a quasigroup which satisfies an irreducible balanced identity is isotopic to a *group*.

Let

$$(x_1, x_2, \dots, x_k) = ((\dots((x_1x_2)x_3)\dots))x_k,$$
$$[x_1x_2\dots x_k] = x_1(x_2(\dots(x_{k-2}(x_{k-1}x_k))\dots))$$

and $m|n$ means that m is a divisor of n. By $|\varphi|$ we denote the order of the automorphism φ and let S_Q denotes the set of all one-to-one mappings of Q onto Q.

A mapping $\gamma \in S_Q$ is called a *quasiautomorphism* of a quasigroup $Q(\cdot)$ if there exist one-to-one mappings $\alpha, \beta \in S_Q$ such that

$$\gamma(xy) = \alpha x \cdot \beta y.$$

According to **Lemma 2.5** [7] if γ is a quasiautomorphism of a group $Q(+)$, then

$$\gamma x = R_s\gamma_1 x = L_s\gamma_2 x,$$

where γ_1, γ_2 are automorphisms of $Q(+)$;

$$R_s x = x + s, \quad L_s x = s + x.$$

V.D.Belousov in [1,p.79] has proved the following important for us statement, which can be formulated as follows

Theorem 2 [1]. *Let $Q(\cdot)$ be a LT-quasigroup:*

$$xy = \varphi x + \beta y,$$

φ is an automorphism of the group $Q(+)$ of the order m, θ is a permutation of the set $M = \{0,1,\dots,n\}$, where $m|n$, satisfying the conditions:

(1) $\theta 0 \neq 0$,

(2) $\theta n \neq n$,

(3) $\theta i \equiv i (\mathrm{mod}\, m)$

for each $i \in M$. Then the following irreducible balanced identity of kind 2

$$(xy_0y_1\cdots y_{n-1}y_n) = (xy_{\theta 0}y_{\theta 1}\cdots y_{\theta(n-1)}y_{\theta n}) \qquad (10)$$

is satisfied in $Q(\cdot)$.

Conversely, if the identity (10) *holds in a quasigroup $Q(\cdot)$ for a nonidentity permutation* θ *of M, then $Q(\cdot)$ is a LT-quasigroup:*

$$xy = \varphi x + \beta y,$$

the automorphism φ *has a finite order* m *which is a divisor of* $(\theta i - i)$ *for each* $i = 0,1,\dots,n$ *and the permutation* θ *satisfies the conditions* (1), (2), *and* (3).

For our aims the next special case of **Theorem 2** [1] is useful.

Theorem 3 . *Let $Q(\cdot)$ be a LT-quasigroup:*

$$xy = \varphi x + \beta y,$$

$|\varphi| = m$, $m|n$. *Then $Q(\cdot)$ satisfies the following irreducible balanced identity of kind 2:*

$$(xy_0y_1\dots y_{n-1}y_n) = (xy_ny_1\dots y_{n-1}y_0). \tag{11}$$

Conversely, if a quasigroup $Q(\cdot)$ satisfies the identity (11), *then $Q(\cdot)$ is a LT-quasigroup:*

$$xy = \varphi x + \beta y,$$

and the order m *of the automorphism is a divisor of* n.

For the proof it is enough to observe that the identity (11) is (10) if $\theta = (0n)$, where $(0n)$ is a transposition (a cycle of the lenght two). Evidently, $\theta = (0n)$, satisfies each of conditions (1), (2), (3).

Remark, that the case $m = n$ corresponds to the identity (11) of a "minimal lenght".

The analogue of **Theorem 2** [1] is true for RT-quasigroups if we take the identity

$$[y_ny_{n-1}\dots y_1y_0x] = [y_{\theta n}y_{\theta(n-1)}\dots y_{\theta 1}y_{\theta 0}x]$$

instead of (10), but we shall formulate and prove the analog of **Theorem 3** changing a little the outline of the proof of the corresponding statement from [1].

Theorem 4. *Let $Q(\cdot)$ be a RT-quasigroup:*

$$xy = \alpha x + \psi y,$$

$|\psi| = k$, $k|l$. *Then the following irreducible balanced identity of kind 2:*

$$[y_ly_{l-1}\dots y_1y_0x] = [y_0y_{l-1}\dots y_1y_lx] \tag{12}$$

is satisfied in $Q(\cdot)$.

Conversely, if the identity (12) *is satisfied in a quasigroup $Q(\cdot)$ for some* $l \geq 1$, *then $Q(\cdot)$ is a RT-quasigroup:*

$$xy = \alpha x + \psi y,$$

and the order k *of the automorphism* ψ *is a divisor of* l.

Proof. Let $Q(\cdot)$ be a RT-quasigroup:

$$xy = \alpha x + \psi y,$$

$|\psi| = k$, $k|l$. Then

$$[y_l y_{l-1} \ldots y_1 y_0 x] = y_l(y_{l-1} \ldots (y_1(y_0 x)) \ldots) =$$
$$= \alpha y_l + \psi\alpha y_{l-1} + \psi^2 \alpha y_{l-2} + \ldots + \psi^l \alpha y_0 + \psi^{l+1} x =$$
$$= \alpha y_l + \psi\alpha y_{l-1} + \psi^2 \alpha y_{l-2} + \ldots + \alpha y_0 + \psi x =$$
$$= y_0(y_{l-1} \ldots (y_1(y_l x)) \ldots) = [y_0 y_{l-1} \ldots y_1 y_l x].$$

Conversely, let the identity (12) be satisfied in a quasigroup $Q(\cdot)$ for some $l \geq 1$. By **Theorem 3** from [1] $Q(\cdot)$ is isotopic to a group $Q(+)$:

$$xy = \lambda x + \delta y \qquad (13)$$

where $\lambda, \delta \in S_Q$. That is why from (12) we have

$$[y_l y_{l-1} \ldots y_1 y_0 x] = y_l [y_{l-1} \ldots y_1 y_0 x] =$$
$$= \lambda y_l + \delta[y_{l-1} \ldots y_1 y_0 x] = \lambda y_0 + \delta[y_{l-1} \ldots y_1 y_l x].$$

Fix x and all $y_j, j \neq 0, l$, in this equality:

$$\lambda y_l + \delta_1 y_0 = \lambda y_0 + \delta_1 y_l$$

for some $\delta_1 \in S_Q$. But by **Lemma 11** from [1] a group $Q(+)$ is abelian if the equality

$$\alpha x + \beta y = \gamma y + \delta x$$

is satisfied in $Q(+)$ for some $\alpha, \beta, \gamma, \delta \in S_Q$.

Next show that δ from (13) is a quasiautomorphism of the abelian group $Q(+)$. The identity (12) means that

$$y_l(y_{l-1} \ldots (y_1(y_0 x)) \ldots) = y_0(y_{l-1} \ldots (y_1(y_l x)) \ldots). \qquad (14)$$

Let $l \geq 3$, then (14) can be written as follows

$$\lambda y_l + \delta(\lambda y_{l-1} + \delta[y_{l-2} \ldots y_1 y_0 x]) =$$
$$= \lambda y_0 + \delta(\lambda y_{l-1} + \delta[y_{l-2} \ldots y_1 y_l x]).$$

Put in this equality

$$x = \lambda y_0 = y_1 = y_{l-2} = 0,$$

where 0 is the identity element of $Q(+)$, then

$$\lambda y_l + \delta_1 y_{l-1} = \delta(\lambda y_{l-1} + \delta_2 y_l)$$

for the corresponding $\delta_1, \delta_2 \in S_Q$. Hence, δ is a quasiautomorphism of $Q(+)$.

Let now $l = 2$, then (14) implies

$$\lambda y_2 + \delta(\lambda y_1 + \delta(y_0 x)) = \lambda y_0 + \delta(\lambda y_1 + \delta(y_2 x)).$$

Put here $\lambda y_0 = x = 0$, then

$$\lambda y_2+\delta_3 y_1=\delta(\lambda y_1+\delta_4 y_2)$$

for $\delta_3,\delta_4 \in S_Q$. At last, let $l=1$, then from (14) we have

$$\lambda y_1+\delta(y_0 x)=\lambda y_0+\delta(y_1 x),$$

or

$$\lambda y_1+\delta' x=\delta(y_1 x),$$

if we put $\lambda y_0=0$.

Thus, in all cases we obtain that δ is a quasiautomorphism of $Q(+)$. According to **Lemma 2.5** [7]

$$\delta x=s+\psi x,$$

where ψ is an automorphism of $Q(+)$, $s\in Q$. Hence,

$$xy=\lambda x+\delta y=\alpha x+\psi y, \tag{15}$$

where

$$\alpha x=\lambda x+s.$$

Using (15) in (14) we have

$$\alpha y_l+\psi\alpha y_{l-1}+\psi^2\alpha y_{l-2}+\ldots+\psi^{l-1}\alpha y_1+\psi^l\alpha y_0+\psi^{l+1}x=$$
$$=\alpha y_0+\psi\alpha y_{l-1}+\psi^2\alpha y_{l-2}+\ldots+\psi^{l-1}\alpha y_1+\psi^l\alpha y_l+\psi^{l+1}x$$

whence

$$\alpha y_l+\psi^l\alpha y_0=\alpha y_0+\psi^l\alpha y_l,$$
$$\psi^l(\alpha y_0-\alpha y_l)=\alpha y_0-\alpha y_l.$$

Therefore, $\psi^l x=x$ for every $x\in Q$, so the order $|\psi|$ of the automorphism ψ is a divisor of l. This completes the proof.

Theorems 3 and **4** imply

Corollary 2. *Let $Q(\cdot)$ be a T-quasigroup:*

$$xy=\varphi x+c+\psi y,$$

$|\varphi|=m$, $|\psi|=k$, $m|n$, $k|l$. *Then the identities* (11),(12) *are satisfied in $Q(\cdot)$. Conversely, if the identities* (11) *and* (12) *hold for certain $n,l\geq 1$ in quasigroup $Q(\cdot)$, then $Q(\cdot)$ is a T-quasigroup:*

$$xy=\varphi x+c+\psi y,$$

$|\varphi|\mid n$ *and* $|\psi|\mid l$.

Proof. Since every T-quasigroup is a LT-quasigroup and a RT-quasigroup, the first statement follows at once from **Theorems 3** and **4.** Conversely, according to **Theorem 4** if (12) is satisfied in a quasigroup $Q(\cdot)$ for some $l\geq 1$, then $Q(\cdot)$ is a RT-quasigroup:

$$xy = \lambda x + \delta y = \alpha x + \psi y,$$

(see (15)) and $|\psi|$ is a divisor of l. Next, using the equalities (11) and (13), we can prove that λ is a quasiautomorphism of $Q(+)$:

$$\lambda x = \varphi x + t,$$

where $t \in Q$, $\varphi \in AutQ(+)$ and $|\varphi| \mid n$. The proof is similar to that of the case for in **Theorem 4**. Thus,

$$xy = \lambda x + \delta y = \varphi x + t + s + \psi y = \varphi x + c + \psi y,$$

where $c = t+s$, $|\varphi| \mid n$, $|\psi| \mid l$. This completes the proof.

3. Some subvarieties of the varieties of LT- (RT-) quasigroups and T-quasigroups.

The above proved results present the posibility to pick out some varieties of primitive LT-quasigroups, RT-quasigroups and T-quasigroups, which are characterized by irreducible balanced identities of kind 2 and depend on the orders of their determining automorphisms.

We begin with the following **Lemma** which means that the order of a determining automorphism φ (ψ) of a LT-quasigroup (RT-quasigroup) $Q(\cdot)$ is its invariant and does not depend on a group over which $Q(\cdot)$ is left (right) linear.

Lemma 1.

(i) *Let $Q(\cdot)$ be a LT-quasigroup and*

$$xy = \varphi x + \beta y = \overline{\varphi} x \circ \overline{\beta} y,$$

where φ $(\overline{\varphi})$ is an automorphism of the abelian group $Q(+)$ $(Q(\circ))$, $\beta, \overline{\beta} \in S_Q$. Then $\varphi x = R_a \overline{\varphi} R_a^{-1} x$ for certain $a \in Q$ $(R_a x = x + a)$, i.e. $|\varphi| = |\overline{\varphi}|$.

(ii) *Let $Q(\cdot)$ be a RT-quasigroup and*

$$xy = \alpha x + \psi y = \overline{\alpha} x \circ \overline{\psi} y,$$

where $\psi \in AutQ(+)$, $\overline{\psi} \in AutQ(\circ)$. Then $\psi y = R_a \overline{\psi} R_a^{-1} y$ for some $a \in Q$, i.e. $|\psi| = |\overline{\psi}|$.

Proof. Let

$$xy = \varphi x + \beta y = \overline{\varphi} x \circ \overline{\beta} y,$$

$$\varphi \in AutQ(+), \ \overline{\varphi} \in AutQ(\circ).$$

In this case the group $Q(\circ)$ is principally isotopic to the group $Q(+)$. By **Albert's Theorem** $Q(\circ)$ is isomorphic to $Q(+)$. Moreover, there exists such an element $a \in Q$ that

$$R_a(x \circ y) = R_a x + R_a y, \quad R_a x = x + a$$

(see the proof of **Albert's Theorem** in [7], p.17). Hence, using the equality

$$R_a^{-1} x = x - a,$$

we have

$$xy = \overline{\varphi} x \circ \overline{\beta} y = R_a^{-1}(R_a \overline{\varphi} x + R_a \overline{\beta} y) =$$
$$= R_a \overline{\varphi} R_a^{-1}(x + a) + \overline{\beta} y = R_a \overline{\varphi} R_a^{-1} x + \overline{\beta}_1 y$$

($\overline{\beta}_1 y = R_a \overline{\varphi} 0 + \overline{\beta} y$, **0** is the identity element of $Q(+)$), since

$$\varphi_1 = R_a \overline{\varphi} R_a^{-1}$$

is an automorphism of $Q(+)$. Thus,

$$xy = \varphi x + \beta y = \varphi_1 x + \overline{\beta}_1 y$$

whence by $x = 0$ have

$$\beta = \overline{\beta}_1, \varphi = \varphi_1, |\varphi| = |\varphi_1| = |\overline{\varphi}|.$$

The second part of **Lemma 1** is proved analogously.

Corollary 3. *If $Q(\cdot)$ is a T-quasigroup and*

$$xy = \varphi x + c + \psi y = \overline{\varphi} x \circ \overline{c} \circ \overline{\psi} y,$$

then

$$\varphi x = R_a \overline{\varphi} R_a^{-1},$$
$$\psi y = R_a \overline{\psi} R_a^{-1},$$

i.e. $|\varphi| = |\overline{\varphi}|, |\psi| = |\overline{\psi}|.$

The **proof** follows immediately from **Lemma 2.**

Now let m,n be natural numbers. Denote by $\mathfrak{R}_m^l$ ($\mathfrak{R}_n^r$) the class of all LT-quasigroups (RT-quasigroups) with determining automorphisms whose orders are devisors of m (of n). In other words, a LT-quasigroup (a RT-quasigroup) $Q(\cdot)$ lies in $\mathfrak{R}_m^l$ ($\mathfrak{R}_n^r$) iff $xy = \varphi x + \beta y$ ($xy = \alpha x + \psi y$) for certain abelian group $Q(+)$, its automorphism φ (ψ) such that $\varphi^m = \varepsilon$ ($\psi^n = \varepsilon$), i.e. $|\varphi| \mid m$ ($|\psi| \mid n$). Here ε is the identity mapping of Q.

By $\mathfrak{R}_{m,n}$ we denote the class of all T-quasigroups with a pair (φ, ψ) of the determining automorphisms such that

$$\varphi^m = \psi^n = \varepsilon.$$

Hence, a T-quasigroup $Q(\cdot)$ belongs to $\Re_{m,n}$ iff

$$xy = \varphi x + c + \psi y,$$

$$|\varphi| \mid m \quad \text{and} \quad |\psi| \mid n$$

for some abelian group $Q(+)$.

From **Lemma 1** it follows at once that

$$\Re^l_m \cap \Re^l_n = \Re^l_{(m,n)} \qquad (\Re^r_m \cap \Re^r_n = \Re^r_{(m,n)})$$

where (m,n) is the greatest common divisor of m,n. In particular, if p,q are prime numbers, then

$$\Re^l_p \cap \Re^l_q = \Re^l_1 \qquad (\Re^r_p \cap \Re^r_q = \Re^r_1).$$

Next we prove

Lemma 2. $\Re_{m,n} = \Re^l_m \cap \Re^r_n$.

Proof. It is clear, that

$$\Re_{m,n} \subseteq \Re^l_m \cap \Re^r_n.$$

Let $Q(\cdot)$ occurs in $\Re^l_m$ and $\Re^r_n$. Then there exists abelian groups $Q(+)$ and $Q(\circ)$, their automorphisms φ and $\overline{\psi}$, such that

$$\varphi^m = \overline{\psi}^n = \varepsilon$$

and

$$xy = \varphi x + \beta y = \alpha x \circ \overline{\psi} y \tag{16}$$

for some $\alpha, \beta \in S_Q$. In this case there exists such an element $a \in Q$ that

$$R_a(x \circ y) = R_a x + R_a y$$

(see the proof of **Lemma 1**). Hence, from (16) by $x = 0$ we have

$$\beta y = \alpha 0 \circ \overline{\psi} y = R_a^{-1}(R_a \alpha 0 + R_a \overline{\psi} y) =$$

$$= -a + a + \alpha 0 + R_a \overline{\psi} R_a^{-1}(a + y) = c + \psi y,$$

where

$$c = \alpha 0 + R_a \overline{\psi} 0,$$

since $\psi = R_a \overline{\psi} R_a^{-1}$ is an automorphism of $Q(+)$. Thus,

$$|\psi| = |\overline{\psi}|,$$

$$xy = \varphi x + c + \psi y,$$

and $Q(\cdot) \in \Re_{m,n}$, as required.

Now denote by $\overline{\mathfrak{R}}_m^l$, $\overline{\mathfrak{R}}_n^r$, $\overline{\mathfrak{R}}_{m,n}$ the classes of corresponding primitive LT-quasigroups, RT-quasigroups and T-quasigroups.

Theorem 5.

(i) $\overline{\mathfrak{R}}_m^l$ *is a variety of primitive LT-quasigroups characterized by the identity*

$$(xy_0y_1\cdots y_m)=(xy_my_1y_2\cdots y_{m-1}y_0). \tag{17}$$

(ii) $\overline{\mathfrak{R}}_n^r$ *is a variety of primitive RT-quasigroups characterized by the identity*

$$[y_ny_{n-1}\cdots y_1y_0x]=[y_0y_{n-1}\cdots y_1y_nx]. \tag{18}$$

(iii) $\overline{\mathfrak{R}}_{m,n}$ *is a variety of primitive T-quasigroups characterized by the identities* (17) *and* (18).

Proof.

(i) Let $Q(\cdot)\in\mathfrak{R}_m^l$:

$$xy=\varphi x+\beta y,\qquad |\varphi|\mid m,$$

then $Q(\cdot)$ satisfies (17) by the first part of **Theorem 2.** Conversely, if $Q(\cdot)$ satisfies (17), then it is a LT-quasigroup by the second part of **Theorem 2** and

$$xy=\varphi x+\beta y,\qquad |\varphi|\mid m,$$

i.e. $Q(\cdot)\in\mathfrak{R}_m^l$.

(ii) follows similarly from **Theorem 3.**

(iii) is a consequence of **Lemma 2,** (i) and (ii).

Next we consider some special cases of the above varieties.

The variety $\overline{\mathfrak{R}}_1^l$ ($\overline{\mathfrak{R}}_1^r$) includes all quasigroups such that

$$xy=x+\beta y\quad (xy=\alpha x+y),\qquad \alpha,\beta\in S_Q$$

over all abelian groups $Q(+)$ (Q is a nonfixed set). These varieties are characterized by the identities

$$xy_0\cdot y_1=xy_1\cdot y_0\quad (y_1\cdot y_0x=y_0\cdot y_1x),$$

respectively.

The variety $\overline{\mathfrak{R}}_2^l$ ($\overline{\mathfrak{R}}_2^r$) includes all quasigroups from $\overline{\mathfrak{R}}_1^l$ ($\overline{\mathfrak{R}}_1^r$) and quasigroups of the form

$$xy=\varphi x+\beta y,\qquad |\varphi|=2$$

$$(xy=\alpha x+\psi y,\qquad |\psi|=2)$$

If $Q(\cdot)\in\overline{\mathfrak{R}}_2^l$ ($Q(\cdot)\in\overline{\mathfrak{R}}_2^r$), then $Q(\cdot)$ satisfies the identity

$$(xy_0\cdot y_1)y_2=(xy_2\cdot y_1)y_0,$$

$$(y_2(y_1\cdot y_0x)=y_0(y_1\cdot y_2x)),$$

and conversely.

Let p,q be simple numbers. Then $\mathfrak{R}_p^l$ ($\mathfrak{R}_q^r$) contains all quasigroups from $\overline{\mathfrak{R}}_1^l$ ($\overline{\mathfrak{R}}_1^r$) and all LT-quasigroups (RT-quasigroups) with the determining automorphisms of the order p (of the order q). If $Q(\cdot)\in\overline{\mathfrak{R}}_{p,q}$, then it has one of the next forms:

$$xy = \varphi x + c + \psi y, \quad |\varphi|=p, \quad |\psi|=q,$$
$$xy = \varphi x + c + y, \quad |\varphi|=p,$$
$$xy = x + c + \psi y, \quad |\psi|=q,$$
$$xy = x + c + y.$$

Finally we note that the variety of all abelian groups is contained in every variety from $\overline{\mathfrak{R}}_m^l$, $\overline{\mathfrak{R}}_n^r$, $\overline{\mathfrak{R}}_{m,n}$ for any m,n.

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