Quasigroups and related systems, vol. **1**, no. 1(1), 1994

TRANSVERSALS IN GROUPS.1. ELEMENTARY PROPERTIES

Evgenii A. Kuznetsov

Abstract

In this work the elementary properties of transversals in groups are studied.

1. Introduction. Necessary definitions and notations.

The present work deals with the properties of specific sets of representatives of left (right) cosets in groups to its subgroups. These sets are called left (right) transversals in groups to its subgroups. They were introduced in [1] and studied in [1,2,6] etc.

We shall use the following notations:

Λ is an index set (Λ contains a distinguished element 1); left (right) cosets in a group G to its subgroup H are numbered by the indexes from Λ;

${}_iH$ is the i-th left coset in a group G to its subgroup H;

H_i is the i-th right coset in a group G to its subgroup H;

e is the unit of group;

$Core_G(H)$ is the maximal subgroup contained in H and normal in G;

$St_a(K)$ is the stabilizor of an element a in a permutation group K.

Definition 1. Let G be a group and H a subgroup in G. A complete system T of representatives of the left (right) cosets in G to H ($e = t_1 \in T$) is called a *left* (*right*) *transversal in G to H*.

Definition 2. A left transversal T in G to H which is also a right transversal in G to H is called a *two-sided* transversal in G to H.

Remark. If we study a two-sided transversal T in G to H then we shall consider that

$$t_i \in {}_iH \cap H_i$$

for any $i \in \Lambda$ and $t_i \in T$. We can always succeed in this by a corresponding renumeration of cosets.

Let T be a transversal (left or right) in G to H. We can introduce correctly the following operations on Λ:

$$i * j = v \Leftrightarrow t_i t_j = t_v h, \quad h \in H \tag{1}$$

if T is a left transversal, and

$$i \circ j = w \Leftrightarrow t_i t_j = h t_w, \quad h \in H \tag{2}$$

if T is a right transversal.

Lemma 1. *The system* $< \Lambda, *, 1 >$ *is a right quasigroup with the unit* 1.

Proof. Let arbitrary elements $a, b \in \Lambda$ be given. Let us consider the following equation:

$$a * x = b. \tag{3}$$

We have from the **Definition 1**: there exists an element $u \in \Lambda$ such that

$$t_a^{-1} t_b = t_u h, \quad h \in H.$$

Then

$$t_a t_u = t_b h^{-1}, \tag{4}$$

i.e.

$$a * u = b.$$

So a solution of the equation (3) exists. Assume it is not unique. Then there exists an element $v \in \Lambda$ such that

$$a * v = b \quad \& \quad v \neq u. \tag{5}$$

Therefore we have

$$t_a t_v = t_b h_1, \quad h_1 \in H. \tag{6}$$

From (4) and (6) it follows that

$$t_a t_u h = t_a t_v h_1^{-1},$$
$$t_v = t_u h_2,$$

and then $v = u$, because only one element from T lies in every coset in G to H. We have a contradiction with (5). So u is the unique solution of the equation (3), i.e. the system $< \Lambda, * >$ is a right quasigroup.

We have for any $a \in \Lambda$

$$a*1=c,$$
$$t_a e=t_c h,\ \ h\in H,$$
$$t_c=t_a h^{-1},\ \ h\in H,$$

and then $c=a$, i.e. $a*1=a$. We can prove that $1*a=a$ for any $a\in\Lambda$ analogously. The proof is complete.

Lemma 1*. *The system* $<\Lambda,\circ,1>$ *is a left quasigroup with the unit* 1.

The **proof** is similar to the proof of **Lemma 1**.

Definition 3. Let T be a left (right) transversal in G to H. If the system $<\Lambda,*,1>$ ($<\Lambda,\circ,1>$) is a loop, then T is called a *left (right) loop transversal in* G *to* H. If $<\Lambda,*,1>$ (respectively, $<\Lambda,\circ,1>$) is a group, then T is called a *left (right) group transversal in* G *to* H.

The next **Lemma** reduces the investigation of transversals in groups to the case when $Core_G(H)=<e>$ (for loop transversals an analogous result was proved in [2]).

Lemma 2. *Let* T *be a left transversal in* G *to* H *and*

$$\varphi: G\to G'=G/Core_G(H)$$

the natural homomorphism. Then we have:

1. *The set*

$$T'=\{\varphi(t_x)|t_x\in T, x\in\Lambda\}$$

is a left transversal in G' *to* $H'=H/Core_G(H)$*;*

2. $$<\Lambda,\bullet,1>\cong<\Lambda,*,1>,$$

where "$\bullet$" is the operation corresponding to the transversal T'*;*

Proof.

1. Let us denote: $t_x'=\varphi(t_x)$. Then we have:

$$(t_x')^{-1}t_y'\in H',$$
$$(\varphi(t_x))^{-1}\varphi(t_y)\in\varphi(H),$$
$$\varphi(t_x^{-1}t_y)\in\varphi(H),$$
$$(t_x^{-1}t_y)\cdot Core_G(H)=h_0\cdot Core_G(H),\ \ h_0\in H,$$
$$t_x^{-1}t_y\in H,$$

i.e. $x=y$, because $Core_G(H)\subseteq H$. Since for any $g'\in G'$ there exists $g\in G$ such that $g'=\varphi(g)$, then

$$g=t_u h_0,$$
$$g'=\varphi(g)=\varphi(t_u h_0)=\varphi(t_u)\varphi(h_0)=t_u' h_0',$$

where $h_0'\in H'$. It means T' is a left transversal in G' to H'.

2. We have:

$$x\bullet y=u,$$
$$t_x' t_y'=t_u' h',\quad h'\in H',$$
$$\varphi(t_x)\varphi(t_y)=t_u' h',\quad h'\in H',$$
$$t_u' h'=\varphi(t_x t_y)=\varphi(t_{x*y}h_1)=t_{x*y}'\cdot h_1',\quad h',h_1'\in H',$$
$$x*y=u,$$

i.e. for any $x,y\in\Lambda$

$$x*y=x\bullet y.$$

It means that

$$<\Lambda,\bullet,1>\equiv<\Lambda,*,1>.$$

Lemma 2*. *Let T be a right transversal in G to H and*

$$\varphi:G\to G'=G/Core_G(H)$$

the natural homomorphism. Then we have:

1. *The set*

$$T'=\{\varphi(t_x)|t_x\in T,x\in\Lambda\}$$

is a right transversal in G' to $H'=H/Core_G(H)$;

2.
$$<\Lambda,\times,1>\equiv<\Lambda,o,1>,$$

where "$\times$" is the operation corresponding to transversal T'.

The **proof** is analogous to that of **Lemma 2**.

Lemma 3. *In notations of* **Lemma 2** (*or* **Lemma** 2*):

$$Core_{G'}(H')=<e>.$$

Proof. Let us assume

$$Core_{G'}(H')=M_0\neq<e>.$$

The complete inverse-image $\varphi^{-1}(M_0)=M_1$ is a subgroup in G. We have:

$$e\in M_0\ \Rightarrow\ Core_G(H)=Ker\varphi=\varphi^{-1}(e)\subset\varphi^{-1}(M_0)=M_1,\qquad (*)$$
$$M_0\subseteq H'\ \Rightarrow\ M_1=\varphi^{-1}(M_0)\subseteq\varphi^{-1}(H')=H.\qquad (**)$$

Let g be an arbitrary element from G and

$$M_g = gM_1g^{-1}.$$

Then

$$\varphi(M_g) = \varphi(gM_1g^{-1}) = \varphi(g)\varphi(M_1)(\varphi(g))^{-1} = g'M_0g'^{-1} = M_0,$$

because M_0 is a normal subgroup in G'. Therefore we have for any element $g \in G$:

$$gM_1g^{-1} = M_g = \varphi^{-1}(M_0) = M_1,$$

i.e. M_1 is a normal subgroup in G. This fact is in a contradiction with (*) and (**) (see above); therefore we have:

$$Core_{G'}(H') = \langle e \rangle.$$

The proof is complete.

Let us consider the permutation representations $\hat{G}$ and $\check{G}$ of group G by the left and right cosets to subgroup H. Let T be a left transversal in G to H; then by the definition:

$$\hat{g}(x) = y \Leftrightarrow gt_xH = t_yH.$$

Analogously, if T is a right transversal in G to H, then by the definition:

$$\check{g}(x) = y \Leftrightarrow Ht_xg = Ht_y.$$

It is well known (see [4]) that

$$\hat{G} \cong \check{G} \cong G / Core_G(H).$$

Lemma 4. *If T is an arbitrary left transversal in G to H then:*

1. *For any* $h \in H$: $\hat{h}(1) = 1$;
2. *For any* $x, y \in \Lambda$:

$$\hat{t}_x(y) = x * y, \quad \hat{t}_x^{-1}(y) = x \backslash y,$$

$$\hat{t}_1(x) = \hat{t}_x(1) = x, \quad \hat{t}_x^{-1}(1) = x \backslash 1, \quad \hat{t}_x^{-1}(x) = 1,$$

(where $x * y = z \Leftrightarrow x \backslash z = y$*);*

3.[1] *The following conditions are equivalent:*

a. *T is a left loop transversal in G to H;*

b. *The set $\hat{T} = \{\hat{t}_x\}_{x \in \Lambda}$ is a sharply transitive set of permutations on Λ.*

Proof. **1.** It is trivial.

2. We have:

$$\hat{t}_x(y) = w \Leftrightarrow t_x t_y H = t_w H \Leftrightarrow w = x * y;$$

$$\hat{t}_x^{-1}(y) = z \Leftrightarrow t_x^{-1} t_y H = t_z H \Leftrightarrow t_y H = t_x t_z H \Leftrightarrow y = x * z \Leftrightarrow z = x \backslash y.$$

Therefore we have as a corollary:

$$\hat{t}_1(x) = 1 * x = x, \quad \hat{t}_x(1) = x * 1 = x,$$

$$\hat{t}_x^{-1}(1) = x \backslash 1, \quad \hat{t}_x^{-1}(x) = x \backslash x = 1,$$

3. We have the following sequence of equivalent assertions:

(T is a left loop transversal in G to H) $\Leftrightarrow$

(the system $<\Lambda, *, 1>$ is a loop) $\Leftrightarrow$

(T is a left transversal in G to H and the equation $x * a = b$ has the unique solution in Λ for any given $a, b \in \Lambda$) $\Leftrightarrow$

(the equation $\hat{t}_x(a) = b$ has the unique solution in Λ for any given $a, b \in \Lambda$) $\Leftrightarrow$

(the set $\{\hat{t}_x\}_{x \in \Lambda}$ is a sharply transitive set of permutations on Λ).

Lemma 4*. *If T is an arbitrary right transversal in G to H then:*

1. *For any* $h \in H$: $\check{h}(1) = 1$;
2. *For any* $x, y \in \Lambda$:

$$\check{t}_x(y) = y \circ x, \quad \check{t}_x^{-1}(y) = y // x,$$

$$\check{t}_1(x) = \check{t}_x(1) = x, \quad \check{t}_x^{-1}(1) = 1 // x, \quad \check{t}_x^{-1}(x) = 1,$$

(where $y // x = z \Leftrightarrow z \circ x = y$*);*

3. *The following conditions are equivalent:*
   - **a.** *T is a right loop transversal in G to H;*
   - **b.** *The set $\check{T} = \{\check{t}_x\}_{x \in \Lambda}$ is a sharply transitive set of permutations on Λ.*

The **proof** is analogous to that of **Lemma 4**.

2. Two-sided transversals in groups.

Lemma 5. *Let T be a two-sided transversal in G to H. Then for any $x,y\in\Lambda$:*

1. $y\backslash(x\backslash 1)=(x\circ y)\backslash 1$,

where "\" is the left inverse operation of "$*$";

2. $(1//y)//x=1//(x*y)$,

where "//" is the right inverse operation of "$\circ$".

Proof. **1.** Using **Lemma 4** we have:

$$x\circ y=z\Leftrightarrow ht_x=t_xt_y, h\in H\Leftrightarrow t_y^{-1}t_x^{-1}h=t_z^{-1}, h\in H\Leftrightarrow$$

$$\Leftrightarrow \hat{t}_y^{-1}\hat{t}_x^{-1}(1)=\hat{t}_z^{-1}(1)\Leftrightarrow \hat{t}_y^{-1}(x\backslash 1)=z\backslash 1\Leftrightarrow y\backslash(x\backslash 1)=z\backslash 1.$$

Therefore

$$y\backslash(x\backslash 1)=(x\circ y)\backslash 1.$$

2. It is proved analogously.

Corollary. *For any two-sided transversal in G to H*

$$x*y=1\Leftrightarrow x\circ y=1.$$

The **proof** is obvious.

Lemma 6. *Let T be a two-sided transversal in G to H. Then the following conditions are equivalent:*

1. *For any $x,y\in\Lambda$:*

$$x*y=x\circ y;$$

2. *The system $<\Lambda,*,1>$ is a WIP-loop;*

3. *The system $<\Lambda,\circ,1>$ is a WIP-loop;*

Proof. The definition of *WIP*-loop one can find in [5].

1.$\Rightarrow$2. Let us have for any $x,y\in\Lambda$

$$x*y=x\circ y;$$

Then the system $<\Lambda,*,1>$ is a left and right quasigroup with the unit 1 simultaneously, i.e. $<\Lambda,*,1>$ is a loop. Moreover, using **Lemma 5** we have:

$$y\backslash(x\backslash 1)=(x*y)\backslash 1\Leftrightarrow(x*y)*(y\backslash(x\backslash 1))=1. \qquad (7)$$

Let

$$(x*y)*z=1.$$

Using the identity (7) we obtain

$$z=y\backslash(x\backslash 1)\Rightarrow x*(y*z)=1.$$

Therefore using a characterization of *WIP*-loop (see [5], p.87) we get: system $<\Lambda,*,1>$ is a *WIP*-loop.

2.⇒1. Let the system $<\Lambda,*,1>$ be a *WIP*-loop. Then for any $x,y\in\Lambda$

$$(x*y)\backslash 1=y\backslash(x\backslash 1).$$

Using **Lemma 5** we have for any $x,y\in\Lambda$

$$x*y=x\circ y.$$

1.⇔3. It is proved analogously.

Lemma 7. *Let T be a left transversal in G to H. Then following conditions are equivalent:*

1. *T is a two-sided transversal in G to H;*

2. *The equation $x*a=1$ has the unique solution in Λ for any given $a\in\Lambda$.*

Proof. **1.⇒2.** Let T be a two-sided transversal in G to H. Then using the **corollary** to **Lemma 5** we have for any given $a\in\Lambda$

$$x*a=1\Leftrightarrow x\circ a=1.$$

The equation $x\circ a=1$ has the unique solution in Λ for any given $a\in\Lambda$. Then the equation $x*a=1$ satisfies the same property.

2.⇒1. Let the condition 2 holds. Then the mapping α which is defined by

$$\alpha:\Lambda\to\Lambda,$$
$$\alpha(x)=u\Leftrightarrow u*x=1,$$

is a permutation on Λ. Then we have for any $x\in\Lambda$:

$$t_{\alpha(x)}t_x=e\cdot h\in H,$$

$$t_{\alpha(x)}=ht_x^{-1}\in Ht_x^{-1},$$

$$Ht_{\alpha(x)}=Ht_x^{-1}.$$

Therefore we get: firstly

$$(Ht_{\alpha(x)})\cap(Ht_{\alpha(y)})\neq\varnothing,$$

$$(Ht_x^{-1})\cap(Ht_y^{-1})\neq\varnothing,$$

$$(t_xH)\cap(t_yH)\neq\varnothing\Rightarrow x=y,$$

secondly, for any $g\in G$ there exist $t_u\in T$ and $h_0\in H$ such that:

$$g^{-1}=t_uh_0,$$

$$g = h_0^{-1} t_u^{-1} = h_0^{-1} h^{-1} t_{\alpha(u)} = h' t_{\alpha(u)}.$$

It means that the set

$$T_1 = \{t_{\alpha(x)} | x \in \Lambda\} \equiv T$$

is a right transversal in G to H; i.e. T is a two-sided transversal in G to H. The proof is complete.

Lemma 7*. *Let T be a right transversal in G to H. Then following conditions are equivalent:*

1. *T is a two-sided transversal in G to H;*

2. *The equation $a \circ x = 1$ has the unique solution in Λ for any given $a \in \Lambda$.*

The **proof** is analogous to that of **Lemma 7**.

Remark. O.Ore has proved in [3]: if the index *(G:H)* is finite, then there exists a two-sided transversal in G to H.

Lemma 8. *Let T be a two-sided transversal in G to H and $Core_G(H) = <e>$. Then for any $h \in H$*

$$h = \bigcap_{u \in \Lambda} (t_{\hat{h}(u)} H t_u^{-1}),$$

$$h = \bigcap_{u \in \Lambda} (t_u^{-1} H t_{\check{h}(u)}).$$

Proof. We will prove the first equality; the second equality can be proved by the same way. For any $u \in \Lambda$ we get:

$$\hat{h}(u) = v \Leftrightarrow h t_u H = t_v H \Leftrightarrow h \in (t_v H t_u^{-1}),$$

$$h \in (t_{\hat{h}(u)} H t_u^{-1}).$$

Therefore we obtain

$$h \in \bigcap_{u \in \Lambda} (t_{\hat{h}(u)} H t_u^{-1}).$$

Let us assume that

$$h_1 \in \bigcap_{u \in \Lambda} (t_{\hat{h}(u)} H t_u^{-1}).$$

Then we have for any $u \in \Lambda$

$$h_1 \in (t_{\hat{h}(u)} H t_u^{-1}) \Leftrightarrow h_1 t_u H = t_{\hat{h}(u)} H \Leftrightarrow$$

$$\Leftrightarrow \hat{h}_1(u) = \hat{h}(u) \Leftrightarrow (h_1^{-1} h) \in Core_G(H) = \langle e \rangle \Rightarrow h_1 = h.$$

It means that

$$h = \bigcap_{u \in \Lambda} (t_{\hat{h}(u)} H t_u^{-1}).$$

The proof is complete.

3. Loop and group transversals in groups.

Lemma 9. *Let T be a left (right) transversal in G to H. Then the following conditions are equivalent:*

1. *The system $\langle \Lambda, *, 1 \rangle$ ($\langle \Lambda, \circ, 1 \rangle$) is a loop;*
2. *T is a left (right) transversal in G to $\pi H \pi^{-1}$ for any $\pi \in G$;*
3. *The set $\pi T \pi^{-1}$ is a left (right) transversal in G to H for any $\pi \in G$.*

The **proof** is contained in [1,2].

As we can see from the next **Lemma** the simplest example of two-sided transversals are loop transversals.

Lemma 10. *The following propositions are equivalent:*

1. *T is a left loop transversal in G to H;*
2. *T is a right loop transversal in G to H;*

Proof. 1.⇒2. Let T be a left loop transversal in G to H. Using **Lemma 9** we have for any $x, y \in \Lambda$ $(x \neq y)$:

$$(t_x^{-1} t_y) \notin (\pi H \pi^{-1}) \qquad (*)$$

for any $\pi \in G$. Assume that T is not a right loop transversal in G to H. Then by **Lemma 9** there exist $\pi_0 \in G$ and $x_0, y_0 \in \Lambda$ such that

$$t_{x_0} = \pi_0 h \pi_0^{-1} t_{y_0}, \quad h \in H, x_0 \neq y_0.$$

Then we get:

$$t_{x_0}^{-1} t_{y_0} = t_{y_0}^{-1} \pi_0 h^{-1} \pi_0^{-1} t_{y_0} = \pi_1 h^{-1} \pi_1^{-1},$$

where $\pi_1 = t_{y_0}^{-1}\pi_0$. The last equality contradicts (*). It means, that T is a right loop transversal in G to H.

2.⇒1. The proof is analogous.

Lemma 11. *Let T be a left (right) group transversal in G to H. Then for any $x,y \in \Lambda$*

$$x * y = x \circ y.$$

Proof. Let T be a left (right) group transversal in G to H. By **Lemma 10** T is a two-sided transversal in G to H. Then using **Lemma 6** we have for any $x,y \in \Lambda$

$$x * y = x \circ y,$$

because any group is a *WIP*-loop. The proof is complete.

Let us introduce the next notations:

$$h_{(x*y)} = t_{x*y}^{-1} t_x t_y,$$

if T be a left transversal; and

$$h_{(x \circ y)} = t_x t_y t_{x \circ y}^{-1},$$

if T be a right transversal.

Lemma 12. *Let $Core_G(H) = \langle e \rangle$ and T be a left (right) transversal in G to H. Then the following conditions are equivalent:*

1. *T is a group transversal in G to H;*

2. *For any $x,y \in \Lambda$*

$$h_{(x*y)} = e \quad (h_{(x \circ y)} = e).$$

3. *For any $x,y \in \Lambda$*

$$t_x t_y = t_{x*y} \quad (t_x t_y = t_{x \circ y}).$$

Proof. We shall prove these equivalences for a left transversal T in G to H. The proof in the case of a right transversal in G to H is analogous.

1.⇒2. Let T be a left group transversal in G to H. Using **Lemma 4** we have for any $x,y \in \Lambda$:

$$\hat{h}_{(x*y)}(u) = \hat{t}_{x*y}^{-1} \hat{t}_x \hat{t}_y(u) = \hat{t}_{x*y}^{-1}(x*(y*u)) =$$
$$= (x*y) \backslash (x*(y*u)) = (x*y) \backslash ((x*y)*u) = u,$$

i.e. $\hat{h}_{(x*y)} = \mathbf{id}$. It means that

$$h_{(x*y)} \in Core_G(H) = \langle e \rangle \Rightarrow h_{(x*y)} = e.$$

2.⇒3. It is evident.

3.⇒1. Let us have

$$t_x t_y = t_{x*y}$$

for any $x, y \in \Lambda$. Then

$$t_{x*(y*u)} = t_x t_{y*u} = t_x t_y t_u = t_{x*y} t_u = t_{(x*y)*u},$$

i.e.

$$x*(y*u) = (x*y)*u,$$

and therefore the system $\langle \Lambda, *, 1 \rangle$ is a group. It means that T is a group transversal in G to H.

The proof is complete.

4. Connection between different transversals of the same subgroup in a group.

We shall consider below that $Core_G(H) = \langle e \rangle$.

Let T be an arbitrary two-sided transversal in G to H. It is obvious that any left transversal L in G to H may be represented by T in the following way:

$$l_{x'} = t_x h_x^{(1)}, \quad h_x^{(1)} \in H, \quad x \in \Lambda, \tag{8}$$

and any right transversal R in G to H may be represented by T in the following way:

$$r_{x''} = h_x^{(2)} t_x, \quad h_x^{(2)} \in H, \quad x \in \Lambda, \tag{9}$$

Remark 1. If we pass to the permutation representation $\hat{G}$ (in the case of a left transversal L) or $\check{G}$ (in a case of the right transversal R), we obtain

$$x' = \hat{l}_{x'}(1) = \hat{t}_x \hat{h}_x^{(1)}(1) = \hat{t}_x(1) = x,$$

$$x'' = \check{r}_{x''}(1) = (1)\check{h}_x^{(2)} \check{t}_x = \check{t}_x(1) = x.$$

Remark 2. If a two-sided transversal S may be represented by a transversal T by formulas (8) and (9), then

$$s_x = t_x h_x^{(1)} = h_{x'}^{(2)} t_{x'}. \tag{10}$$

Remark 3. The following equalities are obvious:

$$h_1^{(1)} = h_1^{(2)} = e, \quad \overset{\wedge(2)}{h_{x'}}(x') = x, \quad \overset{\vee(1)}{h_x}(x) = x'.$$

Let "$\otimes$" and "$\oplus$" be notations of the operations on Λ which correspond to the new transversals L and R.

Lemma 13. *The following assertions are true:*

1. $x \otimes y = x * (\overset{\wedge(1)}{h_x}(y)).$

2. $x \oplus y = (\overset{\vee(2)}{h_y}(x)) o y.$

Proof. 1. We have:

$$s_x s_y = s_{x\otimes y} h, \quad h \in H,$$

$$t_x h_x^{(1)} t_y h_y^{(1)} = t_{x\otimes y} h_{x\otimes y}^{(1)} h, \quad h \in H,$$

$$\hat{t}_x \overset{\wedge(1)}{h_x} \hat{t}_y \overset{\wedge(1)}{h_y}(1) = \hat{t}_{x\otimes y} \overset{\wedge(1)}{h_{x\otimes y}} \hat{h}(1),$$

$$x \otimes y = x * (\overset{\wedge(1)}{h_x}(y)).$$

2. The proof is analogous to that of 1.

Lemma 14. *Let T be an arbitrary two-sided transversal in G to H. Then the following statements are true:*

1. *If a left transversal L can be represented by a transversal T by the formula* (8), *then*

$$t_x h_x^{(1)} t_x^{-1} = \bigcap_{u\in\Lambda} (t_{x\otimes u} H t_{x*u}^{-1}).$$

2. *If a right transversal R can be represented by a transversal T by the formula* (9), *then*

$$t_x^{-1} h_x^{(2)} t_x = \bigcap_{u\in\Lambda} (t_{uox}^{-1} H t_{u\oplus x}).$$

Proof. **1.** Using **Lemma 8** we have:

$$h_x^{(1)} = \bigcap_{u\in\Lambda} (t_{\Lambda^{(1)}_{h_x}(u)} \; H t_u^{-1}).$$

Therefore by **Lemma 13**

$$t_x h_x^{(1)} t_x^{-1} = \bigcap_{u\in\Lambda} (t_x t_{\Lambda^{(1)}_{h_x}(u)} \; H t_u^{-1} t_x^{-1}) =$$

$$= \bigcap_{u\in\Lambda} (t_{\Lambda^{(1)}_{x*h_x}(u)} \; H t_{x*u}^{-1}) = \bigcap_{u\in\Lambda} (t_{x\otimes u} H t_{x*u}^{-1}).$$

2. The proof is analogous to that of **1.**

5. Structural theorem.

Lemma 15. *Let K be a subgroup in G and $H \subseteq K \subset G$. Then:*

1. *If T is a left (right) transversal in G to H, then there exists a right (left) subquasigroup $<\Lambda_1, *, 1>$ $(<\Lambda_1, \circ, 1>)$ in the right (left) quasigroup $<\Lambda, *, 1>$ $(<\Lambda, \circ, 1>)$; moreover, if $(G{:}H) = |\Lambda| < \infty$, then $|\Lambda_1|$ divides $|\Lambda|$;*

2. *If T is a two-sided transversal in G to H, then all assertions from* **1** *take place;*

3. *If T is a loop transversal in G to H then:*

$<\Lambda_1, *, 1>$ is a subloop in the loop $<\Lambda, *, 1>$;

$<\Lambda_1, \circ, 1>$ is a subloop in the loop $<\Lambda, \circ, 1>$,

and if $|\Lambda| < \infty$, then $|\Lambda_1|$ divides $|\Lambda|$.

Proof. **1.** Let us consider a partition of the subgroup K on the left cosets of the subgroup H:

$$K = \bigcup_{i\in\Lambda_1} {}_iH = \bigcup_{i\in\Lambda_1} k_i H.$$

It is evident that this partition can be completed to a partition of G on the left cosets of H:

$$G = \bigcup_{i\in\Lambda} {}_iH.$$

Therefore there exists a left subtransversal $T_1 \subset K$ in a left transversal T, which is indexed by elements of Λ_1. All products of elements from T_1 lie in K and so the system $<\Lambda_1,*,1>$ is a subsystem in $<\Lambda,*,1>$.

The proof for the system $<\Lambda_1,\circ,1>$ is analogous.

If $(G:H)=|\Lambda|=m_0<\infty$, then:

$$|\Lambda_1|=(K:H)=m_1<\infty,$$
$$(G:K)=m_2<\infty.$$

We obtain:

$$(G:H)=(G:K)\cdot(K:H),$$
$$m_0=m_2\cdot m_1,$$

i.e. $|\Lambda_1|$ divides $|\Lambda|$.

2. It is an easy corollary of **1.**

3. If T is a loop transversal in G to H then the system $<\Lambda,*,1>$ is a loop. By the point **1** of this **Lemma** the system $<\Lambda_1,*,1>$ is a subsystem in a loop $<\Lambda,*,1>$, i.e. $<\Lambda_1,*,1>$ is a subloop in $<\Lambda,*,1>$. Analogously for the system $<\Lambda_1,\circ,1>$.

Corollary. *Let a loop transversal in G to H does not exist. Then a loop transversal in G to H for any group* $G^* \supseteq G$ *does not exist too.*

Proof. It is an easy corollary of **Lemma 15.**

6. A criterion of loop transversal existence in a group.

Lemma 16. *Let* $Core_G(H)=<e>$, $H^*=St_1(S_d)$, $d=|\Lambda|=(G:H)$. *Then the following assertions are equivalent:*

1. *There exists a loop transversal in G to H;*
2. *There exists a set* $\{h_x\}_{x\in\Lambda}$ *satisfying the following conditions:*

a. $\hat{h}_1=\mathbf{id}$ *and for any* $x\in\Lambda\setminus\{1\}$

$$\hat{h}_x\in((1x)\cdot\hat{G})\cap H^*;$$

b. *for any* $u\in\Lambda$

$$\bigcap_{x\in\Lambda}\{(1u)(1x)\hat{h}_x(1u)H^*(1x)\}\neq\varnothing.$$

3. *There exists a set* $\{h^{(u)}\}_{u\in\Lambda}$ *satisfying the following conditions:*

a. $\hat{h}^{(1)}=\mathbf{id}$ *and* $\hat{h}^{(u)}\in H^*$ *for any* $u\in\Lambda$;

b. *for any* $x\in\Lambda$

$$\bigcap_{u\in\Lambda}\{\hat{G}\cap((1u)\hat{h}^{(u)}(1x)H^*(1u))\}\neq\varnothing.$$

4.
$$K(G)\neq\varnothing,$$

where $\hat{h}^{(1)}=\mathbf{id}$ *and*

$$K(G)=\bigcup_{(h^{(1)},\ldots,h^{(d)})\in(H^*)^d}\ \bigcup_{\alpha\in S_d}\prod_{x\in\Lambda}\{\hat{G}\cap(\bigcap_{u\in\Lambda}((1u)\hat{h}^{(u)}(1\alpha(x))H^*(1u)))\}.$$

Proof.

1.⇒2. Let T be a loop transversal in G to H, $Core_G(H)=<e>$ and $(G:H)=d$. Let us consider the loop $L=<\Lambda,*,1>$ corresponding to the transversal T and take left and right translations of the loop L:

$$L_u(x)=u*x=\hat{t}_u(x),$$

$$R_u(x)=x*u=\hat{t}_x(u).$$

Since $Core_G(H)=<e>$, then $\hat{G}\cong G$ and the degree of permutations from $\hat{G}$ is equal d. Sets

$$L^*=\{L_u|u\in\Lambda\},\qquad R^*=\{R_u|u\in\Lambda\}$$

are loop transversals in G to H, because L is a loop. By **Lemma 4**, L^* and R^* are sharply transitive sets of permutations on Λ. Therefore again by **Lemma 4** L^* and R^* are loop transversals in S_d to $H^*=St_1(S_d)$. Then for any $u\in\Lambda$ there exists an unique element $h_{(u)}\in H^*$ such that

$$L_u h_{(u)}=R_{\varphi(u)},$$

where φ is a permutation on Λ. Therefore for any $u\in\Lambda$ there exists an unique element $h_{(u)}\in H^*$ such that

$$L_u\hat{h}_{(u)}(x)=R_{\varphi(u)}(x) \qquad (*)$$

for any $x\in\Lambda$. If $x=1$, then we have from (*):

$$u=L_u(1)=R_{\varphi(u)}(1)=\varphi(u).$$

Therefore the identity (*) may be written in the following form

$$\hat{t}_u\hat{h}_{(u)}(x)=\hat{t}_x(u)$$

for any $x \in \Lambda$. Then we have for any $x \in \Lambda$:

$$t_u h_{(u)} t_x H^* = t_x t_u H^*,$$

$$h_{(u)} \in (t_u^{-1} t_x t_u H^* t_x^{-1}),$$

and therefore

$$h_{(u)} \in \bigcap_{x \in \Lambda} (t_u^{-1} t_x t_u H^* t_x^{-1}). \tag{11}$$

Let us consider the following two-sided transversal P_0 in S_d to H^*:

$$P_0 = \{(1x) | x \in \Lambda\},$$

where *(1x)* is a transposition of S_d. Since T is a two-sided transversal (see **Lemma 10**), then there exists a set $\{h_x\}_{x \in \Lambda}$ (see (8)), such that

$$\hat{h}_1 = \mathbf{id};$$

$$\hat{t}_x = (1x)\hat{h}_x, \quad x \neq 1. \tag{11'}$$

We have for any $x \in \Lambda \setminus \{1\}$:

$$H^* \supset \hat{h}_x = ((1x)\hat{t}_x) \in ((1x) \cdot \hat{G}),$$

$$\hat{h}_x \in ((1x) \cdot \hat{G}) \cap H^*.$$

Using this and (11)-(11') we have for any $u \in \Lambda$:

$$h_u' = (h_u h_{(u)}) \in \bigcap_{x \in \Lambda} ((1u)(1x)\hat{h}_x (1u) H^* (1x)). \tag{12}$$

By **Lemma 8** (since $Core_G(H) = \langle e \rangle$) the intersection in (12) consists of an unique element (if this intersection exists). Then the identity (12) may be written in a following form:

$$\bigcap_{x \in \Lambda} \{(1u)(1x)\hat{h}_x (1u) H^* (1x)\} \neq \varnothing.$$

2.⇒3. Let conditions of **2.** hold. Then by **2.b.** and (12) we have for any $u \in \Lambda$:

$$(1u)(1x)\hat{h}_x (1u)\hat{h}_1^{(x)} (1x) = \hat{h}^{(u)}$$

for any $x \in \Lambda \setminus \{1\}$ and some $\hat{h}_1^{(x)} \in H^*$. In particular, from **2.b.** we get:

$$\hat{h}^{(1)} = \bigcap_{x \in \Lambda} ((1x)\hat{h}_x H^* (1x)) = \{\mathbf{id}\}.$$

Therefore we obtain:

$$\hat{h}_x = ((1x)(1u)\hat{h}^{(u)} (1x)(\hat{h}_1^{(x)})^{-1} (1u)) \in ((1x)(1u)\hat{h}^{(u)} (1x) H^* (1u)).$$

Using **2.a.** we have for any $x \in \Lambda \setminus \{1\}$ and $u \in \Lambda$:

$$\hat{h}_x \in H^* \cap ((1x)\hat{G}) \cap ((1x)(1u)\hat{h}^{(u)}(1x)H^*(1u)),$$

and therefore

$$\hat{h}_x \in \bigcap_{u\in\Lambda} \{H^* \cap ((1x)\hat{G}) \cap ((1x)(1u)\hat{h}^{(u)}(1x)H^*(1u))\}. \tag{13}$$

Using **Lemma 8** and (13) we get

$$\hat{h}_x = \bigcap_{u\in\Lambda} \{((1x)\hat{G}) \cap ((1x)(1u)\hat{h}^{(u)}(1x)H^*(1u))\},$$

because $\hat{h}^{(1)} = \mathbf{id}$. Therefore we have for any $x \in \Lambda$:

$$\bigcap_{u\in\Lambda} \{\hat{G} \cap ((1u)\hat{h}^{(u)}(1x)H^*(1u))\} = ((1x)\hat{h}_x) \neq \varnothing.$$

3.⇒4. Let all conditions of **3.** hold. Then we have for any $\alpha \in S_d$ and $x \in \Lambda$:

$$\bigcap_{u\in\Lambda} \{\hat{G} \cap ((1u)\hat{h}^{(u)}(1\alpha(x))H^*(1u))\} \neq \varnothing.$$

Therefore for the set $\{\hat{h}^{(u)}\}_{u\in\Lambda}$ we obtain

$$\bigcup_{\alpha\in S_d} \prod_{x\in\Lambda} \{\bigcap_{u\in\Lambda} (\hat{G} \cap ((1u)\hat{h}^{(u)}(1\alpha(x))H^*(1u)))\} \neq \varnothing.$$

Then

$$K(G) \neq \varnothing,$$

where *K(G)* is the set defined in the condition **4.** of this **Lemma.**

4.⇒1. Let conditions of **4.** hold. We know (see **Lemma 8**) that the intersection

$$\bigcap_{u\in\Lambda} \{(1u)\hat{h}^{(u)}(1x)H^*(1u)\}$$

is either empty or contains an unique element g_x. Therefore we have:

$$g_x = \bigcap_{u\in\Lambda} \{(1u)\hat{h}^{(u)}(1x)H^*(1u)\},$$

$$g_x \in \{(1u)\hat{h}^{(u)}(1x)H^*(1u)\} \quad \text{for any } u \in \Lambda,$$

$$g_x(1u)H^* = (1u)\hat{h}^{(u)}(1x)H^* \quad \text{for any } u \in \Lambda,$$

$$\hat{g}_x(u) = u \bullet \hat{h}^{(u)}(x) \quad \text{for any } u \in \Lambda,$$

where operation "•" corresponds to the transversal

$$P_0 = \{(1x) | x \in \Lambda\}.$$

So the following two conditions are equivalent:

a) for any $\alpha \in S_d$

$$\prod_{x \in \Lambda} \{\hat{G} \cap (\bigcap_{u \in \Lambda} ((1u) \hat{h}^{(u)} (1\alpha(x)) H^{*}(1u)))\} \neq \varnothing;$$

b) for any $x \in \Lambda$ the mapping

$$\gamma_x(u) = u \bullet \hat{h}^{(u)}(x)$$

is a permutation on Λ.

Let us define the following operation on Λ:

$$x \cdot y = \gamma_x(y) = y \bullet \hat{h}^{(y)}(x)$$

and prove that the system $<\Lambda, \cdot, 1>$ is a loop.

We have for the operation "•" (see (1)):

$$x \bullet y = \begin{cases} x, & if \quad y = 1; \\ 1, & if \quad y = x; \\ y, & if \quad y \neq 1, x; \end{cases}$$

Therefore

$$x \cdot 1 = 1 \bullet \hat{h}^{(1)}(x) = 1 \bullet x = x;$$

$$1 \cdot x = x \bullet \hat{h}^{(x)}(1) = x \bullet 1 = x.$$

We have for given $a, b \in \Lambda$:

$$a \cdot x = b \Leftrightarrow \gamma_a(x) = b \Leftrightarrow x = \gamma_a^{-1}(b).$$

Since γ is a permutation on Λ, such an element x in Λ is unique. Finally, we get for arbitrary given $a, b \in \Lambda$:

$$x \cdot a = b,$$

$$a \bullet \hat{h}^{(a)}(x) = b,$$

$$\hat{h}^{(a)}(x) = \begin{cases} 1, & if \quad b = a; \\ a, & if \quad b = 1; \\ b, & if \quad \hat{h}^{(a)}(x) \neq 1, a; \end{cases}$$

$$x = \begin{cases} 1, & if \quad b = a; \\ (\hat{h}^{(a)})^{-1}(a), & if \quad b = 1; \\ (\hat{h}^{(a)})^{-1}(b), & if \quad b \neq 1, a; \end{cases}$$

So the system $<\Lambda,\cdot,1>$ is a loop.

Finally we can see:

$$\hat{g}_a(x) = x \bullet \hat{h}^{(x)}(a) = a \cdot x = L_a(x),$$

where L_a is a left translation on $<\Lambda,\cdot,1>$. Since $\hat{g}_a \in \hat{G}$, then $L_a \in \hat{G}$, i.e. the set

$$L = \{L_a | a \in \Lambda\}$$

is a loop transversal in G to H. The proof is complete.

Lemma 17. *In notations of* **Lemma 16** *the following assertions is true for any* $\pi \in S_d$

$$\pi K(G)\pi^{-1} = K(\pi G \pi^{-1}).$$

Proof. Using the proof of **Lemma 16** we have:

$$K(G) = \bigcup_L \bigcup_{\alpha \in S_d} \{\prod_{x \in \Lambda} (\hat{G} \cap L_{\alpha(x)})\},$$

where L is an arbitrary loop on Λ. Then we have for any $\pi \in S_d$:

$$\pi K(G)\pi^{-1} = \bigcup_L \bigcup_{\alpha \in S_d} \{\pi(\prod_{x \in \Lambda} (\hat{G} \cap L_{\alpha(x)}))\pi^{-1}\} =$$

$$= \bigcup_L \bigcup_{\alpha \in S_d} \{\prod_{x \in \Lambda} ((\pi \hat{G} \pi^{-1}) \cap (\pi L_{\alpha(x)} \pi^{-1}))\} =$$

$$= \bigcup_{L'} \bigcup_{\alpha' \in S_d} \{\prod_{x \in \Lambda} ((\pi \hat{G} \pi^{-1}) \cap L'_{\alpha'(x)})\} = K(\pi \hat{G} \pi^{-1}),$$

where L'_a is a left translation in a some loop L'. The last equality is based on the following sequence of statements:

T is a loop transversal in G to H $\Leftrightarrow$

for any $\alpha \in G$ the set $(\alpha T \alpha^{-1})$ is a left transversal in G to H $\Leftrightarrow$

$$\text{for any } \pi \in S_d \text{ and } \alpha \in G \text{ the set } (\pi\alpha T(\pi\alpha)^{-1}) \text{ is}$$

$$\text{a left transversal in } (\pi \hat{G} \pi^{-1}) \text{ to } (\pi \hat{H} \pi^{-1}) \Leftrightarrow$$

$$\text{for any } \pi \in S_d \text{ and } \alpha_1 \in (\pi G \pi^{-1}) \text{ the set } (\alpha_1 (\pi T \pi^{-1}) \alpha_1^{-1}) \text{ is}$$

$$\text{a left transversal in } (\pi \hat{G} \pi^{-1}) \text{ to } (\pi \hat{H} \pi^{-1}) \Leftrightarrow$$

$$T = (\pi T \pi^{-1}) \text{ is a loop transversal in } (\pi \hat{G} \pi^{-1}) \text{ to } (\pi \hat{H} \pi^{-1}).$$

The proof is complete.

References

1. *Baer R.* Nets and groups. 1. Trans. Amer. Math. Soc., v.**46**(1939), p.110-141.

2. *Johnson K.W.* *S*-rings over loops, right mapping groups and transversals in permutation groups. Math. Proc. Camb. Phil. Soc., v.**89**(1981), p.433-443.

3. *Ore O.* On coset representatives in groups. Proc. Amer. Math. Soc., v.**9**(1958), p.665-670.

4. *Kargapolov M.I., Merzlyakov Y.I.* Foundations of group theory. (Russian), Ed. 3, Moscow, "Nauka", 1982.

5. *Belousov V.D.* Foundations of quasigroup and loop theory. (Russian), Moscow, "Nauka", 1967.

6. *Niemenmaa M., Kepka T.* On multiplication groups of loops. J. of Algebra, v.**135**(1990), p.112-122.

Kuznetsov E.A. **Ph.D.**

department of quasigroup theory,

Institute of Mathematics,

Academy of Sciences of Moldova,

5, Academiei str.,

Kishinau, 277028,

Moldova.

Received August 15, 1993