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On loops with universal elasticity

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Abstract

A property P of a loop $Q(\cdot)$ is called universal for $Q(\cdot)$ if it holds in every loop isotopic to $Q(\cdot)$ ([1,2]). Loops with universal law of elasticity are considered in this article. Necessary and sufficient conditions for a commutative IP-loop with universal elasticity to be a Moufang loop are proved.

Loops with universal law of elasticity $x \cdot yx = xy \cdot x$ were mentioned in [2] and partially studied in [6]. It is our purpose in this paper to continue the study of algebraic properties of loops with universal elasticity. It was shown in [6] that the identity of elasticity $x \cdot yx = xy \cdot x$ is universal for a loop $Q(\cdot)$ if and only if one of identities

$$x\backslash[(xy/b)(a\backslash xb)] = a\backslash[(ay/b)(a\backslash xb)] \tag{1}$$

or

$$[(bx/a)(b\backslash yx)]/x = [(bx/a)(b\backslash ya)]/a\,, \tag{2}$$

holds in the primitive loop $Q(\cdot, /, \backslash)$. So, the identities (1) and (2) are equivalent in $Q(\cdot, /, \backslash)$. More, they are symmetric (dual). If $Q(\cdot)$ is an IP-loop then each of the identities (1) and (2) is equivalent in $Q(\cdot)$ to the following identity:

$$[x(az \cdot y) \cdot zx]a = x[az \cdot (y \cdot zx)a]\,, \tag{3}$$

which is universal for a loop $Q(\cdot)$ if and only if $Q(\cdot)$ is a Moufang loop.

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It is clear that each Moufang loop is a loop with universal elasticity. More, the class of Moufang loops is strictly contained in that of loops with universal elasticity. The following examples give loops with universal elasticity which are not Moufang.

Example 1. Let $R(+,\cdot)$ be the ring of integers modulo 2 and $Q = R^3$. Define on Q the operation $(\cdot)$ as follows:

$$(i\,,j\,,k)\cdot(p\,,q\,,r) = (i+p\,,j+q\,,k+r+jp+jpq+ijq)\,.$$

$Q(\cdot)$ is a loop with universal elasticity of order 8 and exponent 4 (with $(0,0,0)$ as a neutral element). □

Example 2. Let $Q(+,\cdot)$ be the ring from Example 1 and $Q = R^3$. Define on Q the operation $(\cdot)$:

$$(i\,,j\,,k)\cdot(p\,,q\,,r) = (i+p\,,j+q\,,k+r+ijp+ipq).$$

Then $Q(\cdot)$ is a loop with universal elasticity (not Moufang). □

In what follows will be useful the identity

$$(bx\cdot ay)x\cdot ab = bx\cdot a(yx\cdot ab)\,, \tag{4}$$

which is equivalent to (3). Indeed, making the substitutions:$a\to az^{-1}$, $x\to z^{-1}x$ and after this $z^{-1}\to b$ in (3), we find the identity (4), and analogously, from (4) follows (3).

I. As it was shown in [6], loops with universal elasticity are strong power-associative (i.e. each element generates an associative subloop).

Let $Q(\circ)$ be a strong power-associative loop. Define a new loop $Q(\cdot)$ as follows:

$$x\cdot y = x//y^{(-1)}\,,$$

where "$//$" is the left division in $Q(\circ)$, $y\cdot y^{(-1)} = 1$, 1 is the neutral element of $Q(\cdot)$. Then

$$x\circ y = x/y^{(-1)}\,,$$

where "$/$" is the left division in $Q(\cdot)$.

Proposition 1.
i) $1 = e$, *where* e *is the unit of* $Q(\circ)$,
ii) *if* $x \circ x^{-1} = e$, *then* $x^{-1} = x^{(-1)} = {}^{(-1)}x$,
where ${}^{-1}x \cdot x = x \cdot x^{(-1)} = 1$ *for every* x *in* $Q(\cdot)$.

Proof. i) Indeed, as

$$x = x \circ e = x/e^{(-1)},$$

we have $x = x \cdot e^{(-1)}$. So $e^{(-1)} = 1$, or $e = 1$.

ii) $Q(\circ)$ is strong power-associative and

$$e = x^{-1} \circ x = x^{-1}/x^{(-1)},$$

so

$$x^{-1} = ex^{(-1)} = x^{(-1)}.$$

Now

$${}^{(-1)}x \cdot x = e \implies {}^{(-1)}x = e/x = e \circ x^{-1} = x^{-1}. \qquad \square$$

Proposition 2. *$Q(\cdot)$ is a LIP-loop if and only if the permutation $I : x \to x^{-1}$ is an antiautomorphism of $Q(\circ)$.*

Proof. Let $Q(\cdot)$ be a LIP-loop, i.e. $x^{-1} \cdot xy = y$ for every x, y in Q hence $y/xy = x^{-1}$. Making the substitution $x \to x/y$ and using the strong power-associativity of $Q(\circ)$, we get $y/x = (x/y)^{-1}$, i.e.

$$y \circ x^{-1} = (x \circ y^{-1})^{-1} \quad \text{or} \quad y^{-1} \circ x^{-1} = (x \circ y)^{-1}.$$

The proof is reversible. $\square$

Remind that B_3-loops have been considered in [3, 4] by A.Gwaramija. A loop $Q(\cdot)$ is called a *B_3-loop* (or *a medial Bol loop*) if $Q(\cdot, /, \backslash)$ satisfies the identity

$$x(yz\backslash x) = (x/z)(y\backslash x).$$

Mention that the loops from Examples 1 and 2 are B_3-loops as well. We have below another proof of a Gwaramija's result.

Corollary. (A. Gwaramija [4]) *If* $Q(\cdot)$ *is a left Bol loop, then* $Q(\circ)$ *is a* B_3*-loop.*

Proof. It is known ([2]) that a B_3-loop is a loop for which the identity

$$(xy)^{-1} = y^{-1}x^{-1}$$

is universal and a left Bol loop is a loop with universal LIP-property. Let $Q(\circ)$ be a B_3-loop and consider an arbitrary principal isotope $(*) = (\circ)^{(\alpha,\beta,\epsilon)}$ of $Q(\circ)$. If $Q(\circ)$ is a B_3-loop, then $I(x*y) = Iy * Ix$, or

$$I(\alpha x/I\beta y) = I(\alpha x \circ \beta y) = I(x*y) = Iy * Ix = \alpha Iy \circ \beta Ix = \alpha Iy/I\beta x.$$

So we get

$$I(\alpha x/I\beta y) = \alpha Iy/I\beta Ix\,,$$

or, making the substitution $x \to \alpha^{-1}x$ and after that $x \to x \cdot I\beta y$ in the last equality, we obtain:

$$Ix = \alpha Iy/[I\beta I\alpha^{-1}(x \cdot I\beta y)]\,,$$
$$\alpha Iy = Ix \cdot [I\beta I\alpha^{-1}(x \cdot I\beta y)]\,,$$

or finally,

$$y = Ix \bullet (x \bullet y)\ ,$$

where $(\bullet) = (\cdot)^{(\epsilon, I\beta I\alpha^{-1}, \epsilon)}$. So, if $Q(\cdot)$ is a left Bol loop, then each its principal isotope $(\bullet)$ (defined above) has the LIP-property and this fact implies that every principal isotope $(*) = (\circ)^{(\alpha,\beta,\epsilon)}$ of $Q(\circ)$ satisfies the identity $I(x*y) = Iy * Ix$. □

Proposition 3. $Q(\circ)$ *is a* RIP*-loop if and only if* $Q(\cdot)$ *is a* RIP*-loop.*

Proof. Let $Q(\circ)$ be a RIP-loop. If by "$//$" is denoted the left division in $Q(\circ)$, then we have $x//y = xy^{-1}$. So

$$(y \circ x) \circ x^{-1} = y \implies y//x^{-1} = y \circ x \implies yx = y \circ x$$

for every x, y in Q. Conversely, if $yx \cdot x^{-1} = y$ for every x, y in Q then $y/x^{-1} = u \cdot x$ or $y \circ x = y \cdot x$. So the operations $(\cdot)$ and $(\circ)$ coincide in both cases. □

Let $Q(\circ)$ be a loop with universal elasticity and $Q(\cdot)$ be the loop defined in previous propositions: $x \circ y = x/y^{-1}$ for every $x, y \in Q$ (where $(/)$ is the left division for $(\cdot)$).

Proposition 4. *$Q(\cdot)$ is left alternative if and only if the identity*

$$(x \circ y \circ x^{-1})^2 = (x \circ y) \circ (y \circ x^{-1})$$

holds in $Q(\circ)$.

Proof. If $Q(\circ)$ is left alternative, i.e. $x \cdot xy = x^2 \cdot y$, for every x, y in Q, then the equality

$$x//(x//y^{-1})^{-1} = (x//x^{-1})//y^{-1}$$

is true for every x, y in Q as $x//y = xy^{-1}$. Now using $x//x^{-1} = x^2$ in the previous equality ($Q(\circ)$ is strong power-associative) and replacing x by $x \circ y^{-1}$ we get:

$$(x \circ y^{-1})//x^{-1} = (x \circ y^{-1})^2//y^{-1}$$

or

$$[(x \circ y^{-1})//x^{-1}] \circ y^{-1} = (x \circ y^{-1})^2 \,. \tag{5}$$

But $Q(\circ)$ is a loop with universal elasticity, so it satisfies the identity

$$(x \circ y) \circ x^{-1} = x \circ (y \circ x^{-1})$$

(which is a corollary of (1) for $x = b = e$). From the last identity (using $y \to y^{-1}//x^{-1}$) we get

$$(x \circ y^{-1})//x^{-1} = x \circ (y^{-1}//x^{-1})$$

and using this identity in (5) it follows

$$[x \circ (y^{-1}//x^{-1})] \circ y^{-1} = (x \circ y^{-1})^2 \,,$$

or after replacing y^{-1} by $y \circ x^{-1}$ in the last equality:

$$(x \circ y) \circ (y \circ x^{-1}) = (x \circ y \circ x^{-1})^2 \,.$$

The proof is reversible. □

II. An element a of a loop $Q(\cdot)$ is called *Moufang*, if for every $x, y \in Q$ we have $ax \cdot ya = a(xy \cdot a)$ ([5]). Denote

$$M = \{a \in Q \,:\, ax \cdot ya = a(xy \cdot a) \;\; \forall\, x, y \in Q\}\,.$$

The *Moufang center* of a loop is defined as the set of all elements c such that

$$c^2 \cdot xy = cx \cdot cy$$

for all $x, y \in Q$. It is known ([5]), that the Moufang center of a Moufang loop $Q(\cdot)$ is a commutative subloop of $Q(\cdot)$. More, if N is the nucleus, C is the Moufang center and Z is the center of a Moufang loop, then $N \cap C = Z$. For an arbitrary loop $Q(\cdot)$ the fact that $c \in C$ does not necessarily imply $cx = xc$.

Proposition 5. *The Moufang center $C(\cdot)$ of an IP-loop with universal elasticity $Q(\cdot)$ is a commutative subloop of $Q(\cdot)$.*

Proof. Let $a \in C$ and put $y = e$ in $a^2 \cdot xy = ax \cdot ay$, where e is the unit of $Q(\cdot)$. We get

$$a^2x = a^2(xe) = ax \cdot a = a \cdot xa\,.$$

So

$$a^2 \cdot x = a \cdot ax = a \cdot xa$$

and $ax = xa$ for every $x \in Q$. Here was used the law of left alternativity which holds in IP-loops with universal elasticity as was proved in [6]. We shall prove below that $C(\cdot)$ is a subloop of $Q(\cdot)$. If $a \in C$ then replacing x by au and y by av in the equality $a^2 \cdot xy = ax \cdot ay$ we get

$$a^2(au \cdot av) = a^2u \cdot a^2v\,, \quad \text{or} \quad a^4 \cdot uv = a^2u \cdot a^2v$$

for every $u, v \in Q$. Hence $a^2 \in C$. Let $a, b \in C$. Then

$$a^4(b^2 \cdot xy) = a^4(bx \cdot by) = (a^2 \cdot bx)(a^2 \cdot by) = (ab \cdot ax)(ab \cdot ay)\,,$$

for every $x, y \in Q$. But

$$a^4(b^2 \cdot xy) = a^2b^2 \cdot (a^2 \cdot xy) = a^2b^2 \cdot (ax \cdot ay)\,,$$

for every $x, y \in Q$ and

$$a^2b^2 = a^2 \cdot bb = ab \cdot ab = (ab)^2 \,,$$

so we get

$$(ab)^2(ax \cdot ay) = (ab \cdot ax)(ab \cdot ay) \,,$$

for every $x, y \in Q$, i.e. $ab \in C$. If $a \in C$ then we get

$$a^{-2} \cdot uv = a^{-1}u \cdot a^{-1}v \,,$$

after replacing x by $a^{-1}u$ and y by $a^{-1}v$ in $a^2 \cdot xy = ax \cdot ay$. So, $a^{-1} \in C$ and $C(\cdot)$ is a commutative subloop of $Q(\cdot)$. □

Corollary. *If $Q(\cdot)$ is an IP-loop with universal elasticity then $C \le M \le Q$.* □

The following proposition contains some properties of the Moufang elements in IP-loops with universal elasticity.

Proposition 6. *Let $Q(\cdot)$ be an IP-loop with universal elasticity. Then $a \in M$ if and only if at least one of the equalities*

i)	$(ax \cdot y)x = a \cdot xyx,$	*ii)*	$axa \cdot y = a(x \cdot ay),$
iii)	$x(y \cdot xa) = xyx \cdot a,$	*iv)*	$x \cdot aya = (xa \cdot y)a,$
v)	$xy \cdot ax = x \cdot ya \cdot x,$	*vi)*	$xa \cdot yx = x \cdot ay \cdot x$

holds for every $x, y \in Q$.

Proof. To prove this proposition we need the following identities which are corollaries of (3) and (4):

(a) $\quad (x \cdot ay \cdot x)a = x(a \cdot yx \cdot a) \,,$

(b) $\quad (uv \cdot zu)v = u(vz \cdot uv) \,,$

(c) $\quad (v \cdot uy) \cdot uv = v[u(y \cdot uv)] \,,$

(d) $\quad vu \cdot (yu \cdot v) = [(vu \cdot y)u]v \,.$

Indeed, the identity (a) can be obtained from (3) taking $z = e$; the identity (b) is a corollary of (4) for $x \to b^{-1}u$, $a \to vb^{-1}$, $y = b$ and after this $b^{-1} = z$; the identity (c) was obtained replacing x by z^{-1}, a by uz^{-1} and after that z^{-1} by v in (3); (c) and (d) are symmetric. Consider $c \in M$ and substitute x by c in (a):

$$(c \cdot ay \cdot c)a = c(a \cdot yc \cdot a).$$

Hence

$$(ca \cdot yc)a = c(a \cdot yc \cdot a)$$

or, replacing yc by y:

$$(ca \cdot y)a = c \cdot aya$$

for every $a, y \in Q$, so i) is proved.

Analogously, taking $u = c$ in (b) we get: $czc \cdot v = c(x \cdot cv)$ for every $v, z \in Q$, i.e. ii) holds. For $a = c$ in (a) we have:

$$(x \cdot cy \cdot x)c = x(c \cdot yx \cdot c) = x(cy \cdot xc),$$

so

$$(x \cdot cy \cdot x)c = x(cy \cdot xc),$$

and after replacing cy by y: $xyx \cdot c = x(y \cdot xc)$ for every $x, y \in Q$, i.e. iii) is proved.

Taking $v = c$ in (b) we get

$$(uc \cdot zu)c = u(cz \cdot uc) = u(c \cdot zu \cdot c),$$

so

$$(uc \cdot zu)c = u(c \cdot zu \cdot c)$$

or, replacing zu by z: $(uc \cdot x)c = u \cdot czc$ for every $u, z \in Q$, i.e. iv) is proved.

Substitute now u by c in (c) and using ii) we get:

$$(v \cdot cy)cv = v[c(y \cdot cv)] = v(cyc \cdot v),$$

hence

$$(v \cdot cy) \cdot cv = v(cyc \cdot v),$$

or $vz \cdot cv = v(zc \cdot v)$, where by z was denoted cy, i.e. we get v). Analogously, taking $u = c$ in (d) we have:

$$vc \cdot (yc \cdot v) = [(vc \cdot y)c]v = (v \cdot cyc)v,$$

so

$$vc \cdot (yc \cdot v) = (v \cdot cyc)v,$$

or (putting $yc = z$) $vc \cdot zv = (v \cdot cz)v$, for every $v, z \in Q$, i.e. vi) is proved. In each of this cases the proof is reversible, so Proposition 6 is proved. □

A bijection γ on Q is called a *right (left) pseudo-automorphism* of a quasigroup $Q(\cdot)$ if there exists at least one element $c \in Q$ such that

$$\gamma x \cdot (\gamma y \cdot c) = \gamma(xy) \cdot c\,, \qquad (c \cdot \gamma x)\gamma y = c \cdot \gamma(xy)\,,$$

for every $x, y \in Q$. The element c is called a *companion* of γ ([1,5]). It is known ([1]), that every companion of a pseudo-automorphism in IP-loops is a Moufang element. Let $Q(\cdot)$ be a loop with the law of elasticity. A bijection Θ on Q is called a *semiautomorphism* of $Q(\cdot)$, if

$$\Theta(xyx) = \Theta x \cdot \Theta y \cdot \Theta x$$

for every $x, y \in Q$ and $\Theta e = e$. It is known ([1]) that every pseudo-automorphism of a Moufang loop is its semiautomorphism.

Proposition 7. *Any pseudo-automorphism of an IP-loop $Q(\cdot)$ with universal elasticity is its semiautomorphism.*

Proof. Let φ be a pseudo-automorphism of a loop $Q(\cdot)$ with universal elasticity and c be a right companion of φ. Since

$$\varphi(xy) \cdot c = \varphi x \cdot (\varphi y \cdot c)$$

for every $x, y \in Q$, then

$$\varphi(xyx) \cdot c = \varphi x \cdot [\varphi(yx) \cdot c] = \varphi x \cdot [\varphi y \cdot (\varphi x \cdot c)] = (\varphi x \cdot \varphi y \cdot \varphi x)c$$

because c is a Moufang element and so we can apply here iii) from Proposition 6. Hence $\varphi(xyx) = \varphi x \cdot \varphi y \cdot \varphi x$ for every $x, y \in Q$. The proof is analogous in the case when c is a left companion of φ. Note that $\varphi e = e$ for every pseudo-automorphism of an IP-loop with universal elasticity. Indeed, such loops are left and right alternative, so taking $x = y = e$ in $\varphi(xy) \cdot c = \varphi x \cdot (\varphi y \cdot c)$, we get $\varphi e \cdot c = (\varphi e)^2 c$, thus $e = \varphi e$.

III. Let $Q(\cdot)$ be an IP-loop with universal elasticity and define on Q the operation $(+)$ by

$$x + y = xy^{-1}x$$

for every $x, y \in Q$. Then the groupoid $Q(+)$ is called the *core* of $Q(\cdot)$.

Remind ([1]) that the core of a Moufang loop $Q(\cdot)$ is a left-distributive groupoid and it is a quasigroup if and only if the mapping $x \to x^2$ is a permutation on Q. For our class of loops an analogous proposition is true.

Proposition 8. *The core $Q(+)$ of an IP-loop with universal elasticity is a quasigroup if and only if the mapping $x \to x^2$ is a permutation on Q.*

Proof. Let $Q(+)$ be a quasigroup. Then there exists for each $a \in Q$ an unic element $x \in Q$ such that $x + e = a$, where e is the unity of the loop $Q(\cdot)$. So the equation $xe^{-1}x = x^2 = a$ has an unique solution in $Q(\cdot)$ and consequently, $x \to x^2$ is a permutation on Q.

Conversely, suppose that the mapping $\psi : x \to x^2$ is a permutation on Q. The equation $a + x = b$ where $a, b \in Q$ or $ax^{-1}a = b$ has the unique solution $x = ab^{-1}a$. Consider now the equation $x + a = b$, i.e. $xa^{-1}x = b$, where $a, b \in Q$. The last equation is equivalent to $xa^{-1}x \cdot a^{-1} = ba^{-1}$. Returning to (3) and making the substitution $x = e,\ y = a$, we get

$$[(az \cdot a)z]a = az \cdot (az \cdot a)\,,$$

or, using the alternativity of $Q(\cdot)$,

$$(az \cdot a)z \cdot a = (az)^2 \cdot a\,,$$

so by (a) from Proposition 6:

$$aza \cdot z = a \cdot zaz = (az)^2$$

for every $a, z \in Q$. Now the considered equation $x \cdot a^{-1}xa^{-1} = ba^{-1}$ can be represented as follows:

$$(xa^{-1})^2 = ba^{-1}$$

and it has an unique solution because ψ is a permutation on Q. $\square$

Note that the core of an IP-loop with universal elasticity is a groupoid with the law of elasticity. Indeed, using (c),

$$(x + y) + x = xy^{-1}x \cdot (x^{-1} \cdot xy^{-1}x) = xy^{-1}x \cdot y^{-1}x =$$
$$= x(y^{-1}x)^2 = x(y^{-1}xy^{-1} \cdot x) = x(yx^{-1}y)^{-1}x = x + (y + x)$$

for every $x, y \in Q$.

Proposition 9. *The core* $Q(+)$ *of an* IP*-loop*$Q(\cdot)$ *with universal elasticity is a left-distributive groupoid if and only if the following identity*

$$x(y \cdot xzx \cdot y)x = xyx \cdot z \cdot xyx \tag{6}$$

holds in $Q(\cdot)$.

Proof. Let $Q(+)$ be a left-distributive groupoid, i.e.

$$x + (y + z) = (x + y) + (x + z)\,,$$

or

$$x \cdot y^{-1}zy^{-1} \cdot x = xy^{-1}x \cdot x^{-1}zx^{-1} \cdot xy^{-1}x\,.$$

After replacing y^{-1} by y and z by xzx in the last identity we shall obtain (6). □

Corollary 1. ([1]) *The core of a Moufang loop is a left-distributive groupoid.* □

Corollary 2. *If* $Q(\cdot)$ *is a commutative* IP*-loop with universal elasticity for which the mapping* $x \to x^2$ *is a bijection, then* $Q(\cdot)$ *is a commutative Moufang loop if and only if its core* $Q(+)$ *is a left-distributive quasigroup.*

Proof. Let $Q(+)$ be a left-distributive quasigroup. The identities of alternativity and the identity (d) hold in $Q(\cdot)$ (see [6]). So, from (6) we have:

$$x^2(y^2 \cdot x^2z) = (x^2y)^2z = x^4y^2 \cdot z = x^2y^2x^2 \cdot z\,.$$

For $x^2 \to x$ and $y^2 \to y$ in the last identity we get

$$x(y \cdot xz) = xyx \cdot z\,,$$

thus $Q(\cdot)$ is a commutative Moufang loop. The converse statement is proved in [1]. □

Proposition 10. *The core $Q(+)$ of an IP-loop $Q(\cdot)$ with universal elasticity is right-distributive if and only if the identity*

$$xyx \cdot z \cdot xyx = xzx \cdot yz^{-1}y \cdot xzx \tag{7}$$

holds in $Q(\cdot)$.

Proof follows from the law of right-distributivity. □

Corollary. *Let a quasigroup $Q(+)$ be the left-distributive core of an IP-loop with universal elasticity $Q(\cdot)$. Then $Q(+)$ is right-distributive if and only if the identity*

$$xy^2x = yx^2y \tag{8}$$

holds in $Q(\cdot)$.

Proof. Let $Q(\cdot)$ be a loop with the identity (8). For $y \to zyz$ in (8) we get

$$x(zyz)^2x = zyz \cdot x^2 \cdot zyz$$

or, using the left-distributivity of $Q(+)$ and (8):

$$x(zyz)^2x = z(y \cdot zx^2z \cdot y)z\,.$$

Now we shall apply (8) to the last identity:

$$x(zyz)^2x = z(y \cdot xz^2x \cdot y)z\,,$$

or, after replacing $y \to z^{-1}yz^{-1}$:

$$xy^2x = z(z^{-1}yz^{-1} \cdot xz^2x \cdot z^{-1}yz^{-1})z\,.$$

So

$$z^{-1} \cdot xy^2x \cdot z^{-1} = z^{-1}yz^{-1} \cdot xz^2x \cdot z^{-1}yz^{-1}$$

and, making the substitutions $y \to y^2$, $z \to z^{-1}$ in the last identity and using (8),

$$z \cdot y^2x^2y^2 \cdot z = yz^2y \cdot xz^{-2}x \cdot yz^2y\,.$$

But $Q(+)$ is a left-distributive groupoid, hence taking $z = e$ in (6), we get

$$x \cdot yx^2y \cdot x = (xyx)^2\,.$$

Thus from (8) and the last identity it follows

$$x \cdot xy^2x \cdot x = x^2y^2x^2 = (xyx)^2$$

($(x^py)x^q = x^p(yx^q)$ in all IP-loops) and from these identities, using (8), we get

$$z(yxy)^2z = yz^2y \cdot xz^{-2}x \cdot yz^2y\,,$$

or

$$yxy \cdot z^2 \cdot yxy = yz^2y \cdot xz^{-2}x \cdot yz^2y\,.$$

Making the substitution $z^2 \to z$ in the last identity, we obtain

$$yxy \cdot z \cdot yxy = yzy \cdot xz^{-1}x \cdot yzy\,,$$

i.e. $Q(+)$ is right-distributive.

Conversely, let $Q(+)$ be a right-distributive and left-distributive groupoid. Then the identities (6) and (7) hold in $Q(\cdot)$. Hence

$$x(y \cdot xzx \cdot y)x = xzx \cdot yz^{-1}y \cdot xzx\,,$$

and for $z = e$

$$x \cdot yx^2y \cdot x = x^2y^2x^2 = x \cdot xy^2x \cdot x\,.$$

Thus $yz^2y = xy^2x$, which completes the proof. □

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