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On topological n-ary semigroups

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Abstract

In this note some we describe topologies on n-ary semigroups induced by families of deviations.

1. Introduction

Topological n–groups were investigated by many authors. For example, Čupona proved in [5] that each topological n–group can be embedded into a topological group. Žižović described topological medial n–groups (cf. [20]), topological n–groups with the Baire property (cf. [21]) and proved a topological analog of Hosszú theorem (cf. [19]). Crombez and Six described a fundamental system of open neighborhoods of a fixed element (cf. [4]). Endres proved that every topological n–group is homeomorphic to some canonical topological group (cf. [9]). Topologies induced by norms are considered by Boujuf and Mukhin (cf. [2]). Balci Dervis (cf. [1]) described free topological n–groups. In [12] is described a method of embedding topological abelian n–semigroups in topological n–group.

On the other hand, we known that topological n–semigroups have many properties which are not true for binary semigroups.

In this paper we investigate topologies on n–semigroups and n–groups determined by families of left invariant deviations. We describe

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the conditions under which such topology is compatible with the n–ary operation. We find also the necessary and sufficient conditions for the topologically embedding a semiabelian topological n–semigroup in a topological n–group.

2. Preliminaries

Traditionally in the theory of n-ary groups we use the following abbreviated notation: the sequence $x_i, ..., x_j$ is denoted by x_i^j (for $j < i$ this symbol is empty). If $x_{i+1} = ... = x_{i+k} = x$, then instead of x_{i+1}^{i+k} we write $\overset{(k)}{x}$. Obviously $\overset{(0)}{x}$ is the empty symbol. In this notation the formula

$$f(x_1, ..., x_i, x_{i+1}, ..., x_{i+k}, x_{i+k+1}, ..., x_n)\,,$$

where $x_{i+1} = ... = x_{i+k} = x$, will be written as $f(x_1^i, \overset{(k)}{x}, x_{i+k+1}^n)$.

If $m = k(n-1)+1$, then the m-ary operation g given by

$$g(x_1^{k(n-1)+1}) = \underbrace{f(f(..., f(f}_{k-times}(x_1^n), x_{n+1}^{2n-1}), ...), x_{(k-1)(n-1)+2}^{k(n-1)+1})$$

will be denoted by $f_{(k)}$. In certain situations, when the arity of g does not play a crucial role, or when it will differ depending on additional assumptions, we write $f_{(.)}$, to mean $f_{(k)}$ for some $k = 1, 2, ...$

An n–ary operation f defined on G is called *associative* if

$$f(f(x_1^n), x_{n+1}^{2n-1}) = f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1})$$

holds for all $x_1, x_2, ..., x_{2n-1} \in G$ and $i = 1, 2, ..., n$. The set G together with one associative operation f is called an *n–ary semigroup* (briefly: *n–semigroup*). An n–semigroup (G, f) in which for for all $a_1, a_2, ..., a_n, b \in G$ there exits an uniquely determined $x_i \in G$ such that $f(a_1^{i-1}, x_i, a_{i+1}^n) = b$ is called an *n–group*.

From this definition it follows that a group (a semigroup) is a 2-group (a 2–semigroup) in the above sense. Moreover, it is worthwhile to note that, under the assumption of the associativity of f, it suffices only to postulate the existence of a solution of the last equation at the places $i=1$ and $i=n$ or at one place i other than 1 and n (cf. [13], p.213[17]). This means that an n–group may be considered as an algebra (G,f,f_1,f_n) with one associative n–ary operation f and two n–ary operations f_1,f_n such that

$$f(f_1(a_2^n,b),a_2^n)=f(a_a^n,f_n(a_2^n,b))=b \tag{1}$$

for all $a_2^n,b\in G$.

Following E.L.Post ([13], p.282) the solution of the equation

$$f(x,a,...,a,f(a,...,a))=a$$

is denoted by $a^{[-2]}$. An n–semigroup (G,f) with an unary operation $^{[-2]}:G\rightarrow G$ satisfying some natural identities is an n–group (cf. [16]).

The map $x\mapsto f(a_1^{j-1},x,a_{j+1}^n)$ is called an *j–th n–ary translation* determined by $a_1,...,a_n$. In an n–group each n–ary translation is a bijection.

In an n–group (G,f) for any sequence a_1^{n-2} there exists only one $a\in G$ such that

$$f(x,a_1^{n-2},a)=f(a_1^{n-2},a,x)=f(a,a_1^{n-2},x)=f(x,a,a_1^{n-2})=x$$

for all $x\in G$ (cf. [17]). An element a is called *inverse* for a_1^{n-2}. In the binary case, i.e. in the case $n=2$, when the sequence a_1^{n-2} is empty by the inverse we mean the neutral element of a group (G,f) .

A sequence a_2^n is called a *left (right) neutral sequence* if $f(a_2^n,x)=x$ (respectively $f(x,a_2^n)=x$) holds for all $x\in G$. A left and right neutral sequence is called a *neutral sequence*. In an n–group for every sequence a_1^{n-2} may be extended to a neutral sequence, but there are n–semigroups without left (right) neutral sequences.

Let (G,f) be an n–semigroup and let a_2^{n-1} be fixed. Then $(G,*)$, where

$$x*y=f(x,a_2^{n-1},y) \tag{2}$$

is a semigroup, which is called a *binary retract* of (G,f) and is denoted by $ret_{a_2^{n-1}}(G,f)$. A binary retract of an n–group is a group. Moreover, all binary retracts of a given n–group are isomorphic (cf. [7]), but n–groups with the same retract are not isomorphic, in general.

By so-called Hosszú theorem (cf. [11] or [7]), every n–group (G, f) has the form

$$f(x_1^n) = x_1 * \beta(x_2) * \beta^2(x_3) \ldots * \beta^{n-1}(x_n) * b\,, \tag{3}$$

where a_2^n is a fixed right neutral sequence of (G, f), $(G, *) = ret_{a_2^{n-1}}(G, f)$, $b = f(\overset{(n)}{a_n})$ and $\beta(x) = f(a_n, x, a_2^{n-1})$.

The identical result holds for n–semigroups with a right neutral sequence.

3. Topology

An n–semigroup (G, f) defined on a topological space $(G, \mathcal{T})$ is called a *topological n–semigroup* if the operation f is continuous in all variables together.

A topological n–group is defined as a topological n–semigroup with two additional continuous operations f_1 and f_n satisfying (1) (cf. [5]). A topological n–group may be defined also a topological n–semigroup with additional continuous operation $^{[-2]}$. These definitions are equivalent (cf. [15]).

It is clear that retracts of a topological n–semigroup (n–group) are topological semigroups (groups). Obviously all translations of a topological n–semigroup (n–group) are continuous maps. On the other hand, every n–ary operation which may by written in the form (3), where $*$ and β are continuous, is continuous in all variables together. Thus the following lemma is true.

Lemma 3.1. *Assume that an n–semigroup (G, f) with a topology $\mathcal{T}$ has a right neutral sequence a_2^n. Then $(G, f, \mathcal{T})$ is a topological n–semigroup if and only if $ret_{a_2^{n-1}}(G, f)$ is a topological semigroup and $\beta(x) = f(a_n, x, a_2^{n-1})$ is continuous.* □

Corollary 3.2. *An n–group (G, f) defined on a topological space $(G, \mathcal{T})$ is a topological n–group if and only if there exists a right neutral sequence a_2^n such that $x * y = f(x, a_2^{n-1}, y)$, $\beta(x) = f(a_n, x, a_2^{n-1})$ and $^{[-2]} : x \mapsto x^{[-2]}$ are continuous.* □

Proposition 3.3. *An n–group (G,f) defined on a topological space $(G,\mathcal{T})$ is a topological n–group if and only if there exists a right neutral sequence a_2^n such that $ret_{a_2^{n-1}}(G,f)$ is a topological semigroup, $\beta(x)=f(a_n,x,a_2^{n-1})$ and $s:x\to s(x)$, where $f(s(x),a_2^{n-1},x)=a_n$, are continuous.*

Proof. Let a_2^n be a fixed right neutral sequence on an n–group (G,f). If $(G,*)=ret_{a_2^{n-1}}(G,f)$ is a topological semigroup and $\beta(x)=f(a_n,x,a_2^{n-1})$ is continuous, then (G,f) is a topological n–semigroup by Lemma 3.1.

Moreover, a_n is the neutral element of $(G,*)$ and $s(x)$ is the solution of $f(s(x),a_2^{n-1},x)=a_n$, i.e. $s(x)*x=a_n$ in $(G,*)$. Thus $s(x)$ is the inverse of x in $(G,*)$. Hence $(G,*)$ is a topological group, because $s(x)$ is continuous, by the assumption.

Since $f(z,c_2^n)=f(f(z,a_2^n),c_2^n)=z*f(a_n,c_2^n)$ for all $c_j\in G$, then the solution z of $f(z,c_2^n)=b$ in (G,f) is the solution of $z*f(a_n,c_2^n)=b$ in $(G,*)$, then z continuously depends on b and $f(a_n,c_2^n)$. Thus z is a continuous function of variables $b,c_2,...,c_n$. This, for $b=c_2=...=c_{n-1}=x$, $c_n=f(x,...,x)$, implies that $z=x^{[-2]}$ is a continuous function of x. Thus (G,f) is a topological n–group.

The converse is obvious. □

Corollary 3.4. *Let $\mathcal{T}$ be a locally compact topology on an n–group (G,f) with a right neutral sequence a_2^n. If for every $b\in G$ translations $x\mapsto f(x,a_2^{n-1},b)$, $x\mapsto f(b,a_2^{n-1},x)$ and $x\mapsto f(a_n,x,a_2^{n-1})$ are continuous, then $(G,f,\mathcal{T})$ is a topological n–group.*

Proof. In the group $(G,*)=ret_{a_2^{n-1}}(G,f)$ translations $x\mapsto x*b$ and $x\mapsto b*x$ are continuous for every $b\in G$. Thus, by the theorem of Ellis (cf. Theorem 3 in [8]), $(G,*)$ is a topological group. In this group $s(x)$ defined in the previous Proposition is a continuous operation. Hence (G,f) is a topological n–group. □

4. Deviations

By a *deviation* defined on a nonempty set X we mean every map $\varphi : X \times X \to [0, +\infty)$ such that $\varphi(x,x) = 0$, $\varphi(x,y) = \varphi(y,x)$, and $\varphi(x,y) \le \varphi(x,z) + \varphi(z,y)$ for all $x, y, z \in X$. A deviation φ defined on a semigroup (group) $(G, \cdot)$ is left invariant if $\varphi(cx, cy) = \varphi(x,y)$ for all $c, x, y \in G$. A deviation φ defined on an n–semigroup (G, f) is a *left invariant* if

$$\varphi(f(c_1^{n-1}, x), f(c_1^{n-1}, y)) = \varphi(x,y)$$

for all $x, y, c_1^{n-1} \in G$.

Theorem 4.1 ([2]) **.** *A binary semigroup (group) $(G, \cdot)$ with a topology $\mathcal{T}$ is a topological semigroup (group) if and only if there exists a family Φ of continuous left invariant deviations on G which induces $\mathcal{T}$ and $\varphi_z \in \Phi$ for every $z \in G$ and $\varphi \in \Phi$, where φ_z is defined by $\varphi_z(x,y) = \varphi(xz, yz)$.* □

In the case of an n–semigroup (G, f) every deviation φ on (G, f) induces a new deviation (φ, k, c_2^n) defined by

$$(\varphi, k, c_2^n)(x,y) = \varphi(f(c_2^k, x, c_{k+1}^n), f(c_2^k, y, c_{k+1}^n)),$$

where $c_2^n \in G$ and $k = 1, ..., n$ are fixed.

Theorem 4.2. *Let a_2^n be a right neutral sequence of an n–semigroup (G, f) . If a topology $\mathcal{T}$ on G is induced by the family Φ of deviations such that for all $x, y, z \in G$ and $\varphi \in \Phi$*
(a) $\varphi(f(z, a_2^{n-1}, x), f(z, a_2^{n-1}, y)) = \varphi(x,y)$,
(b) $(\varphi, 1, a_2^{n-1}, z)$, $(\varphi, 2, a_n, a_2^{n-1}) \in \Phi$,
then (G, f) is a topological n–semigroup.

Proof. Let Φ be as in the assumption. By (a) every $\varphi \in \Phi$ is a left invariant deviation on a semigroup $(G, *) = ret_{a_2^{n-1}}(G, f)$. From (b) we obtain

$$\varphi_z(x,y) = \varphi(x * z, y * z) = \varphi(f(x, a_2^{n-1}, z), f(y, a_2^{n-1}, z)) =$$

$$= (\varphi, 1, a_2^{n-1}, z)(x, y)$$

for every $z \in G$, which gives $\varphi_z \in \Phi$. By Theorem 4.1 $(G, *)$ is a topological semigroup.

Let $\varepsilon > 0$. If $x, x_0 \in G$ are such that $(\varphi, 2, a_n, a_2^{n-1})(x, x_0) < \varepsilon$, where $\varphi \in \Phi$, then

$$\varphi(\beta(x), \beta(x_0)) = \varphi(f(a_n, x, a_2^{n-1}), f(a_n, x_0, a_2^{n-1})) =$$
$$= (\varphi, 2, a_n, a_2^{n-1})(x, x_0) < \varepsilon\,,$$

which proves that β is continuous. Lemma 3.1 finish the proof. □

Theorem 4.3. *An n–group (G, f) with a topology $\mathcal{T}$ is a topological n–group if and only if there exists the family Φ of deviations such that a topology $\mathcal{T}$ is induced by Φ and for some right neutral sequence a_2^n of G and for all $x, y, z \in G$, $\varphi \in \Phi$ the conditions (a), (b) from the previous theorem are satisfied.*

Proof. Let $(G, f, \mathcal{T})$ be a topological n–group. Then the retract $(G, *) = ret_{a_2^{n-1}}(G, f)$ is a binary topological group for every choice of $a_2, ..., a_{n-1} \in G$. Thus, by Theorem 4.1, there exists the family Φ of continuous left invariant deviations of $(G, *)$ which induces the topology $\mathcal{T}$. Hence, for all $x, y, z \in G$ and $\varphi \in \Phi$, we have

$$\varphi(f(z, a_2^{n-1}, x), f(z, a_2^{n-1}, y)) = \varphi(z * x,\ z * y) = \varphi(x, y),$$

which proves (a).

Moreover, since for all $a_2, ..., a_{n-1} \in G$ there exista $a_n \in G$ such that a_2^n is a right neutral sequence, then from the above follows

$$\varphi(f(c_1^{n-1}, x), f(c_1^{n-1}, y)) =$$
$$= \varphi(f(c_1^{n-1}, f(a_n, a_2^{n-1}, x)), f(c_1^{n-1}, f(a_n, a_2^{n-1}, y))) =$$
$$= \varphi(f(f(c_1^{n-1}, a_n), a_2^{n-1}, x)), f(f(c_1^{n-1}, a_n), a_2^{n-1}, y))) = \varphi(x, y)$$

for all $c_1, ..., c_{n-1} \in G$.

Thus every $\varphi \in \Phi$ is a left invariant deviation of an n–group (G, f) . Hence also (φ, k, c_2^n) is a left invariant deviation for every $k = 1, 2, ..., n$ and all $c_1, ..., c_{n-1} \in G$. Obviously (φ, k, c_2^n) is also left invariant on $(G,*)$ and $(\varphi,k,c_2^n)\in\Phi$. Therefore $(\varphi,1,a_2^n)$, $(\varphi,2,a_n,a_2^{n-1})\in\Phi$, which proves (b).

Conversely, if a topology $\mathcal{T}$ is induced by the family Φ of deviations satisfying (a) and (b), then, by Theorem 4.1, $(G,*)=ret_{a_2^{n-1}}(G,f)$ is a binary topological group. Similarly as in the proof of Theorem 4.2 from $(\varphi,2,a_n,a_2^{n-1})\in\Phi$ follows that the translation $\beta(x)=f(a_n,x,a_2^{n-1})$ is continuous. Proposition 3.3 completes the proof. □

5. Embedding of topological n–semigroups

The necessary and sufficient conditions for the embedding of topological semigroup in topological group are described by N. J. Rothman (cf. [14]) and F. Christoph (cf. [3]). In this section we give some generalizations of these results.

As it is well known (cf. for example [13] or [6]) an n–semigroup (G,f) is called *semiabelian* or $(1,n)$*–commutative* if

$$f(x,a_2^{n-1},y)=f(y,a_2^{n-1},x)$$

holds for all $x,y,a_2,...,a_{n-1}\in G$, and *cancellative* if

$$f(a_1^{i-1},x,a_{i+1}^n)=f(a_1^{i-1},y,a_{i+1}^n)\implies x=y$$

for all $i=1,2,...,n$ and $x,y,a_1,...,a_n\in G$. Every n–group is obviously cancellative.

Now we use the construction of the quotient n–group presented during the Gomel's algebraic conference (1995) by A. M. Gal'mak and V. V. Mukhin.

Let (G,f) be a cancellative semiabelian n–semigroup. Then the relation

$$\langle x,y\rangle\sim\langle z,t\rangle\iff f_{(2)}(\overset{(n-1)}{y},\overset{(n)}{z})=f_{(2)}(\overset{(n-1)}{t},\overset{(n)}{x})$$

defined on $G\times G$ is an equivalence relation. Indeed, the reflexivity and symmetry are obvious. We prove the transitivity.

Let $\langle x,y\rangle\sim\langle z,t\rangle$ and $\langle z,t\rangle\sim\langle u,v\rangle$. Then

$$f_{(2)}(\overset{(n-1)}{y},\overset{(n)}{z})=f_{(2)}(\overset{(n-1)}{t},\overset{(n)}{x})\quad\text{and}\quad f_{(2)}(\overset{(n-1)}{t},\overset{(n)}{u})=f_{(2)}(\overset{(n-1)}{v},\overset{(n)}{z}).$$

Hence

$$f_{(3)}(\overset{(n-1)}{t}, \overset{(n)}{x}, \overset{(n-1)}{v}) = f_{(3)}(\overset{(n-1)}{y}, \overset{(n)}{z}, \overset{(n-1)}{v}) = f_{(3)}(\overset{(n-1)}{y}, \overset{(n-1)}{v}, \overset{(n)}{z}) =$$
$$= f_{(3)}(\overset{(n-1)}{y}, \overset{(n-1)}{t}, \overset{(n)}{u}) = f_{(3)}(\overset{(n-1)}{t}, \overset{(n-1)}{y}, \overset{(n)}{u}),$$

which by the cancellativity gives $f_{(2)}(\overset{(n-1)}{x}, \overset{(n)}{v}) = f_{(2)}(\overset{(n-1)}{y}, \overset{(n)}{u})$.

Since (G, f) is semiabelian, then

$$f_{(2)}(\overset{(n-1)}{x}, \overset{(n)}{v}) = f_{(2)}(\overset{(n-1)}{v}, \overset{(n)}{x}),$$

and in the consequence

$$f_{(2)}(\overset{(n-1)}{v}, \overset{(n)}{x}) = f_{(2)}(\overset{(n-1)}{y}, \overset{(n)}{u}),$$

which proves the transitivity.

In the set $G^* = G \times G/\sim$ of all equivalence classes $\langle x_i, y_i \rangle$ we define the new n–ary operation

$$f^*(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = \langle f(x_1^n), f(y_1^n) \rangle.$$

If $\langle x_i, y_i\rangle \sim \langle s_i, t_i\rangle$ for all $i = 1, 2, ..., n$, then also

$$f_{(2)}(\overset{(n-1)}{y_i}, \overset{(n)}{s_i}) = f_{(2)}(\overset{(n-1)}{t_i}, \overset{(n)}{x_i})$$

and

$$f(f_{(2)}(\overset{(n-1)}{y_1}, \overset{(n)}{s_1}), ..., f_{(2)}(\overset{(n-1)}{y_n}, \overset{(n)}{s_n})) = f(f_{(2)}(\overset{(n-1)}{t_1}, \overset{(n)}{x_1}), ..., f_{(2)}(\overset{(n-1)}{t_n}, \overset{(n)}{x_n})).$$

But every semiabelian n–semigroup is also medial (see [10]), i.e. it satisfies

$$f(f(x_{11}^{1n}), f(x_{21}^{2n}), \ldots, f(x_{n1}^{nn})) = f(f(x_{11}^{n1}), f(x_{12}^{n2}), \ldots, f(x_{1n}^{nn})).$$

Then the last identity may be written in the form

$$f_{(2)}\Big(\overset{(n-1)}{f(y_1^n)}, \overset{(n)}{f(s_1^n)} \Big) = f_{(2)}\Big(\overset{(n-1)}{f(t_1^n)}, \overset{(n)}{f(x_1^n)} \Big),$$

which proves that

$$\langle f(x_1^n), f(y_1^n) \rangle \sim \langle f(s_1^n), f(t_1^n) \rangle.$$

Hence the operation f^* is well defined. It is clear that this operation is also associative and $(1,n)$–commutative.

Now let

$$x = f_{(\cdot)}\Big(a, \overset{(n-1)(n-2)}{d}, \overset{(n-1)(n-1)}{c}\Big)$$

and

$$y = f_{(\cdot)}\Big(b, \overset{(n-1)(n-1)}{d}, \overset{(n-1)n}{c}\Big),$$

where a,b,c,d are fixed elements from G. Then, using $(1,n)$–commutativity, we obtain

$$f_{(\cdot)}\Big(\underbrace{f(y, \overset{(n-1)}{d}), \ldots, f(y, \overset{(n-1)}{d})}_{(n-1)-times}, \overset{(n)}{a}\Big) =$$

$$= f_{(\cdot)}\Big(\underbrace{\underbrace{b, \overset{(n-1)(n-1)}{d}, \overset{(n-1)n}{c}, \overset{(n-1)}{d}}, \ldots, \underbrace{b, \overset{(n-1)(n-1)}{d}, \overset{(n-1)n}{c}, \overset{(n-1)}{d}}}_{(n-1)-times}, \overset{(n)}{a}\Big) =$$

$$= f_{(\cdot)}\Big(\overset{(n-1)}{b}, \overset{(n)}{a}, \overset{(n-1)^2 n}{d}, \overset{(n-1)^2 n}{c}\Big) = W_1$$

and

$$f_{(\cdot)}\Big(\overset{(n-1)}{b}, \underbrace{f(x, \overset{(n-1)}{c}), \ldots, f(x, \overset{(n-1)}{c})}_{n-times}\Big) =$$

$$f_{(\cdot)}\Big(\overset{(n-1)}{b}, \underbrace{\underbrace{a, \overset{(n-1)(n-1)}{d}, \overset{(n-1)(n-2)}{c}, \overset{(n-1)}{c}}, \ldots, \underbrace{a, \overset{(n-1)(n-1)}{d}, \overset{(n-1)(n-2)}{c}, \overset{(n-1)}{c}}}_{n-times}\Big)$$

$$= f_{(\cdot)}\Big(\overset{(n-1)}{b}, \overset{(n)}{a}, \overset{(n-1)^2 n}{d}, \overset{(n-1)^2 n}{c}\Big) = W_2 .$$

Since $W_1 = W_2$, then

$$f_{(\cdot)}\Big(\underbrace{f(y, \overset{(n-1)}{d}), \ldots, f(y, \overset{(n-1)}{d})}_{(n-1)-times}, \overset{(n)}{a}\Big) = f_{(\cdot)}\Big(\overset{(n-1)}{b}, \underbrace{f(x, \overset{(n-1)}{c}), \ldots, f(x, \overset{(n-1)}{c})}_{n-times}\Big)$$

which proves that

$$\langle\, f(x, \overset{(n-1)}{c}),\, f(y, \overset{(n-1)}{d})\,\rangle = \langle a, b\rangle ,$$

i.e.

$$f^*(\,\langle x,y\rangle,\underbrace{\langle c,d\rangle,\,\dots,\langle c,d\rangle}_{n-1 times})=\langle a,b\rangle\,.$$

Hence for all $\langle a,b\rangle,\langle c,d\rangle\in G^*$ the last equation has the solution $\langle x,y\rangle\in G^*$.

In the similar way we prove that for all $\langle a,b\rangle,\langle c,d\rangle\in G^*$ there exists $\langle x,y\rangle\in G^*$ such that

$$f^*(\underbrace{\langle c,d\rangle,\,\dots,\langle c,d\rangle}_{(n-1)-times},\langle x,y\rangle\,)=\langle a,b\rangle\,.$$

This proves (cf. [18]) that (G^*,f^*) is a semiabelian n–group.

The map $p(x)=\langle x,x\rangle$ is a homomorphic embedding of an n–semigroup (G,f) in an n–group (G^*,f^*). Indeed,

$$p(f(x_1^n))=\langle f(x_1^n)\,,\,f(x_1^n)\rangle= \\ =f^*(\langle x_1,x_1\rangle,\,\dots,\langle x_n,x_n\rangle)=f^*(p(x_1),\,\dots,p(x_n))$$

and $p(x)=p(y)$ implies $\langle x,x\rangle=\langle y,y\rangle$, i.e.

$$f_{(2)}(\overset{(n-1)}{x},\overset{(n)}{y})=f_{(2)}(\overset{(n-1)}{y},\overset{(n)}{x})=f_{(2)}(\overset{(n-1)}{x},\overset{(n-1)}{y},x)\,,$$

which by the cancellativity gives $x=y$. Thus the following lemma is true.

Lemma 5.1. *Every semiabelian cancellative n-semigroup may be embedded into a semiabelian n-group.* □

Lemma 5.2. *If φ is a left invariant deviation of a cancellative semiabelian n–semigroup (G,f), then*

$$\varphi_G(\,\langle x,y\rangle,\langle z,t\rangle\,)=\varphi(f_{(2)}(\overset{(n-1)}{t},\overset{(n)}{x})\,,\,f_{(2)}(\overset{(n-1)}{y},\overset{(n)}{z})\,)$$

is a left invariant deviation on G^* such that $\varphi_G(p(x),p(y))=\varphi(x,y)$.

Proof. From the definition of φ_G follows $\varphi_G(\,\langle x,x\rangle,\langle x,x\rangle\,)=0$ and $\varphi_G(\,\langle x,y\rangle,\langle z,t\rangle\,)=\varphi_G(\,\langle z,t\rangle,\langle x,y\rangle\,)$.

Moreover, if $\langle x,y\rangle\sim\langle u,v\rangle$, where $\langle x,y\rangle,\langle u,v\rangle\in G\times G$, then

$$f_{(2)}(\overset{(n-1)}{v},\overset{(n)}{x})=f_{(2)}(\overset{(n-1)}{y},\overset{(n)}{u})$$

and

$$\varphi_G(\,\langle x,y\rangle,\langle z,t\rangle\,) = \varphi(f_{(2)}(\overset{(n-1)}{t},\overset{(n)}{x})\,,\,f_{(2)}(\overset{(n-1)}{y},\overset{(n)}{z})\,) =$$

$$= \varphi(f_{(3)}(\overset{(n-1)}{v},\overset{(n-1)}{t},\overset{(n)}{x})\,,\,f_{(2)}(\overset{(n-1)}{v},\overset{(n-1)}{y},\overset{(n)}{z})\,) =$$

$$= \varphi(f_{(3)}(\overset{(n-1)}{t},\overset{(n-1)}{v},\overset{(n)}{x})\,,\,f_{(3)}(\overset{(n-1)}{v},\overset{(n-1)}{y},\overset{(n)}{z})\,) =$$

$$= \varphi(f_{(3)}(\overset{(n-1)}{t},\overset{(n-1)}{y},\overset{(n)}{u})\,,\,f_{(3)}(\overset{(n-1)}{v},\overset{(n-1)}{y},\overset{(n)}{z})\,) =$$

$$= \varphi(f_{(3)}(\overset{(n-1)}{y},\overset{(n-1)}{t},\overset{(n)}{u})\,,\,f_{(3)}(\overset{(n-1)}{y},\overset{(n-1)}{v},\overset{(n)}{z})\,) =$$

$$= \varphi(f_{(2)}(\overset{(n-1)}{t},\overset{(n)}{u})\,,\,f_{(2)}(\overset{(n-1)}{v},\overset{(n)}{z})\,) = \varphi_G(\,\langle u,v\rangle,\langle z,t\rangle\,)$$

which proves that φ_G is well defined.

Now, for all $\langle x,y\rangle,\langle z,t\rangle \in G\times G$ we have

$$\varphi_G(\,\langle x,y\rangle,\langle z,t\rangle\,) = \varphi(f_{(2)}(\overset{(n-1)}{t},\overset{(n)}{x})\,,\,f_{(2)}(\overset{(n-1)}{y},\overset{(n)}{z})\,) =$$

$$= \varphi(f_{(3)}(\overset{(n-1)}{v},\overset{(n-1)}{t},\overset{(n)}{x})\,,\,f_{(3)}(\overset{(n-1)}{v},\overset{(n-1)}{y},\overset{(n)}{z})\,) \leq$$

$$\leq \varphi(f_{(3)}(\overset{(n-1)}{v},\overset{(n-1)}{t},\overset{(n)}{x})\,,\,f_{(3)}(\overset{(n-1)}{y},\overset{(n-1)}{t},\overset{(n)}{u})\,)$$

$$+\,\varphi(f_{(3)}(\overset{(n-1)}{y},\overset{(n-1)}{t},\overset{(n)}{u})\,,\,f_{(3)}(\overset{(n-1)}{v},\overset{(n-1)}{y},\overset{(n)}{z})\,) =$$

$$= \varphi(f_{(3)}(\overset{(n-1)}{t},\overset{(n-1)}{v},\overset{(n)}{x})\,,\,f_{(3)}(\overset{(n-1)}{t},\overset{(n-1)}{y},\overset{(n)}{u})\,)$$

$$+\,\varphi(f_{(3)}(\overset{(n-1)}{y},\overset{(n-1)}{t},\overset{(n)}{u})\,,\,f_{(3)}(\overset{(n-1)}{y},\overset{(n-1)}{v},\overset{(n)}{z})\,) =$$

$$= \varphi(f_{(2)}(\overset{(n-1)}{v},\overset{(n)}{x})\,,\,f_{(2)}(\overset{(n-1)}{y},\overset{(n)}{u})\,) + \varphi(f_{(2)}(\overset{(n-1)}{t},\overset{(n)}{u})\,,\,f_{(2)}(\overset{(n-1)}{v},\overset{(n)}{z})\,) =$$

$$= \varphi_G(\,\langle x,y\rangle,\langle u,v\rangle\,) + \varphi_G(\,\langle u,v\rangle,\langle z,t\rangle\,).$$

Hence φ_G is a deviation on G^*.

To prove that φ_G is left invariant observe that for all $i=1,...,n-1$, and $a_i,b_i,a_{n-1},x,y,u,v\in G$ we have

$$\varphi_G\Big(f(\langle a_1,b_1\rangle,...,\langle a_{n-1},b_{n-1}\rangle,\langle x,y\rangle),\ f(\langle a_1,b_1\rangle,...,\langle a_{n-1},b_{n-1}\rangle,\langle u,v\rangle)\Big)$$

$$= \varphi_G\Big(\langle f(a_1^{n-1}, x), f(b_1^{n-1}, y)\rangle, \langle f(a_1^{n-1}, u), f(b_1^{n-1}, v)\rangle \Big) =$$

$$= \varphi\Big(f_{(2)}(\underbrace{f(b_1^{n-1}, v), ..., f(b_1^{n-1}, v)}_{(n-1)-times}, \underbrace{f(a_1^{n-1}, x), ..., f(a_1^{n-1}, x)}_{n-times}),$$

$$f_{(2)}(\underbrace{f(b_1^{n-1}, y), ..., f(b_1^{n-1}, y)}_{(n-1)-times}, \underbrace{f(a_1^{n-1}, u), ..., f(a_1^{n-1}, u)}_{n-times})\Big).$$

By the associativity and $(1, n)$–commutativity of f, the last formula may be written in the form

$$\varphi\Big(f_{(.)}(\dots, \overset{(n-1)}{v}, \overset{(n)}{x}), f_{(.)}(\dots, \overset{(n-1)}{y}, \overset{(n)}{u})\Big),$$

which, together with the fact that φ is left invariant, implies

$$\varphi\Big(f_{(2)}(\overset{(n-1)}{v}, \overset{(n)}{x}), f_{(2)}(\overset{(n-1)}{y}, \overset{(n)}{u})\Big) = \varphi_G(\langle x, y\rangle, \langle u, v\rangle).$$

This proves that φ_G is a left invariant deviation on G^*.

Moreover

$$\varphi_G(p(x), p(y)) = \varphi_G(\langle x, x\rangle, \langle y, y\rangle) = \varphi\Big(f_{(2)}(\overset{(n-1)}{y}, \overset{(n)}{x}), f_{(2)}(\overset{(n-1)}{x}, \overset{(n)}{y})\Big)$$

$$= \varphi\Big(f_{(2)}(\overset{(n-1)}{y}, \overset{(n-1)}{x}, x), f_{(2)}(\overset{(n-1)}{x}, \overset{(n-1)}{y}, y)\Big) =$$

$$= \varphi\Big(f_{(2)}(\overset{(n-1)}{y}, \overset{(n-1)}{x}, x), f_{(2)}(\overset{(n-1)}{y}, \overset{(n-1)}{x}, y)\Big) = \varphi(x, y),$$

which completes our proof. □

Theorem 5.3. *A cancellative semiabelian n–semigroup (G, f) with a topology $\mathcal{T}$ may be topologically embedded in a topological n–group if and only if a topology $\mathcal{T}$ is induced by a some family of left invariant deviations defined on G.*

Proof. If a cancellative semiabelian n–semigroup (G, f) with a topology $\mathcal{T}$ is topologically embedded in a topological n–group (H, f) with a topology $\mathcal{T}_H$, then $\mathcal{T}_H$ is induced by some family Φ of deviations such that

$$\varphi(f(z, a_2^{n-1}, x), f(z, a_2^{n-1}, y)) = \varphi(x, y),$$

where $x, y, z \in H$ and $a_2, ..., a_n$ is a right neutral sequence of an n–group H (Theorem 4.3). Since in an n–group H for all $a_2, ..., a_{n-1} \in H$ there exists $a_n \in H$ such that $a_2, ..., a_n$ is a right neutral sequence, then in the above formula all $x, y, z, a_2, ..., a_{n-1}$ are arbitrary. This proves that all $\varphi \in \Phi$ are left invariant deviations.

Conversely, if a topology $\mathcal{T}$ on a cancellative semiabelian n–semigroup (G, f) is induced by a some family Φ of left invariant deviations, then every φ_G defined in Lemma 5.2 is a left invariant deviation on G^*. By Theorem 4.3 the family $\{\varphi_G\}_{\varphi\in\Phi}$ induces on G^* the topology $\mathcal{T}_G$ such that G^* is a topological n–group and $p(x) = \langle x, x\rangle$ is a topological embedding of $(G, f, \mathcal{T})$ in $(G^*, f^*, \mathcal{T}_G)$. □

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