Quasigroups and Related Systems, **4** (1997), 77–87

# On ordered $n-$groups

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**Abstract**

Among the results of the paper is the following proposition. Let $(Q,\{\cdot,\varphi,b\})$ be an arbitrary $nHG-$algebra associated to the $n$-group $(Q,A)$, where $n\geq 3$. If $\leq$ is a partial order defined on $Q$, then, $(Q,A,\leq)$ is an ordered $n-$group iff $(Q,\cdot,\leq)$ is an ordered group and for every $x,y\in Q$ the following implication holds $x\leq y\Longrightarrow\varphi(x)\leq\varphi(y)$.

## 1. Preliminaries

**Definition 1.1.** Let $n\geq 2$ and let $(Q,A)$ be an $n-$groupoid. Then:

$(a)$ $(Q,A)$ is an $n-$*semigroup* iff for every $i,j\in\{1,\ldots,n\}$, $i<j$ the following law (called the $(i,j)-$*associativity*) holds

$$A(x_1^{i-1},A(x_i^{i+n-1}),x_{i+n}^{2n-1})=A(x_1^{j-1},A(x_j^{j+n-1}),x_{j+n}^{2n-1}),$$

$(b)$ $(Q,A)$ is an $n-$*quasigroup* iff for every $i\in\{1,\ldots,n\}$ and for every $a_1^n\in Q$ is exactly one $x_i\in Q$ such that

$$A(a_1^{i-1},x_i,a_i^{n-1})=a_n,$$

$(c)$ $(Q,A)$ is a *Dörnte $n$–group* (briefly: *$n$-group*) iff is an $n$-semigroup and an $n-$quasigroup.

1991 Mathematics Subject Classification: 20N15

Keywords: $n-$semigroup, $n-$quasigroup, $n-$group, $\{1,n\}-$neutral operation, $nHG-$algebra.

A notion of an $n-$group was introduced by W. Dörnte in [2] as a generalization of the notion of a group.

**Proposition 1.2.** [10] *Let $n \geq 2$ and let $(Q, A)$ be an $n-$groupoid. Then the following statements are equivalent:*

$(i)$ $(Q, A)$ *is an $n-$group,*

$(ii)$ *there are mappings $^{-1}$ and $\mathbf{e}$ respectively of the sets $Q^{n-1}$ and $Q^{n-2}$ into the set $Q$ such that in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $< n, n-1, n-2 >$ the following laws hold:*

$(a)$ $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))\,,$

$(b)$ $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x\,,$

$(c)$ $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})\,,$

$(iii)$ *there are mappings $^{-1}$ and $\mathbf{e}$ respectively of the sets $Q^{n-1}$ and $Q^{n-2}$ into the set $Q$ such that in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $< n, n-1, n-2 >$ the following laws hold:*

$(\overline{a})$ $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})\,,$

$(\overline{b})$ $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x\,,$

$(\overline{c})$ $A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2}).$ □

**Remark 1.3.** $\mathbf{e}$ is an $\{1, n\}-$neutral operation of $n-$grupoid $(Q, A)$ iff algebra $(Q, \{A, \mathbf{e}\})$ of type $< n, n-2 >$ satisfies the laws $(b)$ and $(\overline{b})$. The notion of $\{i, j\}-$neutral operation $(i, j \in \{1, \ldots, n\}, i < j)$ of an $n-$groupoid is defined in a similar way (cf. [6]). In every $n-$groupoid there is at most one $\{i, j\}-$neutral operation. A $\{1, n\}-$neutral operation there exists in every $n-$group, but there are $n-$groups without $\{i, j\}-$neutral operations with $\{i, j\} \neq \{1, n\}$ (cf. [9]). Operation $^{-1}$ is a generalization of the inverse operation in a group. In fact, if $(Q, A)$ is an $n-$group, $n \geq 2$, then for every $a \in Q$ and for every sequence $a_1^{n-2}$ over $Q$ is

$$(a_1^{n-2}, a)^{-1} = \mathsf{E}(a_1^{n-2}, a, a_1^{n-2})\,,$$

where $\mathsf{E}$ is an $\{1, 2n-1\}-$neutral operation of the $(2n-1)-$group $(Q, \overset{2}{A})$ defined by $\overset{2}{A}(x_1^{2n-1}) = A(A(x_1^n), x_{n+1}^{2n-1})$ (cf. [7]). Obviously, for $n = 2$, $a^{-1} = \mathsf{E}(a)$; $a^{-1}$ is the inverse element of the element $a$ with respect to the neutral element $\mathbf{e}(\emptyset)$ of the group $(Q, A)$.

**Theorem 1.4.** (Hosszú–Gluskin Theorem) (cf. [5], [4])
*For every $n-$group $(Q, A)$, $n \geq 3$, there is an algebra $(Q, \{\cdot, \varphi, b\})$ such that the following statements hold:*

1° $(Q, \cdot)$ *is a group,*
2° $\varphi \in Aut(Q, \cdot)$,
3° $\varphi(b) = b$,
4° *for every* $x \in Q$, $\varphi^{n-1}(x) \cdot b = b \cdot x$,
5° *for every* $x_1^n \in Q$, $A(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \ldots \cdot \varphi^{n-1}(x_n) \cdot b$. □

**Definition 1.5.** [8] We say that an algebra $(Q, \{\cdot, \varphi, b\})$ is a *Hosszú–Gluskin algebra* of order $n$ ($n \geq 3$) (briefly: *$nHG$–algebra*) iff it satisfies $1° - 4°$ from the above theorem. If it satisfies also 5°, then we say that an $nHG$– algebra $(Q, \{\cdot, \varphi, b\}))$ is *associated* to the $n-$group $(Q, A)$.

**Proposition 1.6.** [8] *Let $n \geq 3$, let $(Q, A)$ be an $n-$group, and* $\mathbf{e}$ *its $\{1, n\}-$ neutral operation. Further on, let $c_1^{n-2}$ be an arbitrary sequence over $Q$ and let for every $x, y \in Q$*

$$B_{(c_1^{n-2})}(x, y) = A(x, c_1^{n-2}, y),$$
$$\varphi_{(c_1^{n-2})}(x) = A(\mathbf{e}(c_1^{n-2}), x, c^{n-2}) \quad \textit{and}$$
$$b_{(c_1^{n-2})} = A(\mathbf{e}(c_1^{n-2}), \mathbf{e}(c_1^{n-2}), \ldots, \mathbf{e}(c_1^{n-2})).$$

*Then, the following statements hold*
*(i)* $(Q, \{B_{(c_1^{n-2})}, \varphi_{(c_1^{n-2})}, b_{(c_1^{n-2})})\}$ *is an $nHG-$algebra associated to the $n-$group $(Q, A)$ and*
*(ii)* $\mathcal{C}_A = \{(Q, \{B_{(c_1^{n-2})}, \varphi_{(c_1^{n-2})}, b_{(c_1^{n-2})}\}) : c_1^{n-2} \in Q\}$ *is the set of all $nHG-$algebras associated to the $n-$group $(Q, A)$.* □

**Proposition 1.7.** [8] *Let $(Q, A)$ be an $n$–group,* $\mathbf{e}$ *its $\{1, n\}$–neutral operation and $n \geq 3$. Then for every $a_1^{n-2} \in Q$ and every $1 \leq i \leq n-2$ there is exactly one $x_i \in Q$ such that* $\mathbf{e}(a_1^{i-1}, x_i, a_i^{n-3}) = a_{n-2}$. □

## 2. Main results

**Definition 2.1.** Let $(Q, A)$ be an $n-$group, $n \geq 2$. If $\leq$ is a partial order on $Q$ such that

$$x \leq y \;\Rightarrow\; A(z_1^{i-1}, x, z_i^{n-1}) \leq A(z_1^{i-1}, y, z_i^{n-1}) \tag{1}$$

for all $x, y, z_1, ..., z_{n-1} \in Q$ and $i \in \{1, 2, ..., n-1\}$, then, we say that $(Q, A, \leq)$ is an *ordered $n-$group.*

Note that in the case $n = 2$ $(Q, A, \leq)$ is an ordered group in the sense of [3].

**Theorem 2.2.** *Let $\leq$ be a partial order on $Q$. Also, let $n \geq 3$ and let $(Q, A)$ be an $n-$group. In addition, let $(Q, \{\cdot, \varphi, b\}\}$ be an arbitrary $nHG-$algebra associated to the $n-$group $(Q, A)$. Then, $(Q, A, \leq)$ is an ordered $n-$group iff for all $x, y, z \in Q$ the following two formulas hold*

$$x \leq y \;\Rightarrow\; xz \leq yz \;\wedge\; zx \leq zy \tag{2}$$

$$x \leq y \;\Rightarrow\; \varphi(x) \leq \varphi(y)). \tag{3}$$

*Proof.* Let $(Q, A, \leq)$ be an ordered $n-$group and let $n \geq 3$. Also, let $\mathbf{e}$ be an $\{1, n\}-$neutral operation of the $n-$group $(Q, A)$. In addition, let $(Q, \{\cdot, \varphi, b\})$ be an arbitrary $nHG-$algebra associated to the $n-$group $(Q, A)$. Then, by Proposition 1.6, there is at least one sequence $c_1^{n-2}$ over $Q$ such that for every $x, y \in Q$ the following two equalities hold:

$$x \cdot y = A(x, c_1^{n-2}, y)\,,$$

$$\varphi(x) = A(\mathbf{e}(c_1^{n-2}), x, c_1^{n-2}).$$

Hence, by Definition 2.1, we conclude that the formulas (2) and (3) hold in $(Q, \{\cdot, \varphi, b\})$.

Conversely, let $(Q, \{\cdot, \varphi, b\})$ be an arbitrary $nHG-$algebra associated to the $n-$group $(Q, A)$. Also, let $\leq$ be a partial order on $Q$. Assume that an $nHG-$ algebra $(Q, \{\cdot, \varphi, b\})$ satisfies (2) and (3). Then, for every $x, y, z_1^{n-2} \in Q$ and $i \in \{1, 2, ..., n\}$ it satisfies also (1).

Indeed, for $1 \leq i \leq n-1$ $x \leq y$ implies $\varphi^{i-1}(x) \leq \varphi^{i-1}(y)$, and in the consequence

$$z_1 \cdot \ldots \cdot \varphi^{i-2}(z_{i-1}) \cdot \varphi^{i-1}(x) \leq z_1 \cdot \ldots \cdot \varphi^{i-2}(z_{i-1}) \cdot \varphi^{i-1}(y)\,,$$

which gives

$$z_1 \cdot \ldots \cdot \varphi^{i-2}(z_{i-1}) \cdot \varphi^{i-1}(x) \cdot \varphi^{i}(z_i) \cdot \ldots \cdot b \cdot z_{n-1} \leq \\ z_1 \cdot \ldots \cdot \varphi^{i-2}(z_{i-1}) \cdot \varphi^{i-1}(y) \cdot \varphi^{i}(z_i) \cdot \ldots \cdot b \cdot z_{n-1}\,.$$

Hence, by Definition 1.5, we conclude that (1) holds.

The cases $i = 1$ and $i = n$ are obvious. □

**Example 2.3.** Let $(Z, +)$ be the additive group of all integers, and let $\leq$ by the natural order defined on $Z$. Then $Z$ with the ternary operation $A$ defined by

$$A(x, y, z) = x + (-y) + z$$

is a 3–group.

Moreover, $(Z, \{+, \varphi, 0\})$, where $\varphi(x) = -x$, is an $nHG-$algebra associated to a 3–group $(Z, A)$.

Since for every $x, y \in Z$ $x \leq y$ implies $\varphi(y) \leq \varphi(x)$, we conclude (by Theorem 2.2) that $(Z, A, \leq)$ is not an ordered $3-$group. □

**Example 2.4.** Let $(Z, +, \leq)$ be as in the previous example. Let

$$B(x_1^n) = x_1 + x_2 + \ \ldots\ + x_n + 2$$

for every $x_1^n \in Z,\ n \geq 3$. Then, $(Z, B)$ is an $n-$group with $(Z, \{+, id, 2\})$ as its associated $nHG-$algebra. Obviously $(Z, B, \leq)$ is an ordered $n-$group.

Moreover, $(Z, C, \leq)$ and $(Z, D, \leq)$ where

$$C(x_1^n) = x_1 + x_2 + \ \ldots\ + x_n\,,$$

$$D(x_1^n) = x_1 + x_2 + \ \ldots\ + x_n + (-2)$$

are ordered $n-$groups as well. □

**Theorem 2.5.** *Let* $(Q, \leq)$ *be a chain. Also, let* $(Q, A)$ *be an* $n-$*group,* ${}^{-1}$ *its inverse operation,* $\mathbf{e}$ *its* $\{1, n\}-$*neutral operation and* $n \geq 3$. *Moreover, let* $a$ *be an arbitrary element of the set* $Q$ *and* $a_1^{n-2}$ *be an sequence over* $Q$ *such that* $\mathbf{e}(a_1^{n-2}) = a$. *Then*

(*i*) $(\{x : a \leq x\}, A)$ *is an* $n-$*subsemigroup of the* $n-$*group* $(Q, A)$ *iff* $a \leq A(\overset{n}{a})$,

(*ii*) $(\{x : (a_1^{n-2}, A(\overset{n}{a}))^{-1} \leq x\}, A)$ *is an* $n-$*subsemigroup of the* $n-$*group* $(Q, A)$ *iff* $A(\overset{n}{a}) \leq a$,

(*iii*) *let* $a \leq A(\overset{n}{a})$ *and let* $c$ *be an arbitrary element of the set* $Q$ *such that* $a \leq c$. *Then* $(\{x : c \leq x\}, A)$ *is an* $n-$*subsemigroup of the* $n-$*group* $(Q, A)$,

(*iv*) *let* $A(\overset{n}{a}) \leq a$ *and let* $c$ *be an arbitrary element of the set* $Q$ *such that* $(a_1^{n-2}, A(\overset{n}{a}))^{-1} \leq c$. *Then* $(\{x : c \leq x\}, A)$ *is an* $n-$*subsemigroup of the* $n-$*group* $(Q, A)$.

*Proof.* 1) Let $a$ be an arbitrary element of the set $Q$. Also let $a_1^{n-2}$ be an sequence over $Q$ such that $\mathbf{e}(a_1^{n-2}) = a$. Moreover, let

(a) $x \cdot y = A(x, a_1^{n-2}, y)$,

(b) $\varphi(x) = A(a, x, a_1^{n-2})$,

(c) $b = A(\overset{n}{a})$,

(d) $x^{\mathbf{-1}} = (a_1^{n-2}, x)^{-1}$

for all $x, y \in Q$. Then:

$1^o$ $(Q, \{\cdot, \varphi, b\})$ is an $nHG-$algebra associated to $(Q, A)$,

$2^o$ $a = \mathbf{e}(a_1^{n-2})$ is a neutral element of the group $(Q, \cdot)$,

$3^o$ ${}^{\mathbf{-1}}$ is an inverse operation of the group $(Q, \cdot)$.

By Theorem 2.2 and $1^o$, we conclude that

$4^o$ $(Q, \cdot, \leq)$ is a linearly ordered group,

$5^o$ $x \leq y \Rightarrow \varphi(x) \leq \varphi(y)$ for all $x, y \in Q$.

2) Assume now that $(\{x : a \leq x\}, A)$ is an $n-$subsemigroup of the $n-$group $(Q, A)$. Then for all $x_1^n \in Q$ from $x_1^n \in \{x : a \leq x\}$ follows $A(x_1^n) \in \{x : a \leq x\})$, whence we conclude that $a \leq A(\overset{n}{a})$.

Conversely, let $a \le A(\overset{n}{a})$. Hence, by $4^o$ and $5^o$, we conclude that for every sequence $x_1^n$ over $Q$ the following implications hold:

$$\bigwedge_{i=1}^{n} x_i \in \{x : a \le x\} \Rightarrow a \le x_1 \cdot \varphi(x_2) \cdot ... \cdot \varphi^{n-1}(x_n) \cdot b \Rightarrow a \le A(x_1^n),$$

i.e.

$$(\forall x_i \in Q)_1^n \, (\bigwedge_{i=1}^{n} x_i \in \{x : a \le x\} \Rightarrow A(x_1^n) \in \{x : a \le x\}).$$

3) Let $(\{x : (a_1^{n-2}, A(\overset{n}{a}))^{-1} \le x\}, A)$ be an $n-$subsemigroup of the $n-$group $(Q, A)$. Then for all

$$\bigwedge_{i=1}^{n} x_i \in \{x : b^{-1} \le x\} \Rightarrow A(x_1^n) \in \{x : b^{-1} \le x\}$$

by (c), (d). Whence, by $4^o$, $\varphi(b) = b$, $\varphi(b^{-1}) = b^{-1}$ we conclude that

$$b^{-1} \le A(b^{-1}, b^{-1}, ..., b^{-1}) = b^{-1} \cdot \varphi(b^{-1}) \cdot ... \cdot \varphi^{n-2}(b^{-1}) \cdot b \cdot b^{-1}$$
$$= b^{-1} \cdot b^{-1} \cdot ... \cdot b^{-1} \cdot b \cdot b^{-1},$$

i.e. $b^{n-2} \le a$. Hence $b \le a$ by $4^o$.

On the other hand, if $A(\overset{n}{a}) \le a$, then, by (c),(d) and $1^\circ$-$4^\circ$, we have $a \le b^{-1}$, whence, by $1^\circ$ and $\varphi(b^{-1}) = b^{-1}$, we obtain

$$\begin{array}{l} b^{-1} \le b^{-1} \le b^{-1} \\ a \le b^{-1} \le \varphi(b^{-1}) \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ a \le b^{-1} \le \varphi^{n-2}(b^{-1}) \\ b \le b \le b \\ b^{-1} \le b^{-1} \le b^{-1} \; . \end{array}$$

Hence, by $4^\circ$, $1^\circ$ and 1.5, we conclude that

$$b^{-1} \le b^{-1} \cdot \varphi(b^{-1}) \cdot ... \cdot \varphi^{n-2}(b^{-1}) \cdot b \cdot b^{-1} = A(b^{-1}, b^{-1}, \dots, b^{-1}),$$

i.e.

$$b^{-1} \le A(b^{-1}, b^{-1}, \dots, b^{-1}),$$

whence, by (i), we see that $(\{x : b^{-1} \le x\}, A)$ is an $n$-subsemigroup of the $n$-group $(Q, A)$.

4) Let $a \le A(\overset{n}{a}) = b$. Also let $c$ be an arbitrary element of the set $Q$ such that $a \le c$. Since $a \le b$, then

$$\text{(a)} \qquad c \cdot \varphi(c) \cdot \ \ldots \ \cdot \varphi^{n-1}(c) \cdot a \ \leq \ c \cdot \varphi(c) \cdot \ \ldots \ \cdot \varphi^{n-1}(c) \cdot b.$$

By $1^\circ, 2^\circ, 5^\circ$ and $a \leq c$, we obtain: $c \leq c$, $a \leq \varphi(c)$, ... , $a \leq \varphi^{n-1}(c)$, whence, by $2^\circ, 4^\circ$ and $5^\circ$, we conclude that

$$\text{(b)} \qquad c \ \leq c \ \cdot \varphi(c) \cdot \ \ldots \ \cdot \varphi^{n-1}(c) = c \cdot \varphi(c) \cdot \ \ldots \ \cdot \varphi^{n-1}(c) \cdot a.$$

By (a) and (b), we conclude that

$$c \ \leq \ c \cdot \varphi(c) \cdot \ \ldots \ \cdot \varphi^{n-1}(c) \cdot b,$$

i.e. $c \ \leq \ A(\overset{n}{c})$. Hence, by (i) $(\{x : c \leq x\}, A)$ is an $n$-subsemigroup of the $n$-group $(Q, A)$.

5) Let $A(\overset{n}{a}) \leq a$. Also let $c$ be an arbitrary element of the set $Q$ such that $b^{-1} \leq c$. Hence, by $1^\circ$, 1.5, $2^\circ$, $4^\circ$ and $5^\circ$, we conclude

$$\begin{aligned} c = c \cdot a \cdot \ \ldots \ \cdot a \cdot b \cdot b^{-1} &= c \cdot \varphi(a) \cdot \ \ldots \ \cdot \varphi^{n-2}(a) \cdot b \cdot b^{-1} \\ &\leq c \cdot \varphi(b^{-1}) \cdot \ \ldots \ \cdot \varphi^{n-2}(b^{-1}) \cdot b \cdot b^{-1} \\ &\leq c \cdot \varphi(c) \cdot \ \ldots \ \cdot \varphi^{n-2}(c) \cdot b \cdot c \\ &= A(\overset{n}{c}), \end{aligned}$$

whence, by (i) we prove that $(\{x : c \leq x\}, A)$ is an $n$-subsemigroup of the $n$-group $(Q, A)$. □

**Remark 2.6.** The above theorem describes so-called the *right cone* (cf. [3]), i.e. the set $K_r(c) = \{x : c \leq x\}$. The analogous result holds for the *left cone* $K_l(c) = \{x : x \leq c\}$.

# 3. Four propositions more

**Proposition 3.1.** *If* $(Q, A, \leq)$ *is an ordered* $n-$*group* $(n \geq 2)$*, then*

$$(\forall x \in Q) \ (\forall y \in Q) \ (\forall z_j \in Q)_1^{n-1}$$
$$\bigwedge_{i=1}^{n} (x \leq y \iff A(z_1^{i-1}, x, z_i^{n-1}) \leq A(z_1^{i-1}, y, z_i^{n-1}) ).$$

*Proof.* We prove only $\Leftarrow$ since the implication $\Rightarrow$ is obvious.
1) In the case $i = 1$, $A(x, a_1^{n-2}, a) \ \leq \ A(y, a_1^{n-2}, a)$ implies

$$A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \ \leq \ A(A(y, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}),$$

and in the consequence
$A(x,a_1^{n-2},A(a,a_1^{n-2},(a_1^{n-2},a)^{-1})) \le A(y,a_1^{n-2},A(a,a_1^{n-2},(a_1^{n-2},a)^{-1}))$,
which gives
$A(x,a_1^{n-2},\mathbf{e}(a_1^{n-2})) \le A(y,a_1^{n-2},\mathbf{e}(a_1^{n-2}))$. Hence $x \le y$.

2) The case $i=n$ may be proved analogously.

3) Let now $i \in \{2,\dots,n-1\}$. Then

$$A(a_1^{i-1},x,a_i^{n-1}) \le A(a_1^{i-1},y,a_i^{n-1}) \Rightarrow$$
$$A(b_i^{n-1},A(a_1^{i-1},x,a_i^{n-1}),b_1^{i-1}) \le A(b_i^{n-1},A(a_1^{i-1},y,a_i^{n-1}),b_1^{i-1}) \Rightarrow$$
$$A(A(b_i^{n-1},a_1^{i-1},x),a_i^{n-1},b_1^{i-1}) \le A(A(b_i^{n-1},a_1^{i-1},y),a_i^{n-1},b_1^{i-1}) \Rightarrow$$
$$A(b_i^{n-1},a_1^{i-1},x) \le A(b_i^{n-1},a_1^{i-1},y) \Rightarrow x \le y. \qquad \square$$

**Proposition 3.2.** *Let $(Q,A,\le)$ be an ordered $n-$group and let $n \ge 2$. Also, let $^{-1}$ be an inverse operation of the $n-$group $(Q,A)$. Then*

$$(\forall x,y \in Q)\,(\forall a_j \in Q)_1^{n-1} \quad x \le y \Leftrightarrow (a_1^{n-1},y)^{-1} \le (a_1^{n-1},x)^{-1}.$$

*Proof.* $x \le y \Leftrightarrow A((a_1^{n-2},x)^{-1},a_1^{n-2},x) \le A((a_1^{n-2},x)^{-1},a_1^{n-2},y) \Leftrightarrow$
$\mathbf{e}(a_1^{n-2}) \le A((a_1^{n-2},x)^{-1},a_1^{n-2},y) \Leftrightarrow A(\mathbf{e}(a_1^{n-2}),a_1^{n-2},(a_1^{n-2},y)^{-1}) \le$
$\le A(A((a_1^{n-2},x)^{-1},a_1^{n-2},y),a_1^{n-2},(a_1^{n-2},y)^{-1}) \Leftrightarrow$
$(a_1^{n-2},y)^{-1} \le A((a_1^{n-2},x)^{-1},a_1^{n-2},A(y,a_1^{n-2},(a_1^{n-2},y)^{-1})) \Leftrightarrow$
$(a_1^{n-2},y)^{-1} \le A((a_1^{n-2},x)^{-1},a_1^{n-2},\mathbf{e}(a_1^{n-2})) \Leftrightarrow$
$(a_1^{n-2},y)^{-1} \le (a_1^{n-2},x)^{-1}$. $\square$

**Proposition 3.3.** *Let $(Q,A,\le)$ be an ordered $n-$group and let $n \ge 3$. Also, let $\mathbf{e}$ be an $\{1,n\}-$neutral operation of the $n-$group $(Q,A)$. Then*

$$(\forall x \in Q)\,(\forall y \in Q)\,(\forall a_j \in Q)_1^{n-3}$$
$$\bigwedge_{i=1}^{n-2} (\, x \le y \Leftrightarrow \mathbf{e}(a_1^{i-1},y,a_i^{n-3}) \le \mathbf{e}(a_1^{i-1},x,a_i^{n-3})\,).$$

*Proof.* Since $A(a,x_1^{n-2},b) = A(A(a,y_1^{n-2},(y_1^{n-2},\mathbf{e}(x_1^{n-2}))^{-1}),y_1^{n-2},b)$ by Theorem 4 from [7], then

$$x \le y \Leftrightarrow A(a, a_1^{i-1}, x, a_i^{n-3}, b) \le A(a, a_1^{i-1}, y, a_i^{n-3}, b) \Leftrightarrow$$
$$A(A(a, c_1^{n-2}, (c_1^{n-2}, \mathbf{e}(a_1^{i-1}, x, a_i^{n-3}))^{-1}), c_1^{n-2}, b) \le$$
$$A(A(a, c_1^{n-2}, (c_1^{n-2}, \mathbf{e}(a_1^{i-1}, y, a_i^{n-3}))^{-1}), c_1^{n-2}, b) \Leftrightarrow$$
$$A(a, c_1^{n-2}, (c_1^{n-2}, \mathbf{e}(a_1^{i-1}, x, a_i^{n-3}))^{-1}) \le$$
$$A(a, c_1^{n-2}, (c_1^{n-2}, \mathbf{e}(a_1^{i-1}, y, a_i^{n-3}))^{-1}) \Leftrightarrow$$
$$(c_1^{n-2}, \mathbf{e}(a_1^{i-1}, x, a_i^{n-3}))^{-1} \le (c_1^{n-2}, \mathbf{e}(a_1^{i-1}, y, a_i^{n-3}))^{-1} \Leftrightarrow$$
$$\mathbf{e}(a_1^{i-1}, y, a_i^{n-3}) \le \mathbf{e}(a_1^{i-1}, x, a_i^{n-3}). \qquad \square$$

**Proposition 3.4.** *Let $(Q, A, \le)$ be an ordered $n-$group and let $n \ge 3$. Also, let $^{-1}$ be an inverse operation of the $n-$group $(Q, A)$. Then*

$$(\forall x \in Q)\,(\forall y \in Q)\,(\forall b \in Q)(\forall a_j \in Q)_1^{n-3}$$
$$\bigwedge_{i=1}^{n-2} (\, x \le y \;\Rightarrow\; (a_1^{i-1}, y, a_i^{n-3}, b)^{-1} \le (a_1^{i-1}, x, a_i^{n-3}, b)^{-1}\,).$$

*Proof.* Since $x \le y$ implies

$$\mathsf{E}(a_1^{i-1}, y, a_i^{n-3}, b, a_1^{i-1}, y, a_i^{n-3}) \le \mathsf{E}(a_1^{i-1}, y, a_i^{n-3}, b, a_1^{i-1}, x, a_i^{n-3})$$

and

$$\mathsf{E}(a_1^{i-1}, y, a_i^{n-3}, b, a_1^{i-1}, x, a_i^{n-3}) \le \mathsf{E}(a_1^{i-1}, x, a_i^{n-3}, b, a_1^{i-1}, x, a_i^{n-3}),$$

then from the transitivity of $\le$ follows that $x \le y$ implies

$$\mathsf{E}(a_1^{i-1}, y, a_i^{n-3}, b, a_1^{i-1}, y, a_i^{n-3}) \le \mathsf{E}(a_1^{i-1}, x, a_i^{n-3}, b, a_1^{i-1}, x, a_i^{n-3}).$$

This completes the proof because

$$(a_1^{i-1}, z, a_i^{n-3}, b)^{-1} = \mathsf{E}(a_1^{i-1}, z, a_i^{n-3}, b, a_1^{i-1}, z, a_i^{n-3}). \qquad \square$$

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Received September 20 1997 and in revised form March 29, 1999

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