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Centrally isotopic quasigroups

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Abstract

Relation of central isotopy between usual quasigroups is considered. The connection of this relation with central isotopy relation of equasigroups and with the abelian subgroup of the multiplication group of a quasigroup corresponding to the centre congruence of this quasigroup is established.

1. Introduction

In the work [10] and in Chapter III of [8] J.D.H.Smith has considered the relation of central isotopy between equasigroups (i.e. primitive quasigroups). This relation is tighter than isotopy but looser than isomorphism. In the base of this relation lies the concept of the centre congruence of an equasigroup $Q(\cdot,\backslash,/)$, introduced in the same works. In the articles [2, 3] the concept of the h-centre Z_h for an usual quasigroup $Q(\cdot)$ where h is an arbitrary fixed element of Q was introduced and it was proved that the h-centre defines a normal congruence which is called the centre congruence of $Q(\cdot)$ and does not depend on the element h. Finally, in article [5] a proof was given that the centre congruence of a quasigroup $Q(\cdot)$ coincides with the centre congruence of the equasigroup $Q(\cdot,\backslash,/)$, corresponding to $Q(\cdot)$. Thus, it was established that the h -centre is an inner characterization of the centre congruence of an equasigroup.

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In this article we consider the relation of central isotopy between usual quasigroups, establish its connection with the relation of central isotopy of the corresponding equasigroups and also with the abelian subgroup Γ of the multiplication group of a quasigroup $Q(\cdot)$. This subgroup was picked out in [4] and corresponds to the centre congruence of a quasigroup ($\Gamma h = Z_h$ for any $h \in Q$).

2. Preliminaries

An algebra $Q(\cdot, \setminus, /)$ with the three binary operations $\cdot, \setminus, /$ satisfying the identities

$$(xy)/y = x, \quad x \setminus (xy) = y, \quad (x/y)y = x, \quad x(x \setminus y) = y$$

is called an *equasigroup* [6] (or a *primitive quasigroup* [1]).

A groupoid $Q(\cdot)$ is called a *quasigroup* if each of the equations $ax = b,\ xa = b$ has a unique solution for any $a, b \in Q$. The equasigroup $Q(\cdot, \setminus, /)$ corresponds to quasigroup $Q(\cdot)$ where

$$x \setminus y = z \Longleftrightarrow xz = y, \quad x/y = z \Longleftrightarrow zy = x.$$

The quasigroups $Q(\setminus)$ and $Q(/)$ are called the *right inverse* and the *left inverse* quasigroups for $Q(\cdot)$ respectively.

A quasigroup $Q(\cdot)$ is said to be *isotopic* to a quasigroup $P(\circ)$ if there exist three bijections $\alpha, \beta, \gamma : Q \to P$ such that $\alpha a \circ \beta b = \gamma(ab)$ for all $a, b \in Q$. The ordered triple $T = (\alpha, \beta, \gamma)$ is called an *isotopy.*

According to [10], two equasigroups $Q(\cdot, \setminus, /)$ and $P(\circ, \setminus\setminus, //)$ are called *isotopic* if for each operation ω from $\circ, \setminus\setminus, //$ ($\cdot, \setminus, /$, respectively) there exist bijections θ_1, θ_2 and $\theta_3 : Q \to P$ such that

$$\theta_1 a \omega \theta_2 b = \theta_3(a \omega b)$$

for all $a, b \in Q$.

It is easy to see that if equasigroups $Q(\cdot, \setminus, /)$ and $P(\circ, \setminus\setminus, //)$ are isotopic then the pairs of the quasigroups $Q(\cdot)$ and $P(\circ)$, $Q(\setminus)$ and $P(\setminus\setminus)$, $Q(/)$ and $P(//)$ are isotopic, and conversely, isotopy of any such pair implies isotopy of corresponding equasigroups (it suffices to make the suitable permutation in the triplet of bijections).

According to [8], a congruence α of an equasigroup $Q(\cdot,\backslash,/)$ is central iff the diagonal $\hat{Q} = \{(q,q) \mid q \in Q\}$ is a normal subquasigroup on α, i.e. if it is a class of some congruence V on α. In his case the congruence α is considered as a subquasigroup of the direct product $(Q \times Q)(\cdot,\backslash,/)$ containing the diagonal $\hat{Q}$. This congruence V on α is said to centre α [6].

By Theorem III.3.10 from [8] an equasigroup $Q(\cdot,\backslash,/)$ has a unique maximal central congruence called the *centre congruence* $\zeta(Q)$ (or $\zeta(\cdot,\backslash,/)$) of $Q(\cdot,\backslash,/)$. Thus, the centre congruence $\zeta(Q)$ of an equasigroup $Q(\cdot,\backslash,/)$ is the (unique) maximal subquasigroup in $(Q{\times}Q)(\cdot,\backslash,/)$ containing the diagonal $\hat{Q}$ as a normal subquasigroup.

Let V be the congruence centering the centre congruence of an equasigroup $P(\circ,\backslash\backslash,//)$.

Definition 1. An equasigroup $Q(\cdot,\backslash,/)$ is said *to be a central isotope of an equasigroup* $P(\circ,\backslash\backslash,//)$ iff there is a bijection $\theta : Q \to P$, called a *central shift*, such that for each of the operations $\circ,\backslash\backslash,//$ (correspondingly, $\cdot,\backslash,/$) denoted ω, there is an element $(p_\omega,\bar{p}_\omega)$ of $\zeta(P)$ such that

$$(p_\omega,\bar{p}_\omega)V(\theta(q_1\omega q_2),\theta q_1\omega\theta q_2) \tag{1}$$

for each pair q_1,q_2 of elements of Q.

In [8] the following properties of central isotopy were proved.

- Centrally isotopic equasigroups are isotopic (Proposition III.4.2.).
- Isomorphic equasigroups are centrally isotopic (Proposition III.4.3).
- A central shift $\theta : Q \to P$ mapping an idempotent of Q to an idempotent of P is an isomorphism (Proposition III.4.4).
- Central isotopy is an equivalence relation. Further, if $\theta : Q \to P$ is a central shift and the centre congruence $\zeta(Q)$ of Q is centered by W, then $\hat{\theta}\zeta(Q) = \zeta(P)$ and $\overline{\theta}W$ center $\zeta(P)$, where

$$\hat{\theta} : Q \times Q \to P \times P; \quad (q_1,q_2) \mapsto (\theta q_1,\theta q_2)$$

and

$$\overline{\theta} : (Q \times Q) \times (Q \times Q) \to (P \times P) \times (P \times P)\,; \\ ((q_1,q_2),(q_3,q_4)) \mapsto ((\theta q_1,\theta q_2),(\theta q_3,\theta q_4))\,.$$

(Theorem III.4.5).

- Centrally isotopic quasigroups have isomorphic multiplication groups (Proposition III.4.6).

In articles [2, 3] the concept of h-centre Z_h of a quasigroup $Q(\cdot)$ where h is an arbitrary fixed element in Q was introduced. It was also proved that the h-centre Z_h defines the same normal congruence $\theta_z(\cdot)$ on $Q(\cdot)$ by any $h \in Q$. This congruence is called the centre congruence of the quasigroup $Q(\cdot)$. Remind that each congruence of an equasigroup $Q(\cdot, \backslash, /)$ is a normal congruence in $Q(\cdot)$ and conversely.

Let the centre congruence $\zeta(Q)$ of an equasigroup $Q(\cdot, \backslash, /)$ be denoted by $\zeta(\cdot, \backslash, /)$. By Theorem 1 from [5]

$$\zeta(\cdot, \backslash, /) = \theta_z(\cdot) = \theta_z(\backslash) = \theta_z(/),$$

where $Q(\backslash)$ and $Q(/)$ are the right inverse and the left inverse quasigroups for $Q(\cdot)$. Thus, the h-centre Z_h is the $\zeta(\cdot, \backslash, /)$-class containing the element h.

Let $G_{(\cdot)}$ be the multiplication group of a quasigroup $Q(\cdot)$, i.e. the group generated by its translations L_a, R_a ($L_a x = ax$, $R_a x = xa$) for all $a \in Q$. In [4] it was picked out an abelian normal subgroup Γ in $G_{(\cdot)}$ corresponding to the centre congruence and acting sharply transitively on each h-centre, $h \in Q$. The subgroup Γ is characterized by means of the groups of left and right regular mappings of a quasigroup $Q(\cdot)$, in the sense of [9]. Recall these concepts.

Let $Q(\cdot)$ be a quasigroup. A mapping λ (ρ) of the set Q onto Q is called left (right) regular if there is a mapping λ^* (ρ^*) such that

$$\lambda x \cdot y = \lambda^*(xy), \quad x \cdot \rho y = \rho^*(xy) \tag{2}$$

for each $x, y \in Q$. The mappings λ, λ^*, ρ, ρ^* are permutations on Q and λ^* (ρ^*) is called a conjugate to λ (ρ). The set of all left (right) regular mappings of a quasigroup (the set of all mappings conjugate to them) forms the group Λ, respectively Λ^* (R, correspondingly, R^*). These groups are subgroups of the multiplication group $G_{(\cdot)}$ of $Q(\cdot)$ since

$$\lambda^* = R_x^{-1} \lambda R_x = L_{\lambda x} L_x^{-1}, \quad \rho^* = L_x \rho L_x^{-1} = R_{\rho x} R_x^{-1} \tag{3}$$

for each $x \in Q$.

Let $Core_G(H)$ be the maximal normal subgroup of a group G which lies in a subgroup H. By Theorem 1 from [4] $Z_h = \Gamma h$ where

$$\Gamma = Core_G(\Lambda \cap R) = Core_G(\Lambda^* \cap R^*)\,,$$

Z_h is the h-centre of a quasigroup $Q(\cdot)$, G is the multiplication group of $Q(\cdot)$.

In view of Corollary 3 from [4]

$$\Gamma = \{R_x R_y^{-1} \mid (x,y) \in \theta_z(\cdot)\} = \{L_x L_y^{-1} \mid (x,y) \in \theta_z(\cdot)\}$$
$$= \{R_x R_h^{-1} \mid x \in Z_h\} = \{L_x L_h^{-1} \mid x \in Z_h\}$$

for any arbitrary fixed h in Q.

3. Isotopic quasigroups and the subgroup Γ

Let an equasigroup $Q(\cdot, \backslash, /)$ be a central isotope of $P(\circ, \backslash\backslash, //)$. Then condition (1) means that for all q_1, q_2 of Q the pairs of the form $(\theta(q_1 q_2), \theta q_1 \circ \theta q_2)$ lie in the same class of the congruence V centering the centre congruence $\zeta(P)$ of the equasigroup $P(\circ, \backslash\backslash, //)$. Analogously, in the same class by V all pairs $(\theta(q_1/q_2), \theta q_1 // \theta q_2)$ (all pairs $(\theta(q_1 \backslash q_2), \theta q_1 \backslash\backslash \theta q_2)$) are contained. But, as it was noted above, the diagonal $\hat{P}$ is one of the classes of the congruence V, so all classes of V have the form

$$\hat{P}(a_1, b_1) = \{(p,p)(a_1, b_1) \mid p \in P, (a_1, b_1) \in \zeta(P)\}.$$

Thus

$$(\theta(q_1 q_2), \theta q_1 \circ \theta q_2) = (p,p) \circ (a_1, b_1)$$

and

$$\theta(q_1 q_2) = p \circ a_1, \quad \theta q_1 \circ \theta q_2 = p \circ b_1$$

for all $q_1, q_2 \in Q$. From these equalities it follows that

$$R_{b_1}^{-1}(\theta q_1 \circ \theta q_2) = R_{a_1}^{-1} \theta(q_1 q_2) = p,$$

or

$$(\theta q_1 \circ \theta q_2) = R_{b_1} R_{a_1}^{-1} \theta(q_1 q_2) \tag{4}$$

where $R_a x = x \circ a$, i.e.

$$T_1 = (\theta, \theta, R_{b_1} R_{a_1}^{-1} \theta) \tag{5}$$

is an isotopy of the quasigroups $Q(\cdot)$ and $P(\circ)$. In the same way we establish that

$$T_2 = (\theta, \theta, R_{b_2} R_{a_2}^{-1} \theta) \tag{6}$$

is an isotopy of $Q(\backslash)$ and $P(\backslash\backslash)$, $R_a x = x \backslash\backslash a$, and

$$T_3 = (\theta, \theta, R_{b_3} R_{a_3}^{-1} \theta) \tag{7}$$

is an isotopy of $Q(/)$ and $P(//)$, $R_a x = x//a$, for some (a_2, b_2), (a_3, b_3) in $\zeta(P)$. Therefore, if an equasigroup $Q(\cdot, \backslash, /)$ is a central isotope of $P(\circ, \backslash\backslash, //)$ and θ is a central shift of this central isotopy, then there are the three isotopies (5), (6), (7) of the quasigroups $Q(\cdot)$ and $P(\circ)$ ($Q(\backslash)$ and $P(\backslash\backslash)$, $Q(/)$ and $P(//)$ correspondingly) for some (a_1, b_1), (a_2, b_2), (a_3, b_3) in $\zeta(P)$.

Let θ_z^P denote the centre congruence of a quasigroup $P(\circ)$. It is natural to give the following

Definition 2. A quasigroup $Q(\cdot)$ is said *to be a central isotope of a quasigroup* $P(\circ)$ iff there are a bijection $\theta : Q \to P$, a pair $(a, b) \in \theta_z^P$ such that

$$R_b R_a^{-1} \theta(q_1 q_2) = \theta q_1 \circ \theta q_2 \tag{8}$$

for all $q_1, q_2 \in Q$, where $R_a x = x \circ a$.

In this case $T = (\theta, \theta, R_b R_a^{-1} \theta)$ is an isotopy between $Q(\cdot)$ and $P(\circ)$.

Let $\Gamma_{(\circ)}$ be the subgroup of the multiplication group $G_{(\circ)}$ of a quasigroup $P(\circ)$ corresponding to the centre congruence θ_z^P of $P(\circ)$ (in the sense of Theorem 1 from [4]). Then we can give the following statement that is equivalent to Definition 2 (see also Corollary 3 in [4]).

Proposition 1. *A quasigroup $Q(\cdot)$ is a central isotope of a quasigroup $P(\circ)$ iff there are a bijection $\theta : Q \to P$ and $\alpha \in \Gamma_{(\circ)}$ such that*

$$\alpha\theta(q_1 q_2) = \theta q_1 \circ \theta q_2 \tag{9}$$

for all $q_1, q_2 \in Q$. □

A substitution α on P (a bijection θ) we shall call a *central torsion* (a *central shift*) of the central isotopy defined by (9). By a *central torsion* of a quasigroup $P(\circ)$ we mean its isotope $P(*)$, where $x * y = \alpha^{-1}(x \circ y)$ for all $x, y \in P$, $\alpha \in \Gamma_{(\circ)}$.

From Proposition 1 we have

Corollary 1. *If a quasigroup $Q(\cdot)$ is a central isotope of $P(\circ)$, then $Q(\cdot) \cong P(*)$, where $P(*)$ is a central torsion of $P(\circ)$.*

Proof. Indeed, let $x * y = \alpha^{-1}(x \circ y)$, $\alpha \in \Gamma_{(\circ)}$, for all $x, y \in P$. From (9) it follows that

$$q_1 q_2 = \theta^{-1}\alpha^{-1}(\theta q_1 \circ \theta q_2) = \theta^{-1}(\theta q_1 * \theta q_2)$$

i.e. $Q(\cdot) \cong P(*)$. □

Thus, a central isotopy is a sequential taking of a central torsion and an isomorphism.

Now we shall prove the following

Theorem 1. *An equasigroup $Q(\cdot, \backslash, /)$ is centrally isotopic to an equasigroup $P(\circ, \backslash\backslash, //)$ iff the quasigroup $Q(\cdot)$ is centrally isotopic to $P(\circ)$.*

Proof. Let an equasigroup $Q(\cdot, \backslash, /)$ be a central isotope of $P(\circ, \backslash\backslash, //)$, then as it was shown above, the quasigroup $Q(\cdot)$ is a central isotope of $P(\circ)$ (see (4)).

Conversely, let a quasigroup $Q(\cdot)$ be centrally isotopic to a quasigroup $P(\circ)$, i.e.

$$\theta q_1 \circ \theta q_2 = R_b R_a^{-1} \theta(q_1 q_2) \tag{10}$$

for all $q_1, q_2 \in Q$ where θ is a central shift, $(a, b) \in \theta_z^P$. But then for all $q_1, q_2 \in Q$

$$R_a^{-1}\theta(q_1 q_2) = R_b^{-1}(\theta q_1 \circ \theta q_2) = p \in P$$

and

$$\theta(q_1 q_2) = p \circ a, \quad \theta q_1 \circ \theta q_2 = p \circ b.$$

Whence,

$$(\theta(q_1 q_2), \theta q_1 \circ \theta q_2) = (p, p) \circ (a, b) \in \hat{P} \circ (a, b).$$

It means that all pairs of such form lie in the same class of the congruence V centering the centre congruence of the equasigroup $P(\circ, \backslash\backslash, //)$ since from $(a, b) \in \theta_z^P$ it follows that $(a, b) \in \zeta(\cdot, \backslash, /)$ (see Theorem 1 in [5]). Hence, the condition of Definition 1 is satisfied for the operations $(\cdot)$ and $(\circ)$.We shall show that this condition holds for the operations $(/)$ and $(//)$ ($(\backslash)$ and $(\backslash\backslash)$).

From (10) it follows that

$$R_aR_b^{-1}(\theta q_1 \circ \theta q_2) = \theta(q_1q_2) \tag{11}$$

for all $q_1, q_2 \in Q$. But $(a, b) \in \theta_z^P$, so by Theorem 1 from [4]

$$R_aR_b^{-1} \in \Gamma_{(\circ)} \subseteq \Lambda^* \cap R^*.$$

If $\lambda^* \in \Gamma_{(\circ)}$, then by (3)

$$\lambda = R_x^{-1}\lambda^* R_x \in \Gamma_{(\circ)}$$

for any $x \in P$ since $\Gamma_{(\circ)}$ is a normal subgroup in the multiplication group $G_{(\circ)}$ of $P(\circ)$. So $\lambda = R_cR_d^{-1}$ for some pair $(c, d) \in \theta_z^P$ by Corollary 3 in [4]. Now by the definition of a left regular mapping (see (2)) we get

$$R_aR_b^{-1}(\theta q_1 \circ \theta q_2) = R_cR_d^{-1}\theta q_1 \circ \theta q_2$$

for all $q_1, q_2 \in Q$. Taking into account (11), we have

$$\theta(q_1q_2) = R_cR_d^{-1}\theta q_1 \circ \theta q_2,$$

i.e. $T_1 = (R_cR_d^{-1}\theta, \theta, \theta)$ is an isotopy between $Q(\cdot)$ and $P(\circ)$. But then $T_1' = (\theta, \theta, R_cR_d^{-1})$ is an isotopy between $Q(/)$ and $P(//)$.

Indeed, if $\alpha x \circ \beta y = \gamma(xy) = \gamma z$, then $\gamma z // \beta y = \alpha x = \alpha(z/y)$, i.e.

$$(\alpha, \beta, \gamma) \to (\gamma, \beta, \alpha).$$

Therefore,

$$R_cR_d^{-1}\theta(q_1/q_2) = \theta q_1 // \theta q_2$$

and, as in the first case, we receive that

$$(\theta(q_1/q_2), \theta q_1 // \theta q_2) = (p_1 \circ d, p_1 \circ c) \in \hat{P} \circ (d, c)$$

for $(d, c) \in \theta_z^P$ and for all $q_1, q_2 \in Q$, i.e. all pairs $(\theta(q_1/q_2), \theta q_1 // \theta q_2)$ lie in the same class. This means that the condition of Definition 1 holds for the operations $(/)$ and $(//)$.

It remains to check this condition for the operations $(\backslash)$ and $(\backslash\backslash)$. Since $\Gamma_{(\circ)}$ is a normal subgroup in $G_{(\circ)}$ and $R_aR_b^{-1} \in \Gamma_{(\circ)} \subseteq R^*$, then $R_aR_b^{-1} = \rho^*$ and $\rho = L_x^{-1}\rho^* L_x \in \Gamma_{(\circ)}$ for any $x \in P$ (see (3)). By Corollary 3 from [4] there exists a pair $(s, t) \in \theta_z^P$ such that $\rho = R_sR_t^{-1}$ and so

$$R_aR_b^{-1}(\theta q_1 \circ \theta q_2) = \theta q_1 \circ R_sR_t^{-1}\theta q_2$$

by the definition of a right regular mapping (see (2)). Taking into account (11), we get

$$\theta(q_1 q_2) = \theta q_1 \circ R_s R_t^{-1} \theta q_2$$

and $T_2 = (\theta, R_s R_t^{-1}\theta, \theta)$ is an isotopy between $Q(\cdot)$ and $P(\circ)$. But then $T_2' = (\theta, \theta, R_s R_t^{-1}\theta)$ is an isotopy between $Q(\backslash)$ and $P(\backslash\backslash)$ and so

$$R_t^{-1}\theta(q_1 \backslash q_2) = R_s^{-1}(\theta q_1 \backslash\backslash \theta q_2).$$

From this equality it follows that

$$(\theta(q_1 \backslash q_2), \theta q_1 \backslash\backslash \theta q_2) \in \hat{P} \circ (t, s), \quad (t, s) \in \theta_z^P.$$

This proves that $Q(\cdot, \backslash, /)$ is an isotope of $P(\circ, \backslash\backslash, //)$ with the central shift θ in the sense of Definition 1. □

From this proof and Proposition 1 the following result follows.

Corollary 2 *The transformation of central isotopy of quasigroups is invariant with respect to parastrophy of quasigroups (i.e. with respect to passage to a conjugate quasigroup).* □

Corollary 3. *If a quasigroup $Q(\cdot)$ is a central isotope of a quasigroup $P(\circ)$, then there exist substitutions $\alpha, \alpha_1, \alpha_2$ from $\Gamma_{(\circ)}$ such that*

$$\alpha\theta(q_1 q_2) = \theta q_1 \circ \theta q_2,$$
$$\theta(q_1 q_2) = \alpha_1 \theta q_1 \circ \theta q_2,$$
$$\theta(q_1 q_2) = \theta q_1 \circ \alpha_2 \theta q_2$$

for all $q_1, q_2 \in Q$. □

Using now Theorem III.4.5 from [8], we get

Corollary 4. *Central isotopy of quasigroups is an equivalence relation.*

Theorem III.4.5 in [8] describes how a central shift acts at the centre congruence. The following statement shows how a central shift and a central torsion act at the Z_h-centres, i.e. at the classes of the centre congruence.

Proposition 2. *Let a quasigroup $Q(\cdot)$ be a central isotope of $P(\circ)$ with a central shift θ and a central torsion α, $Z_h(\cdot), h \in Q$ $(Z_h(\circ), h \in P)$ be the h-centre of $Q(\cdot)$ (of $P(\circ)$). Then*

$$\alpha Z_h(\circ) = Z_h(\circ), \quad \theta Z_h(\cdot) = Z_{\theta h}(\circ).$$

Proof. Let $\alpha \in \Gamma_{(\circ)}$. Taking into account Theorem 1 in [4], we get

$$\alpha Z_h(\circ) = \alpha(\Gamma_{(\circ)} h) = \Gamma_{(\circ)} h = Z_h(\circ),$$

since $\alpha \in \Gamma_{(\circ)}$. Comparing Theorem 1 from [4] and Theorem III.4.5 from [8], we get $\hat{\theta}\theta_z(\cdot) = \theta_z(\circ)$, where

$$\hat{\theta} \ : \ Q \times Q \to P \times P; \quad (q_1, q_2) \mapsto (\theta q_1, \theta q_2).$$

Then for any $h \in Q$

$$\hat{\theta}(Z_h(\cdot), h) = (\theta Z_h(\cdot), \theta h) \in \theta_z(\circ)$$

and so $\theta Z_h(\cdot) \subseteq Z_{\theta h}(\circ)$. But $\theta_z(\cdot) = \hat{\theta}^{-1}\theta_z(\circ)$, $(Z_{\theta h}(\circ), \theta h) \subseteq \theta_z(\circ)$. From these equalities it follows that $\theta^{-1} Z_{\theta h}(\circ) \subseteq Z_h(\cdot)$. Hence, $\theta Z_h(\cdot) = Z_{\theta h}(\circ)$. $\square$

According to Proposition III.4.6 from [8], centrally isotopic equasigroups have isomorphic multiplication groups. It is found that an analogous result is true for subgroups $\Gamma_{(\cdot)}$ and $\Gamma_{(\circ)}$.

Proposition 3. *If a quasigroup $Q(\cdot)$ is centrally isotopic to a quasigroup $P(\circ)$ with a central shift θ, then $\Gamma_{(\cdot)} \cong \Gamma_{(\circ)}$, namely, $\Gamma_{(\cdot)} = \theta^{-1}\Gamma_{(\circ)}\theta$.*

Proof. Let $x \cdot y = \theta^{-1}\alpha^{-1}(\theta x \circ \theta y)$, $\alpha \in \Gamma_{(\circ)}$. Then $R_a = \theta^{-1}\alpha^{-1}\tilde{R}_{\theta a}\theta$ for all $a \in Q$, where R_a $(\tilde{R}_{\theta a})$ is a right translation in $Q(\cdot)$ (in $P(\circ)$). From the last equality we get

$$R_a R_b^{-1} = \theta^{-1}\alpha^{-1}\tilde{R}_{\theta a}\tilde{R}_{\theta b}^{-1}\alpha\theta, \quad a, b \in Q.$$

Using this equality, Corollary 3 from [4] and Theorem III.4.5 from [8], we obtain

$$\Gamma_{(\cdot)} = \theta^{-1}\alpha^{-1}\Gamma_{(\circ)}\alpha\theta = \theta^{-1}\Gamma_{(\circ)}\theta,$$

since $\alpha \in \Gamma_{(\circ)}$. $\square$

The centre $Z_{(\cdot)}$ of a loop $Q(\cdot)$, i.e. the set of $a \in Q$ such that

$$ax \cdot y = a \cdot xy, \quad x \cdot ya = xy \cdot a, \quad ax = xa$$

for all $x, y \in Q$ is a subloop of $Q(\cdot)$ (cf. [7]). Note that $Z_{(\cdot)}$ is also the h-centre for $h = e$, where e is the unit of the loop $Q(\cdot)$ (cf. [2]).

It is known that every quasigroup is isotopic to a loop. This result is not true in the case of central isotopy.

Theorem 2. *A quasigroup $P(\circ)$ is centrally isotopic to a loop iff there exists an element $a \in P$ such that $\tilde{R}_a = \tilde{L}_a \in \Gamma_{(\circ)}$ where $\tilde{R}_a x = x \circ a$, $\tilde{L}_a x = a \circ x$.*

Proof. By Corollary 1 it suffices to consider the case when a central shift is equal to the identity mapping. Let a quasigroup $P(\circ)$ be centrally isotopic to a loop $P(\cdot)$ with a central torsion α. Then $P(\cdot)$ is centrally isotopic to the quasigroup $P(\circ)$ with the central torsion α^{-1}: $xy = \alpha(x \circ y)$, $\alpha \in \Gamma_{(\cdot)}$, since by Proposition 3 $\Gamma_{(\cdot)} = \Gamma_{(\circ)}$. Let e be the unit of the loop $P(\cdot)$, then

$$e \cdot x = x \cdot e = \alpha(e \circ x) = \alpha(x \circ e) = x,$$

i.e. $\tilde{R}_e = \tilde{L}_e = \alpha^{-1} \in \Gamma_{(\circ)}$. Conversely, let there exist an element $a \in P$ in quasigroup $P(\circ)$ such that $\tilde{R}_a = \tilde{L}_a = \alpha \in \Gamma_{(\circ)}$. Then the quasigroup $P(\cdot)$: $xy = \alpha^{-1}(x \circ y)$ is a loop with the unit a. $\square$

Corollary 5. *Every loop $Q(\circ)$ with nontrivial centre $Z_{(\circ)}$ is centrally isotopic to a loop with a nonidentical central torsion.*

Proof. Indeed, if $a \in Z_{(\circ)} \neq \emptyset$, then $R_a = L_a \in \Gamma_{(\circ)}$ since by Corollary 3 from [4] $\Gamma_{(\circ)} = \{R_a R_e^{-1} = R_a,\ a \in Z_{(\circ)} = Z_e\}$ where e is the unit of $Q(\circ)$ and the quasigroup $Q(\cdot)$: $x \cdot y = R_a^{-1}(x \circ y)$ is a loop with the unit a , centrally isotopic to the loop $Q(\circ)$. It means that $Q(\circ)$ is a central isotope of $Q(\cdot)$ with the central torsion $R_a^{-1} \in \Gamma_{(\circ)}$, since in this case $\Gamma_{(\cdot)} = \Gamma_{(\circ)}$ by Proposition 3. $\square$

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