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Some results on the up to fourth length balanced identities

Oleg U. Kirnasovsky

Abstract

In the work the up to fourth length balanced identities having a groupoid signature are classified in the class of all quasigroups: the list consisting of 33 such identities is determined and for each arbitrary identity of this form it is shown which identity from this list is equivalent to the given identity in the class of all quasigroups. From next works of the author it will be follow that these 33 identities are pairwise non-equivalent in the class of all quasigroups.

A term is called *repetition-free*, iff every propositional variable appears in its record at most one time. The number of the propositional variables appearing in the record of this term is called the *length of this term*. A formula $v = w$ is called a *balanced identity*, iff v and w are repetition-free terms containing no propositional constant and consisting of the same propositional variables. The length of this term v (it, of course, is equal to the length of the term w) is called the *length of such balanced identity*. A balanced identity is called a *balanced identity of the first kind*, iff the order of the appearing of all propositional variables in the both sides of the identity is the same (otherwise the balanced identity is called a *balanced identity of the second kind*). Along with selecting and with study of the quasigroup classes determined by one or many identities, in the quasigroup theory, it is observed an aspiration to classify all identities of some type. Thus, for example, in [3] it is given some full classification of the iden-

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tities of the minimal length. Also in the works [1], [5], [7] and [8] some classifications of the balanced identities were studied.

Formulas Φ and Ψ are called *equivalent in a class* K, iff from the the fact that the formula Φ is truth in algebras from the class K it follows that the formula Ψ is truth in all algebras from K and conversely. The lists of identities from the works [1], [5], [7] and [8] are incomplete and contain equivalent identities. The question of the classification of the balanced identities (even, of the first kind only) of one binary operation signature by the equivalence relation in the class of all quasigroups is rather difficult and is unsolved up to now. The question on the full classification of the up to fourth balanced identities was open, although balanced identities on quasigroups were studied in many various works. For example, a result of series of V.D. Belousov's, M.A. Taylor's, J. Duplak's and other's works is the statement: if in a primitive quasigroup an identity of the form

$$\ldots(\ldots x \ldots y \ldots)\ldots z \cdots = \ldots x \ldots(\ldots y \ldots z \ldots)\ldots$$

holds, then the quasigroup is isotopic to some group (see, for example [1], [4], [2], [6] and [10]).

We say that a *class* K *of quasigroups is selected by an identity* $v = w$, iff for each quasigroup $(Q;\cdot)$ the identity $v = w$ holds in $(Q;\cdot)$ iff $(Q;\cdot)$ belongs to the class K. There exist exactly 33 classes of the quasigroups which are selected by the up to fourth length balanced identities. The author find the list consisting of 33 identities selecting these 33 quasigroup classes.

We prove that every up to fourth length balanced identity of a groupoid signature in the class of all quasigroups is equivalent to at least one of the identities from the list (1). In reality, here we obtain a more general result: we indicate for an arbitrary given identity of the considered type, which an identity from the list (1) is equivalent in the class of all quasigroups to the given identity. The author expresses his sincere thanks to Dr. F. Sokhatsky for the permanent attention to the work.

Note that all identities of the list (1) are pairwise-nonequivalent in the class of all quasigroups, that will be following from next works of the author. After that the words “at least” can will be replacing here with “exactly”.

The list of 33 identities selecting these 33 quasigroup classes is as follows:

$$\left.\begin{array}{rrcl rrcl}
1. & x & = & x & & & & \\
2. & xy & = & yx & & & & \\
3. & xy\cdot zt & = & xz\cdot yt & & & & \\
4. & xy\cdot zt & = & ty\cdot zx & & & & \\
5. & xy\cdot zt & = & tz\cdot yx & & & & \\
6. & xy\cdot z & = & xz\cdot y & 6^*. & x\cdot yz & = & y\cdot xz \\
7. & (xy\cdot z)t & = & (xt\cdot z)y & 7^*. & x(y\cdot zt) & = & z(y\cdot xt) \\
8. & xy\cdot z & = & x\cdot zy & & & & \\
9. & (xy\cdot z)t & = & (yt\cdot z)x & & & & \\
10. & xy\cdot z & = & zy\cdot x & 10^*. & x\cdot yz & = & z\cdot yx \\
11. & (xy\cdot z)t & = & (zt\cdot x)y & 11^*. & x(y\cdot zt) & = & z(t\cdot xy) \\
12. & (xy\cdot z)t & = & (ty\cdot z)x & 12^*. & x(y\cdot zt) & = & t(y\cdot zx) \\
13. & (x\cdot yz)t & = & (x\cdot yt)z & & & & \\
14. & (x\cdot yz)t & = & (x\cdot tz)y & 14^*. & x(yz\cdot t) & = & z(yx\cdot t) \\
15. & (xy\cdot z)t & = & y(x\cdot tz) & & & & \\
16. & (x\cdot yz)t & = & (z\cdot tx)y & 16^*. & x(yz\cdot t) & = & z(tx\cdot y) \\
17. & (xy\cdot z)t & = & y(z\cdot tx) & & & & \\
18. & (x\cdot yz)t & = & z(ty\cdot x) & & & & \\
19. & xy\cdot zt & = & xz\cdot ty & & & & \\
20. & xy\cdot zt & = & (x\cdot yt)z & 20^*. & xy\cdot zt & = & y(xz\cdot t) \\
21. & xy\cdot z & = & x\cdot yz & & & & \\
22. & (xy\cdot z)t & = & z(yx\cdot t) & 22^*. & x(y\cdot zt) & = & (x\cdot tz)y \\
23. & (xy\cdot z)t & = & x(yz\cdot t) & 23^*. & x(y\cdot zt) & = & (x\cdot yz)t
\end{array}\right\} \quad (1)$$

It is possible to show that the number of the possible bracketings in a repetition-free term of a length n is equal to a_n, where

$$a_1 = 1, \qquad a_i = \sum_{j=1}^{i-1} a_j a_{i-j}, \quad i \in \mathbb{N}\backslash\{1\},$$

or

$$a_1 = 1, \qquad a_i = \frac{2(2i-3)!}{i!(i-2)!}, \quad i \in \mathbb{N}\backslash\{1\}.$$

Then with such notation the number of the balanced identities of a length n of a groupoid signature and with a fixed set of variables consisting of n variables is equal to $(n! \cdot a_n)^2$. A balanced identity of a binary operation $(\cdot)$ signature is said to be 4-*identity*, iff the identity is written by the variables x, y, z and t. From the above, we have that the possible number of the brackettings in a repetition-free term of the length 4 is equal to 5, and the number of all 4-identities is equal to 14400.

Let *abcdn*, where a, b, c and d are metavariables and n is a natural number, which is not greater than 5, denotes the term

$$\begin{cases} ab \cdot cd, & \text{if} \quad n = 1, \\ (ab \cdot c)d, & \text{if} \quad n = 2, \\ (a \cdot bc)d, & \text{if} \quad n = 3, \\ a(bc \cdot d), & \text{if} \quad n = 4, \\ a(b \cdot cd), & \text{if} \quad n = 5. \end{cases}$$

The number n is called the *bracketting in the term abcdn*. The balanced identity

$$xyztm = abcdn, \tag{2}$$

where metavariables a, b, c and d have values from the set of the variables $\{x; y; z; t\}$, is denoted by *abcdmn*. The substitution

$$\begin{pmatrix} x & y & z & t \\ a & b & c & d \end{pmatrix}$$

is called a *conversion substitution of the identity* (2). We say that *the substitutions* α *on a set of the variables* $\{x; y; z; t\}$ *is greater than* β, i.e. $\alpha > \beta$, iff the word $\langle \alpha x, \alpha y, \alpha z, \alpha t \rangle$ follows lexicographically after the word $\langle \beta x, \beta y, \beta z, \beta t \rangle$ in the ordered alphabet $\langle x, y, z, t \rangle$. A 4-identity is *canonical* iff it has the form *abcdmn*, where either $m = n$ and the conversion substitution of this 4-identity is not greater than the inverse for it substitution, or $m < n$.

Theorem 1. *From every balanced identity of length* 4 *of a groupoid signature by the way of variable renaming and by swapping of the left-hand and right-hand sides it is possible to obtain a unique canonical* 4*-identity. For this it is enough either to rename the variables so that they, in the left-hand side of the identity, follow in the order* x, y, z, t, *or to make the same action after the swapping of the left-hand and right-hand sides of the identity.*

Proof. Since each variable renaming operation commutes with the operation of the swapping of the left-hand and right-hand sides of the identity, it is clear that no canonical 4-identity different from the 4-identities obtained by two given ways can be obtained from a balanced identity of the length 4 of a groupoid signature. Let us prove that among two such 4-identities there is exactly one canonical (or both coincide and are canonical). Let one of them have the form *abcdmn*. Let us denote with metavariables p, q, r and s such value variables x, y, z and t, that

$$\begin{pmatrix} x & y & z & t \\ a & b & c & d \end{pmatrix}^{-1} = \begin{pmatrix} x & y & z & t \\ p & q & r & s \end{pmatrix}. \qquad (3)$$

For obtaining from the 4-identity *abcdmn* the second of these identities, it is enough to swap the left-hand and right-hand sides and after that to rename the variables a, b, c and d to x, y, z and t respectively. Such renaming is equivalent by the formula (3) to the renaming of the variables x, y, z and t on p, q, r and s respectively. Hence, we obtain the 4-identity *pqrsnm*. If $m \neq n$, then it is obviously, that exactly one of the 4-identities *abcdmn* and *pqrsnm* is canonical. Let $m = n$. If

$$\langle a, b, c, d \rangle = \langle p, q, r, s \rangle,$$

i. e., the conversion substitutions of the given identities are involutions, then these two 4-identity coincide and, obviously, are canonical. Two cases remain:

$$\begin{pmatrix} x & y & z & t \\ a & b & c & d \end{pmatrix} > \begin{pmatrix} x & y & z & t \\ p & q & r & s \end{pmatrix}$$

and

$$\begin{pmatrix} x & y & z & t \\ a & b & c & d \end{pmatrix} < \begin{pmatrix} x & y & z & t \\ p & q & r & s \end{pmatrix}$$

It is easy to see, that in the first case the second identity is canonical only, and in the second case the first one is canonical only. □

The canonical 4-identity, obtained by the way described in Theorem 1 from a given balanced identity of length 4 of a groupoid signature is called the *canonical form of this identity.* An identity is called the *canonical form of a balanced identity* $v = w$ *of a length* $n < 4$ *and of* *a groupoid signature* iff it is the canonical form of the 4-identity

$$(\ldots((v \cdot x_1) \cdot x_2) \cdot \ldots) \cdot x_{4-n} = (\ldots((w \cdot x_1) \cdot x_2) \cdot \ldots) \cdot x_{4-n},$$

where $x_1, \ldots, x_{4-n}$ are different propositional variables, which do not appear in the terms v and w. In the quasigroups, such canonical form is equivalent to the identity $v = w$, because both the left and right divisions are fulfilled uniquely. Recall that a groupoid $(Q; \oplus)$ is called the *commutation of a groupoid* $(Q; +)$ iff the identity

$$x \oplus y = y + x$$

holds. Identities I and J of the signature of a groupoid operation $(\cdot)$ are called *conjugate* (denote this by the equalities $I^* = J$ and $J^* = I$) iff one is obtained from other by the replacement of the operation $(\cdot)$ to its commutation $(\circ)$ with respective replacements of all subterms and the subsequent formal renaming of the symbol of the operation $(\circ)$ to the symbol of the operation $(\cdot)$. A class K of groupoids is said to be the *conjugate to a class L of groupoids* (denote such class by L^*) iff for each groupoid from the class K in the class L the commutation of this groupoid is contained and vica versa. A class of groupoids is said to be *self-conjugate* iff it is conjugate to itself. Everywhere in this work by an identity we consider a closed formula, that is a formula that starts with generality quantifiers on all propositional variables appearing in the identity.

Theorem 2. *If $\mathcal{A}$ is a propositional form (that is a boolean function) with n variables, $I_1, \ldots, I_n$ are identities of a binary operation signature, and K is a class of groupoids, then the formula*

$$\mathcal{A}(\mathcal{I}_\infty, \ldots, \mathcal{I}_\backslash) \tag{4}$$

holds in all groupoids of the class K iff the formula

$$\mathcal{A}(\mathcal{I}^*_\infty, \ldots, \mathcal{I}^*_\backslash) \tag{5}$$

holds in all groupoids of the class K^*.

Proof. Since the transformations of the conjugating of identities and of the conjugating of classes of groupoids are involutions, it is enough to prove the theorem in one direction only. Let (4) be hold in every groupoid from the class K. We consider an arbitrary groupoid $(Q;\cdot)$ from K. Let $(Q;\circ)$ be the commutation of this groupoid. Then the formula obtained from (4) by replacement of each subterm uv with $v\circ u$ holds in $(Q;\circ)$. But so obtained formula is transformed to the formula (5) by the formal renaming of the symbol of the operation $(\circ)$ to the symbol of the given signature. Hence, (5) holds in $(Q;\circ)$. But since groupoid $(Q;\cdot)$ is selected from the class K arbitrary, the formula (5) holds in every groupoid from the class K^*. □

Corollary 1. *If K is a self-conjugate class of groupoids, $\mathcal{A}$ is a propositional form (that is a boolean function) with n variables, and $I_1, \ldots, I_n$ are identities of a binary operation signature, then the formulas* (4) *and* (5) *either are fulfilled simultaneously in all groupoids from K, or each of them is not fulfilled at least in some ones.*

Let M_{195} denotes the set of such 195 canonical 4-identities, for which

$$\langle m,n\rangle \notin \{\langle 1,4\rangle;\langle 1,5\rangle;\langle 3,5\rangle;\langle 4,4\rangle;\langle 4,5\rangle;\langle 5,5\rangle\}.$$

Theorem 3. *For every balanced identity I of the length* 4 *of a groupoid signature the canonical form of at least one of the identities I and I^* belongs to the set M_{195}.*

Proof. It is easy to see that two identities conjugated to two balanced identities of length 4 of a groupoid signature with the same canonical form have the same canonical form as well. Therefore, we can consider without loss of generality that I is a canonical 4-identity. Then $\langle m,n\rangle$ may have the values $\langle 1,1\rangle$, $\langle 1,2\rangle$, $\langle 1,3\rangle$, $\langle 1,4\rangle$, $\langle 1,5\rangle$, $\langle 2,2\rangle$, $\langle 2,3\rangle$, $\langle 2,4\rangle$, $\langle 2,5\rangle$, $\langle 3,3\rangle$, $\langle 3,4\rangle$, $\langle 3,5\rangle$, $\langle 4,4\rangle$, $\langle 4,5\rangle$, $\langle 5,5\rangle$ only. The pair $\langle i,j\rangle$, where i and j are brackettings respectively in the left-hand and right-hand sides of the canonical form of the identity I^*, respectively has the values $\langle 1,1\rangle$, $\langle 1,5\rangle$, $\langle 1,4\rangle$, $\langle 1,3\rangle$, $\langle 1,2\rangle$, $\langle 5,5\rangle$, $\langle 4,5\rangle$, $\langle 3,5\rangle$, $\langle 2,5\rangle$, $\langle 4,4\rangle$, $\langle 3,4\rangle$, $\langle 2,4\rangle$, $\langle 3,3\rangle$, $\langle 2,3\rangle$, $\langle 2,2\rangle$. Thence we see that in the occasion, when I does not belong to the set M_{195}, the canonical form of the identity I^* belongs to it. □

	11	12	13	22	23	24	25	33	34
xyzt	1	21	21	1	21	23	21	1	21
xytz	2	8	20	6	8	9	8	13	8
xzyt	3	8	8	6	8	8	9	2	8
xzty	19	8	6	6	8	8	9	9	8
xtyz	—	8	8	—	8	9	8	—	8
xtzy	19	8	8	7	8	11	8	14	8
yxzt	2	11	8	2	8	8	8	6*	8
yxtz	2	8	8	8	8	9	15	8	8
yzxt	19	8	8	8	8	8	9	8	8
yztx	19	8	8	8	8	8	17	8	9
ytxz	19	8	8	8	8	9	8	8	8
ytzx	19	10	8	9	8	8	8	11	9
zxyt	—	11	8	—	2	8	8	—	8
zxty	—	8	8	—	9	8	8	—	8
zyxt	19	8	8	10	2	22	8	10*	8
zytx	19	8	8	8	9	8	8	15	9
ztxy	2	8	8	11	9	11	8	16	8
ztyx	2	10	8	8	9	8	8	8	18
txyz	—	8	8	—	8	2	8	—	8
txzy	—	8	8	—	8	8	8	—	8
tyxz	—	8	11*	—	8	2	8	—	8
tyzx	4	8	8	12	8	8	8	10	2
tzxy	—	8	11*	—	8	8	2	—	8
tzyx	5	8	8	8	8	8	2	8	2

We decompose the set of all 325 canonical 4-identities on 33 classes: C_k and C_k^*, where k is an up to 23 natural number. Moreover, we identify the classes C_k and C_k^* when

$$k \in \{1; 2; 3; 4; 5; 8; 9; 13; 15; 17; 18; 19; 21\}$$

(therefore there are 33 classes). Classes C_k and C_k^* are called *conjugate*. Identified classes are called *self-conjugate*. Determine by the table for each given canonical 4-identity I, to which class the identity is referred: if $I \in M_{195}$ and I has the form $abcdmn$, then the number of *index* of the class is on the intersection of the row with the name "abcd" and of the column with the name "mn" in the table, and if $I \notin M_{195}$, then the identity belongs to the class conjugated to the class having the canonical form of the identity I^*.

Lemma 4. *All identities from the class C_1 hold in every groupoid.*

Proof. Indeed, that follows from the coincidence of the left-hand and right-hand sides of each of these identities. □

Denote by L_c and R_c the left and right translation of the operation $(\cdot)$ by an element c, with ε — the identical substitution for a clear from context set. Let T_a^i denotes one of the translations L_a or R_a of the groupoid $(Q;\cdot)$ in some non-fixed dependence from values of the variable i.

Lemma 5. *If in a quasigroup $(Q;\cdot)$ a 4-identity which can be rewritten by the equality*

$$(T_a^1 T_b^2 c)d = c(T_p^3 T_q^4 d),$$

where a, b, c, d are different variables and p, q are metavariables having different values from the set of the variables $\{a; b\}$, holds, then there exists a substitution α on the set Q such that

$$T_a^1 T_{\alpha a}^2 = \varepsilon, \tag{6}$$

and

$$\begin{cases} T_a^3 T_{\alpha a}^4 = \varepsilon, & \text{if} \quad p = a, \\ T_{\alpha a}^3 T_a^4 = \varepsilon, & \text{if} \quad p = b. \end{cases} \tag{7}$$

Proof. For all $c \in Q$ there exists a bijection between such a and b, that $T_a^1 T_b^2 c = c$. Fix some value of c and denote the respective bijection by α ($b = \alpha a$). Then, with $b = \alpha a$, we have that $T_a^1 T_b^2 c = T_a^1 T_{\alpha a}^2 c = c$, whence

$$T_p^3 T_q^4 d = d, \tag{8}$$

that is $T_p^3 T_q^4 = \varepsilon$. Hence, the respective equality from (7) holds as well. Consider the given identity again, but not fixing c. Let $b = \alpha a$, the equality (8) holds. Thus $T_a^1 T_{\alpha a}^2 c = c$, i. e., (6) holds also. □

Lemma 6. *From every identity of the set $M_{195} \cap (C_2 \cup C_9 \cup C_{19})$ commutativity of the quasigroup $(Q;\cdot)$ follows.*

Proof. Use in identities ytzx11, ytzx22, ztyx23, xtyz24 the equality $t = x$; in zyxt11, zxty23, yxtz24, xzty33 the equality $t = y$; in xtzy11, yzxt11, zytx11 the equality $z = x$; in ytxz11, xytz24, ytxz24 the equalities $t = y = x$; in tyxz24, tzxy24 the equality $t = xy \cdot z$; in xzty11 the equality $z = y$; in yxtz11 the equality $t = z$; in yztx11 the equalities $z = y = x$; in ztyx11 the equalities $z = x$ and $t = y$; in zyxt23 the equality $z = xy$; in tzyx25 the equalities $t = zz$ and $z = xy$; in tyzx34 the equality $t = x \cdot yz$; in tzyx34 the equalities $t = xx$ and $x = yz$; in zytx23 the equalities $t = x$ and $z = xy$; in ztxy23 the equalities $t = y$ and $z = xy$. After that, the obtained identities and xytz11, yxzt11, ztxy11, yxzt22, zxyt23, txyz24, xzyt33 too by the way of cancellations and of replacements are reduced to the identity of commutativity. For the identities xzty25, yzxt25, ytzx34, zytx34 the application of Lemma 5 gives the equality of translations L_u and R_u, that is commutativity. For the identity xzyt25 it gives the equalities $R_u R_{\alpha u} = L_u L_{\alpha u} = \varepsilon$, after that the substitution in the identity of the equalities $z = x$ and $t = \alpha x$ gives $(xy \cdot x)\alpha x = x(x \cdot y\alpha x)$, i. e., $R_{\alpha x} R_x L_x y = L_x(x \cdot y\alpha x)$, whence $y = x \cdot y\alpha x$, and hence,

$$x(\alpha x \cdot y) = L_x L_{\alpha x} y = y = x \cdot y\alpha x,$$

that after cancellation gives commutativity. At last, for the identity yztx34 Lemma 5 gives $L_u R_{\alpha u} = R_u L_{\alpha u} = \varepsilon$, and substitution $x = \alpha t$ gives $(\alpha t \cdot yz)t = y(zt \cdot \alpha t)$, i.e, $R_t L_{\alpha t} L_y z = L_y(zt \cdot \alpha t)$, whence $z = zt \cdot \alpha t$, and therefore $tz = t(zt \cdot \alpha t) = L_t R_{\alpha t}(zt) = zt$. □

Corollary 2. *In the quasigroup every identity from the class C_2 is equivalent to commutativity.*

Proof. Each of these identities follows from the commutativity. On the other hand, by Lemma 6, every of the identities of the set $M_{195} \cap C_2$ implies commutativity, whence by Corollary 1 from each of the rest identities of the class C_2 commutativity follows as well. □

Corollary 3. *In the quasigroup the identities from the class C_9 are pairwise-equivalent.*

Proof. Indeed, by Lemma 6 and Corollary 1, each of these identities implies the commutativity, and all of them are obtained one from other by permutation of the places of the factors in some products (subterms) after respective replacements. □

Recall, that identity

$$ab \cdot cd = ac \cdot bd \tag{9}$$

is called *mediality* and a quasigroup in which it holds is called *medial.*

Corollary 4. *In the quasigroups every identity from the class C_{19} is equivalent to the conjuction of commutativity and mediality.*

Proof. Indeed, by Lemma 6, each of them implies commutativity, and on the other hand, all of them are obtained from (9) by replacements of variables and by permutations of the places of the factors in some products (subterms). □

Lemma 7. *In the quasigroup every identity from the class C_6 is equivalent to the identity*

$$ab \cdot c = ac \cdot b. \tag{10}$$

Proof. Indeed, from the identities xzty13, xytz22, xzyt22, xytz44 by trivial replacements and cancellations the identity (10) is obtained, and from the identity xzty22, by the substitution $t = y$,— the identity, which reduces after replacements and cancellations to (10). On the other hand, the identity (10) is equivalent to the commutation of right translations, whence $(xy \cdot z)t = R_tR_zR_yx = R_yR_tR_zx = (xz \cdot t)y$, that is xzty22 holds. □

Recall that in the quasigroup the solutions of the equations $a \cdot x = b$ and $y \cdot a = b$ are denoted respectively as $x = a\backslash b$ and $y = b/a$. As usual, the *left and right local unit for an element a of a quasigroup* $(Q; \cdot)$ are respectively the elements a/a and $a\backslash a$, which are denoted respectively by f_a and e_a. The left and right units respectively of left and right loops are denoted respectively by f and e.

Lemma 8. *If in a quasigroup* $(Q;\cdot)$ *the formula* $T_u^1 T_v^2 a = T_w^1 a$*, where* a *is a propositional variable,* u*,* v*,* w *are terms of a signature* $(\cdot)$ *which do not contain this variable, and some propositional variable* b *appears in one of the terms* u *and* w *only, and exactly one time, holds, then* $(Q;\cdot)$ *is a left loop when* $T_v^2 = L_v$ *and is a right loop when* $T_v^2 = R_v$*.*

Proof. Due to the uniqueness of the left and right divisions in the quasigroup there exists such value of b that $u = w$. But then $T_v^2 a = a$, whence

$$v = \begin{cases} f_a, & \text{if} \quad T_v^2 = L_v, \\ e_a, & \text{if} \quad T_v^2 = R_v. \end{cases}$$

And since v does not depend on a, then $(Q;\cdot)$ is respectively a left or right loop. □

Lemma 9. *From every identity of the set* $M_{195} \cap C_{21}$ *associativity of the quasigroup* $(Q;\cdot)$ *follows.*

Proof. Indeed, from xyzt12, xyzt23, xyzt34 by trivial replacements and cancellations the identity of associativity is obtained. By Lemma 8 a quasigroup with the identity xyzt13 is a left loop. Then, replacing in xyzt13 the variable x with the left unit we have the identity of associativity. Finally, for a quasigroup with the identity xyzt25 associativity follows from the work [9]. □

Corollary 5. *Every identity from the class* C_{21} *is equivalent to associativity of the quasigroup* $(Q;\cdot)$*.*

Proof. By Lemma 9 and by Corollary 1 each identity from the class C_{21} implies associativity. On the other hand, from associativity all balanced identities of the first kind follow. □

Lemma 10. *A quasigroup with the identity* $yztx22$ *is a group.*

Proof. Let $t = z = x$ in the given identity. After cancellations, we obtain the commutativity. Accounting that, from the given identity we see that $(xy \cdot z)t = x(yz \cdot t)$ for $z = xy\backslash x$ may be rewritten in the form $xt = x(y(xy\backslash x) \cdot t)$. Thus $t = y(xy\backslash x) \cdot t$, whence $y(xy\backslash x) = f_t$. Therefore the given quasigroup is a left loop, and due to commutativity it is a right loop. When $t = e$ the given identity reduces to the identity of associativity. □

Lemma 11. *If in a groupoid with a unit a balanced identity of the second kind holds, then this groupoid is commutative.*

Proof. Indeed, let a and b be variables appeared in the sides of the identity in a different order. Then substituting instead of all other variables the unit we obtain the identity of commutativity. □

Lemma 12. *If in a quasigroup $(Q;\cdot)$ an identity from $M_{195} \cap C_8$ holds then $(Q;\cdot)$ is commutative or associative.*

Proof. First note that it is enough to prove that $(Q;\cdot)$ is a loop or in $(Q;\cdot)$ one of the identities

$$\begin{array}{rclrclrcl} a\cdot bc &=& ca\cdot b, & a\cdot bc &=& ac\cdot b, & a\cdot bc &=& ba\cdot c,\\ ab\cdot c &=& ca\cdot b, & ab\cdot c &=& c\cdot ba, & a\cdot bc &=& b\cdot ca. \end{array}$$

holds. Indeed, by Lemma 11 a loop $(Q;\cdot)$ must be commutative, by Lemma 8 a quasigroup with the first of these six identities is a loop, and other five enumerated identities respectively by the substitutions $a = b$, $a = c$, $b = c$, $c = ab$, $a = b$ and by subsequent trivial replacements and cancellations reduce to the identity of commutativity. Use in identities tyxz12, tyzx12, tzyx22, yztx23, ytzx23, tyzx23, tzyx23, tzyx33 the equality $t = x$; in zyxt12, zytx12, yzxt22, zytx22, xzyt23, tzyx24 the equality $z = x$; in tyzx24, tzxy24, txyz25, txzy25, tyxz25, tyzx25 the equality $t = xy \cdot z$; in xzty12, zxty12, txzy12, xtzy13, zxty13 the equality $y = zt$; in xtzy12, xzty23, xtzy23, txzy23, tzxy23 the equality $t = y$; in tzyx12, zytx13, ztyx13, tyzx13, tzyx13 the equality $x = zt$; in yxtz22, xytz23, xtyz23, ytxz23, yxtz33 the equality $t = z$; in yztx22, ytxz22, xzyt24, ztyx33 the equalities $t = z = x$; in txyz34, txzy34, tyxz34, tzxy34 the equality $t = x \cdot yz$; in yxtz23, yxzt24, zxty24 the equalities $t = z = y$; in xzyt12, txzy24 the equality $z = y$; in tzxy12, ytzx13 the equalities $x = y = zt$; in xtyz12 the equalities $t = y$ and $z = xy$; in yztx12 the equalities $t = y$ and $x = zy$; in ztxy13 the equalities $z = x$ and $y = xt$; in txzy13 the equalities $y = zt$ and $x = t$; in ztyx22 the equalities $t = x$ and $z = y$; in tyxz23 the equalities $t = z = xy$; in yztx33 the equalities $t = y = x$; in ytxz33 the equalities $y = x$ and $t = z$. After that the obtained identities and also xytz12, ztxy12, yztx13, zxyt13, txyz13, yxzt23, yzxt23, txyz23, zxyt24, ztxy25, yzxt33 by cancellations and by replacements reduce to one of such identities, that stipulate commutativity or associativity. The identities zxty13, yxtz33, zytx22, tzyx22 constitutes exception only, which after the described substitutions, cancellations and replacements reduce to the identity

$$a \cdot bc = b \cdot ac \tag{11}$$

(the first two ones) or to

$$ab \cdot c = ac \cdot b$$

(the last two ones). These two obtained identities with $a = f_c$ and with $c = e_a$ respectively reduce after simplifications to the equalities $f_c = f_{bc}$ and $e_a = e_{ab}$, and hence, fulfill in left and right loops only respectively. Use now in zxty13, yxtz33, zytx22, tzyx22 respectively the equalities $x = z = f$, $x = y = f$, $y = t = e$, $y = t = e$, after obtaining everywhere the identity of commutativity. The application of Lemma 5 to the identities xzty24, ytzx24, xytz25, yxzt25, ytxz34, zxty34 gives after simplifications the equality of the translations L_u and R_u, that is commutativity, and to yztx24, yxtz34, ztxy34 one gives respectively the equalities (for some substitution α)

$$R_u L_{\alpha u} = R_{\alpha u} L_u = \varepsilon, \quad L_u R_{\alpha u} = \varepsilon, \quad L_u L_{\alpha u} = R_{\alpha u} R_u = \varepsilon.$$

Use in yztx24 the equality $y = \alpha x$:

$$(x\alpha x \cdot z)t = \alpha x(zt \cdot x) = L_{\alpha x} R_x(zt) = zt,$$

whence $x\alpha x \cdot z = z$, that is $x\alpha x = f_z$, and hence, the quasigroup with the identity yztx24 is a left loop, i. e., $L_f = \varepsilon$, whence $R_f = \varepsilon$, that means, that this quasigroup is a loop as well. Substitute $t = \alpha x$ in yxtz34 $(x \cdot yz)\alpha x = y(x\alpha x \cdot z)$, whence $y(x\alpha x \cdot z) = R_{\alpha x} L_x(yz) = yz$, and therefore $x\alpha x = f_z$, that is $(Q;\cdot)$ is a left loop, whence from yxtz34 when $x = y = f$ we have commutativity. Let $t = \alpha(yz)$ in ztxy34:

$$(x \cdot yz)\alpha(yz) = z \cdot (\alpha(yz) \cdot x)y,$$

that is

$$z \cdot (\alpha(yz) \cdot x)y = R_{\alpha(yz)} R_{yz} x = x,$$

whence

$$z \cdot (\alpha(yz) \cdot (yz \cdot x))y = yz \cdot x,$$

that is

$$yz \cdot x = z(L_{\alpha(yz)} L_{yz} x \cdot y) = z \cdot xy.$$

Applying to all other identities from $M_{195} \cap C_8$ Lemma 8, we obtain the existence of the left or right unit, after what substitution respectively f or e instead of some variables after simplifications leads to one of such identities, that stipulate either commutativity or associativity, or the identities of the type $uf = u$ or $eu = u$, which are equivalent respectively to $f = e_u$ and to $e = f_u$, that is we have that the given quasigroup is a loop. □

Lemma 13. *In a quasigroup $(Q;\cdot)$ an arbitrary fixed identity from the class C_8 holds iff $(Q;\cdot)$ is an abelian group.*

Proof. If $(Q;\cdot)$ is an abelian group then all balanced identities hold in it. But if an identity I from the set $M_{195} \cap C_8$ holds in the quasigroup $(Q;\cdot)$ then by Lemma 12 this quasigroup is commutative or associative (in the last occasion the quasigroup is a group and therefore by Lemma 11 it is commutative as well). But if $(Q;\cdot)$ is commutative then we obtain again a true in $(Q;\cdot)$ identity by permutating in some products (subterms) of the identity I the factor places. Note that we can obtain by such manipulation from I after respective replacements either an identity from the class C_{21}, or yztx22. By Corollary 5 and by Lemma 10 then $(Q;\cdot)$ is a group. □

Lemma 14. *In the quasigroup each identity from the class C_{10} is equivalent to the identity*

$$ab \cdot c = cb \cdot a. \tag{12}$$

Proof. From the identities ztyx12, zyxt22, xtzy44, tyzx33 this identity is obtained by the way of trivial cancellations and replacements. Substitute $x = zt$ in ytzx12. After cancellations and replacements we again have the identity (12). On the contrary, let (12) hold in a quasigroup then $xy \cdot zt = (zt \cdot y)x = (yt \cdot z)x$. □

Lemma 15. *Every quasigroup with* (12) *is medial.*

Proof. Indeed, $xy \cdot zt = (zt \cdot y)x = (yt \cdot z)x = xz \cdot yt$. □

Lemma 16. *In the quasigroup with the identity* (12) *each identity from the classes* C_{11}, C_{22} *and* C_{23} *is equivalent to the identity* (11).

Proof. If we replace in identities yxzt12 and ytzx14 the left-hand sides with $xz \cdot yt$, then by Lemma 15, we obtain identities, which are equivalent to the respective initial ones. Moreover, these obtained identities reduce by variable replacements to zxyt12 and ztyx14 respectively. Further, if replacing similarly in yxzt12 and ytzx14 the left-hand sides with $(zt \cdot y)x$, in xyzt24 and xtzy24 with $(zy \cdot x)t$, and in xyzt24 the right-hand side with $x(tz \cdot y)$, then we obtain ones equivalent to initial identities, because the motivation of the truth of all such changes follows from the identity (12).

From the obtained identities after replacements we obtain respectively ztxy22, zyxt24, zyxt24, ztxy24, xtzy24. If replacing in ytzx33 (by the same considerations as before) the left-hand side with $(t \cdot yz)x$, then after cancellation and replacements we obtain the identity (11). The last one is equivalent to the identity

$$xt \cdot (xy \cdot z) = xy \cdot (xt \cdot z),$$

from which by replacements of the left-hand side with $(xy \cdot z)t \cdot x$, and of the right-hand side with $(xt \cdot z)y \cdot x$ (as before again) after cancellation we have the identity xtzy22, which reduces to ztxy22 after replacement of the rigght-hand side with $(zt \cdot x)y$. It remains to prove that the identities yxzt12 and ytzx14 are equivalent in the class of all quasigroups. Accounting the already proved, that yxzt12 is equivalent to (11), we replace in yxzt12 the right-hand side at first with $tz \cdot yx$, and later with $y(tz \cdot x)$. Hence, from yxzt12 it follows the identity ytzx14.

On the contrary, let ytzx14 hold in a quasigroup with the identity (12), then

$$xy \cdot zt = y(tz \cdot x) = y(xz \cdot t).$$

But also $ty \cdot zx = y(xz \cdot t)$, whence

$$xy \cdot zt = ty \cdot zx.$$

By Lemma 15 the quasigroup is medial, therefore $ty \cdot zx = tz \cdot yx$. Hence,

$$xy \cdot zt = tz \cdot yx = (yx \cdot z)t,$$

that is yxzt12 holds. □

Lemma 17. *In the quasigroup every identity from the class C_{11} implies the identity* (12)*.*

Proof. By Lemma 8 a quasigroup with one of the identities yxzt12, zxyt12, ytzx14 or ztxy24 is a left loop. Substitute in yxzt12 at first $y = t = f$ and after replacements we have that

$$af \cdot bf = ab \cdot f, \tag{13}$$

later substitute $z = f$ and after replacements and cancellation we get

$$ab = ba \cdot f, \tag{14}$$

and finally substitute $y = f$:

$$xf \cdot zt = (fx \cdot z)t = xz \cdot t,$$

whence

$$xz \cdot t = xf \cdot (tz \cdot f) = (x \cdot tz)f = tz \cdot x.$$

From zxyt12 analogously when $x = t = f$ we obtain the identity (14), when $y = t = f$ we obtain the identity $xf \cdot zf = (zx \cdot f)f$, that is the identity (13), when $t = f$ we obtain that $xy \cdot zf = (zx \cdot y)f$, whence

$$(zx \cdot y)f = (yx \cdot f) \cdot zf = (yx \cdot z)f,$$

that is $zx \cdot y = yx \cdot z$. From ytzx14 we have when $x = f$ the identity (14), when $y = t = f$ the identity $af \cdot bf = ba$, when $t = f$ we obtain that $xy \cdot zf = y \cdot zx$, whence

$$(zx \cdot y)f = (yx \cdot f) \cdot zf = z \cdot yx = (yx \cdot z)f,$$

that is (12) holds. From ztxy24 we have when $t = f$ the identity (14), when $z = f$ we have that $(xy \cdot f)t = tx \cdot y$, that is $yx \cdot t = tx \cdot y$. Using in the identities ztyx14, ztxy22 and xtzy24 respectively the equalities $z = xy$, $t = y$ and $x = tz$ we'll obtain after cancellations and replacements the identity (12). At last, substitute $x = t$ in ytzx33: $(t \cdot yz)t = (y \cdot tz)t$, whence $(t \cdot yz)x = (y \cdot tz)x = (x \cdot yz)t$, that is (12) holds again. □

Corollary 6. *In the quasigroups each identity from the class C_{11} is equivalent to the conjuction of the identities* (11) *and* (12)*.*

Proof. It follows from Lemmas 17 and 16. □

Lemma 18. *In the quasigroups from the identity $xytz33$ it follows mediality, and also the conjugate to $xytz33$ identity.*

Proof. Let c be an element of a quasigroup $(Q; \cdot)$ with the identity xytz33, then (due to the uniqueness of the left and right divisions in the quasigroups) there exists a bijection α between such a and b ($b = \alpha a$), for which

$$a \cdot cb = c.$$

Substitute $y = c$ and $z = \alpha x$ in xytz33:

$$(x \cdot c\alpha x)t = (x \cdot ct)\alpha x,$$

whence $ct = (x \cdot ct)\alpha x$, that is

$$b = ab \cdot \alpha a,$$

or also $R_{\alpha a} L_a = \varepsilon$. Let $u = z/\alpha y$, $v = t/\alpha y$, then xytz33 can be rewritten in the form

$$x(y \cdot u\alpha y) \cdot v\alpha y = x(y \cdot v\alpha y) \cdot u\alpha y,$$

that is

$$xL_yR_{\alpha y}u \cdot v\alpha y = xL_yR_{\alpha y}v \cdot u\alpha y,$$

whence

$$xu \cdot v\alpha y = xv \cdot u\alpha y, \tag{15}$$

that is (9) holds. Let also $p = u/\alpha y$, $q = v/\alpha y$, then, when $y = x$, we have from the identity (15) that

$$(y \cdot p\alpha y) \cdot v\alpha y = (y \cdot q\alpha y) \cdot u\alpha y,$$

whence $p \cdot v\alpha y = q \cdot u\alpha y$, or also

$$(p \cdot v\alpha y)(\alpha y \cdot w) = (q \cdot u\alpha y)(\alpha y \cdot w),$$

and from mediality we have that

$$p\alpha y \cdot (v\alpha y \cdot w) = q\alpha y \cdot (u\alpha y \cdot w),$$

that is $u(v\alpha y \cdot w) = v(u\alpha y \cdot w)$, whence it is obtained by variable replacement the identity, which is conjugate to xytz33. □

Corollary 7. *In the quasigroup the identities from the class C_{13} are pairwise-equivalent.*

Proof. Indeed, one of them is the identity xytz33, the second one reduces by variable replacement to the identity conjugated to xytz33, but there are no other identities in this class. The inverse implication follows from Corollary 1 and from involutivity of the transformation of the conjugating of identities. □

Lemma 19. *In the quasigroups every identity from the class C_{15} is equivalent to the conjuction of the identities* (12) *and*

$$a \cdot bc = c \cdot ba. \tag{16}$$

Proof. Since zytx33 after variable replacement becomes conjugate to ytzx44, then by the Corollary 1 it is enough to prove our statement for yxtz25 and for zytx33 only. Let at first the identities (12) and (16) fulfil in a quasigroup. Then

$$(xy \cdot z)t = tz \cdot xy = y(x \cdot tz),$$
$$(x \cdot yz)t = (t \cdot yz)x = (z \cdot yt)x.$$

Now let in the quasigroup yxtz25 hold. When $t = xy$, this identity reduces by replacements and cancellation to (12), whence

$$(xy \cdot z)t = y(x \cdot tz(= tz \cdot xy.$$

Finally, let in quasigroup zytx33 hold. Substitute in the identity the equality $p = x$. After cancellation and replacements we obtain the identity (16). Then from zytx33 we have that

$$(x \cdot yz)t = (z \cdot yt)x = (t \cdot yz)x,$$

which completes the proof. □

Theorem 20. *In each of* 33 *classes* C_k *and* C_k^* *of the partition on the canonical* 4*-identities all identities are pairwise-equivalent.*

Proof. Indeed, for the classes C_1, C_2, C_6, C_8, C_9, C_{10}, C_{11}, C_{13}, C_{15}, C_{19} and C_{21} this follows respectively from Lemma 4, Corollary 2, Lemmas 7 and 13, Corollary 3, Lemma 14, Corollaries 6 and 7, Lemma 19 and Corollaries 4 and 5. The classes C_3, C_4, C_5, C_7, C_{12}, C_{14}, C_{16}, C_{17}, C_{18}, C_{20}, C_{22} and C_{23} consist of one identity each. Finally, for the classes C_k^* we used Corollary 1. □

Theorem 21. *Each balanced identity* I *of a signature with one binary operation only and of the up to fourth length is equivalent in the class of all quasigroups to at least one of* 33 *identities of the list* (1)*, namely: to the identity having in the list* (1) *the same number, as number* k *or* k^* *of the respective class* C_k *or* C_k^**, to which the canonical form of the identity* I *belongs.*

Proof. Without loss of generality, we can consider that I is a canonical 4-identity. But the canonical form of every identity with a number k or k^* in the list (1) respectively belongs to the class C_k or C_k^*, therefore by Theorem 20 the identity I is equivalent to the respective identity from the list (1). □

The variety of quasigroups satisfying the identity with a number k or k^* is denoted by V_k or V_k^*, respectively.

Theorem 22. *The following formulas are true:*

$$V_9 \subset V_2, \tag{17}$$

$$V_{10} \subset V_3, \tag{18}$$

$$V_{13} \subset V_3, \tag{19}$$

$$V_8 = V_2 \cap V_{21}, \tag{20}$$

$$V_{15} = V_{10} \cap V_{10}^*, \tag{21}$$

$$V_{19} = V_2 \cap V_3, \tag{22}$$

$$V_{11} = V_{10} \cap V_{22} = V_{10} \cap V_{23} = V_{10} \cap V_6^*. \tag{23}$$

Proof. By Theorem 21 we can consider each identity from the respective class of the partition on the canonical 4-identities instead of the respective identity of the list (1). Then by Lemma 6 we have the formula (17), by Lemma 15 the formula (18), by Lemma 18 the formula (19), by Lemma 13 the formula (20), by Lemma 19 the formula (21), and by Corollary 4 the formula (22). By Corollary 6 we have

$$V_{11} = V_{10} \cap V_6^*.$$

Thus by Lemma 16 we obtain (23). □

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Department of Algebra
Pedagogical University
Vinnytsia, 287100
Ukraine