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Incidence systems over groups that can be supplemented up to projective planes

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Abstract

In the present article incidence systems over groups, which can be supplemented up to the projective planes by two lines, are studied. All such groups and projective planes are described.

1. Introduction

At present different methods of determination of projective planes over some algebraic systems are known. Algorithms of constructing projective planes over fields, near-fields, semifields [3, 4], complete systems of orthogonal Latin squares [4, 1], ternary systems [3, 4, 1, 6], loop transversals in groups [10] etc. are described. In process of finding the new projective planes researchers more often abandon traditional methods of describing projective planes, and use construction of such incidence systems over universal algebras [2, 5, 11, 13] (in particular, over groups [14, 12]), that can be supplemented up to projective planes in some natural way.

One of the most natural methods of constructing the incidence system over group, that can be supplemented up to projective plane, consists of the following:

"points" of incidence system are all elements of group G,
"lines" of incidence system are left (right) cosets by some

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collection of subgroups of group G, and incidence relation is a belonging relation. This incidence system is supplemented up to projective plane π by the addition of all points of l lines.

It is shown in [14] that if $l = 1$ then the group G is an elementary abelian group and projective plane π is the desarguesian plane. It is shown in [12] that if $l = 2$ then over group G_K of linear transformations of a field K a projective plane can be constructed (by the method mentioned above), which is the desarguesian plane. In general case, if $l = 2$, the problem of describing all groups, that can be supplemented up to projective planes by the method mentioned above, is open. The solution of this problem will be given in the present article.

2. Necessary definitions and notations

Definition 1. A *DK-ternar* is a system $< E, (x, t, y), 0, 1 >$ (where (x, t, y) is a ternary operation on the set E and $0, 1$ are the distinguished elements in E), if the following conditions hold:

1. $(x, 0, y) = x$,
2. $(x, 1, y) = y$,
3. $(x, t, x) = x$,
4. $(0, t, 1) = t$,
5. if a, b, c, d are arbitrary elements from E and $a \neq b$, then the system
$$\begin{cases} (x, a, y) = c \\ (x, b, y) = d \end{cases}$$
has a unique solution in $E \times E$.
6. E is finite or

(a) if a, b, c are arbitrary elements from E and $c \neq 0, (c, a, 0) \neq b$, then the system
$$\begin{cases} (x, a, y) = b \\ (x, t, y) \neq (c, t, 0) \qquad \forall t \in E \end{cases}$$
has a unique solution in $E \times E$.

(b) if a, b are arbitrary elements from E and $b \neq 0$, then the inequality $(a, t, b) \neq (x, t, 0) \quad \forall t \in E$ has a unique solution in E.

Definition 2. A group G is called *sharply double transitive permutation group* on a set E, if the following conditions hold:

1. For any two pairs (a, b) and (c, d) (where $a \neq b, c \neq d$) of elements from E there exists an unique permutation $\alpha \in G$ such that $\alpha(a) = c, \alpha(b) = d$.

2. Set E is finite, or for any elements $a, b \in E$ (where $a \neq b$) there exists an unique fixed-point-free permutation $\alpha \in G$ such that $\alpha(a) = b$.

Definition 3. An *incidence system* $\Sigma(G)$ *of left (right) cosets over the system* Σ *of some subgroups of a group* G is an incidence system $< A, L, I >$ such that:

1. "Points" from A (ΣG*-points*) are all elements of the group G.
2. "Lines" from L (ΣG*-lines*) are left (right) cosets by the subgroups from Σ.
3. An incidence relation is a belonging relation.

3. Main theorems

The main results of the present article are contained in the following two theorems.

Theorem 1. *Let G be a group and Σ be a such system of subgroups of the group G that system $\Sigma(G)$ can be supplemented up to some projective plane π by all points of two lines. Then the following statements are true:*

1. *Group G is isomorphic to a sharply double transitive permutation group on some set E.*
2. *Plane π is a plane dual to the translations plane.*

Theorem 2. *Let conditions of Theorem 1 hold and all subgroups from Σ are centralizators of non-identity elements of group G. Then the following statements are true:*

1. *Group G is isomorphic to the group G_K of linear transformations of some field K.*
2. *Plane π is the desarguesian plane.*

Theorem 2 gives a negative answer to the problem 4.70 from [7].

4. Preliminary statements

Lemma 3. *Let π be an arbitrary projective plane. On plane π coordinates $(a,b),(m),(\infty)$ for points and $[a,b],[m],[\infty]$ for lines (where set E is some set with the distinguished elements $0,1$ and $a,b,m \in E$) can be introduced such that if we define a ternary operation (x,t,y) on the set E by the formula*

$$(x,t,y)=z \quad \overset{def}{\Longleftrightarrow} \quad (x,y)\in[t,z],$$

then system $< E,(x,t,y),0,1 >$ is a DK-ternar.

Proof. See Lemma 1 in [9]. □

Define the following binary operation (x,∞,y) on the set E:

$$(x,\infty,0) \overset{def}{=} x,$$

$$\left\{ \begin{array}{l} (x,\infty,y)=u \\ (x,y)\neq(u,0) \end{array} \right. \quad \overset{def}{\Longleftrightarrow} \quad (x,t,y)\neq(u,t,0) \quad \forall t\in E.$$

According to condition 6(b) of Definition 1, the operation (x,∞,y) is defined correctly.

Lemma 4. *Operation (x,∞,y) satisfies the following conditions:*

1. $\left\{ \begin{array}{l} (x,\infty,y)=(u,\infty,v) \\ (x,y)\neq(u,v) \end{array} \right. \quad \Longleftrightarrow \quad (x,t,y)\neq(u,t,v) \quad \forall t\in E.$
2. $(x,\infty,x)=0.$
3. *if a,b,c are an arbitrary elements from E, then system*

$$\left\{ \begin{array}{l} (x,a,y)=b \\ (x,\infty,y)=c \end{array} \right.$$

has a unique solution in $E\times E$.

Proof. See Lemma 4 in [9]. □

Let $< E,(x,t,y),0,1 >$ be a DK-ternar. Let $(a,b),(m),(\infty)$ be points and $[a,b],[m],[\infty]$ (where $a,b,m\in E$) be lines. We define the following incidence relation I between points and lines:

$$\begin{array}{c}(a,b)\ I\ [c,d] \iff (a,c,b)=d\\ (a,b)\ I\ [d] \iff (a,\infty,b)=d\\ (a)\ I\ [c,d] \iff a=c\\ (a)\ I\ [\infty],\qquad (\infty)\ I\ [d],\qquad (\infty)\ I\ [\infty]\\ (a,b)\ I\ [\infty] \iff (a)\ I\ [d] \iff (\infty)\ I\ [c,d] \iff false.\end{array} \tag{1}$$

Lemma 5. *The incidence system* $< X, L, I >$*, where*

$X = \{(a,b),(m),(\infty) \mid a,b,m \in E\}$,
$L = \{[a,b],[m],[\infty] \mid a,b,m \in E\}$,
I *- incidence relation defined in* (1),

is a projective plane.

Proof. See Lemma 5 in [9]. □

5. Proof of Theorem 1

Let conditions of Theorem 1 hold.

Lemma 6. *All subgroups from the set Σ of subgroups of a group G are exactly all ΣG-lines on a projective plane π, which are incident to ΣG-point E (E is the unit of G).*

Proof. Since the unit of a group is included to any of its subgroup, ΣG-point E is incident to all ΣG-lines-subgroups of the plane π, i.e. it is incident to all subgroups from Σ. Because cosets by the same subgroup in the group G are not intersected, ΣG-point E can not be incident to some ΣG-line, which differs from ΣG-lines-subgroups. □

Define on the plane π coordinates such that the following conditions hold:

1. ΣG-point E has coordinates $(0,1)$.

2. Supplementary lines L_1 and L_2 of the plane π (which is not ΣG-lines) have coordinates $[0]$ and $[\infty]$.

It can be done in the following way, assuming by definition:

$$L_1 = [0], \quad L_2 = [\infty], \quad L_1 \cap L_2 = (\infty).$$

Let M_1, M_2 be arbitrary lines on plane π, which are incident to the ΣG-point E and are not incident to the point (∞). Let

$$O = M_1 \cap L_1, \quad I = M_2 \cap L_1, \quad X = M_1 \cap L_2, \quad Y = M_2 \cap L_2.$$

Points X, Y, O, I are four points in a general position on the plane π (this means that any line contains at most two of these points). Introducing coordinates on π according to Lemma 1 from [9] (see also Lemma 3), we obtain the necessary coordinatization.

According to coordinatization introduced above, we obtain

$$M_1 = [0, 0], \qquad M_2 = [1, 1],$$

and ΣG-lines M_1 and M_2 contain exactly by two supplemented points (i.e. points, which are incident to supplemented lines of the plane π) - points $(0, 0)$ and (0), $(1, 1)$ and (1), correspondingly.

We examine the following two classes of "parallel" lines on the plane π - cosets by subgroups M_1 and M_2:

$$\begin{array}{cccc} M_1^{(0)} = M_1, & M_1^{(1)}, & \ldots & M_1^{(k)}, \quad \ldots \\ M_2^{(0)}, & M_2^{(1)} = M_2, & \ldots & M_2^{(m)}, \quad \ldots \end{array}$$

Since cosets $M_1^{(i)}$ $(M_2^{(j)})$ either don't intersect or coincide, then as lines of π, they can intersect only in supplemented points of the plane π.

Lemma 7. *All lines $M_1^{(i)}$ ($M_2^{(j)}$) intersect in the same supplemented point of the plane π.*

Proof. Assume the contrary, i.e. there exist two different lines $M_1^{(i)}$ and $M_1^{(j)}$ such that

$$M_1 \cap M_1^{(i)} = (0, 0), \quad M_1 \cap M_1^{(j)} = (0).$$

But lines $M_1^{(i)}$ and $M_1^{(j)}$ must intersect in a supplemented point of the plane π too. We have:

$$\begin{array}{rcl} (0, 0) & \in & M_1^{(i)} \implies M_1^{(i)} = [i, 0], \\ (0) & \in & M_1^{(j)} \implies M_1^{(j)} = [0, j]. \end{array}$$

But the line $[i, 0]$ is incident to only two supplemented points of π: the point $(0, 0) = [i, 0] \cap [0]$ and the point $(i) = [i, 0] \cap [\infty]$. Analogously, line $[0, j]$ is incident to only two supplemented points of π: the point $(0) = [0, j] \cap [\infty]$ and the point $(j, j) = [0, j] \cap [0]$. Since

$$(0, 0) = M_1 \cap M_1^{(i)} \neq M_1^{(j)} \cap M_1^{(i)} \neq M_1 \cap M_1^{(j)} = (0),$$

then $(i) = M_1^{(j)} \cap M_1^{(i)} = (j, j)$, which is a contradiction.

For the class of lines $M_2^{(j)}$ the proof is analogous. □

We will suppose below that cosets by subgroups from the set Σ are left cosets. Proof of Theorem 1 in the case of right cosets by subgroups from the set Σ is analogous.

According to Lemma 5, only following four cases may take place:

Case 1. $\bigcap_i M_1^{(i)} = (0), \qquad \bigcap_j M_2^{(j)} = (1).$

Case 2. $\bigcap_i M_1^{(i)} = (0, 0), \quad \bigcap_j M_2^{(j)} = (1, 1).$

Case 3. $\bigcap_i M_1^{(i)} = (0), \qquad \bigcap_j M_2^{(j)} = (1, 1).$

Case 4. $\bigcap_i M_1^{(i)} = (0, 0), \quad \bigcap_j M_2^{(j)} = (1).$

Case 2 is reduced to Case 1 by rearrangement of supplemented lines $L_1 = [0]$ and $L_2 = [\infty]$ before coordinatization of the plane π. Cases 3 and 4 are impossible. Indeed, if Case 3 holds, then

$$\bigcap_i M_1^{(i)} = (0) \implies M_1^{(i)} = [0, i],$$

$$\bigcap_j M_2^{(j)} = (1, 1) \implies M_2^{(j)} = [j, 1].$$

So we obtain $M_1^{(1)} = [0, 1] = M_2^{(0)}$, i.e. for some $g_1, g_2 \in G$

$$g_1 \cdot [0, 0] = g_1 \cdot M_1^{(0)} = g_2 \cdot M_2^{(1)} = g_2 \cdot [1, 1].$$

Whence we obtain $[1, 1] = (g_2^{-1} g_1) \cdot [0, 0]$.

Because $[0,0] \neq [1,1]$, then $(g_2^{-1}g_1) \notin [0,0]$. So we have

$$e \notin (g_2^{-1}g_1) \notin [0,0] = [1,1] \ni e.$$

That is a contradiction. Impossibility of Case 4 is shown analogously.

In Case 1 we have:

$$\begin{array}{l} (0) \in M_1^{(i)} \implies M_1^{(i)} = [0,i], \\ (1) \in M_2^{(j)} \implies M_2^{(j)} = [1,j]. \end{array}$$

For $M_1^{(i)} = [0,i] \doteq A_i$ and $M_2^{(j)} = [1,j] \doteq B_j$ we have

$$A_0 \cap B_1 = [0,0] \cap [1,1] = (0,1)$$

$$A_i \cap B_j = \begin{cases} [0,i] \cap [1,j], & if \quad i \neq j \\ [0,i] \cap [1,i], & if \quad i = j \end{cases} = \begin{cases} (i,j) \in G, & if \quad i \neq j \\ (i,i) \notin G, & if \quad i = j. \end{cases}$$

Now let

$$a_t = \begin{cases} A_0 \cap B_t, & if \quad t \neq 0 \\ c_0 = B_0 \cap A_1, & if \quad t = 0 \end{cases}$$

$$b_t = \begin{cases} B_1 \cap A_t, & if \quad t \neq 1 \\ c_0 = B_0 \cap A_1, & if \quad t = 1. \end{cases}$$

Obviously $A_t = b_t \cdot A_0$ and $B_t = a_t \cdot B_1$.

Lemma 8. *The following statements are true:*

1. $(A_0 \cdot c_0) \cap B_1 = (B_1 \cdot c_0) \cap A_0 = \emptyset$.
2. $(A_0 \cdot c_0) \cap A_0 = (B_1 \cdot c_0) \cap B_1 = \emptyset$.
3. $A_t \cdot c_0 = B_t$ *and* $B_t \cdot c_0 = A_t$ *for* $t \in E$.

Proof. **1**. We prove only the first equality. The proof of the second is analogous. Assume the contrary, i.e. that there exists an element $g_0 \in G$ such that

$$g_0 \in (A_0 \cdot c_0) \cap B_1 = (A_0 \cdot (A_1 \cap B_0)) \cap B_1.$$

Then we obtain

$$\begin{cases} g_0 \in B_1 \\ g_0 \in A_0 \cdot (A_1 \cap B_0) \end{cases} \iff \begin{cases} g_0 = b \in B_1 \\ g_0 \in (a \cdot A_1) \cap (a \cdot B_0) \\ a \in A_0 \end{cases} \iff$$

$$\left\{\begin{array}{l} a \in A_0 \\ B_1 \ni b = (a \cdot A_1) \cap (a \cdot B_0) \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} a \in A_0 \\ a \cdot B_0 = B_1 \end{array}\right. \Longleftrightarrow$$

$$\left\{\begin{array}{l} a^{-1} \in A_0 \\ B_0 = a^{-1} \cdot B_1 \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} a^{-1} \in A_0 \\ a^{-1} \in B_0 \end{array}\right. \Longrightarrow a^{-1} \in A_0 \cap B_0 = \emptyset.$$

We have obtained a contradiction.

2. As in the previous case assume that there exists an element $g_0 \in G$ such that

$$g_0 \in (A_0 \cdot c_0) \cap A_0 = (A_0 \cdot (A_1 \cap B_0)) \cap A_0\,.$$

Then

$$\left\{\begin{array}{ll} g_0 \in A_0, & g_0 = a \cdot c_0 \\ a \in A_0, & c_0 = A_1 \cap B_0 \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} c_0 \in B_0 \\ c_0 = (a^{-1} \cdot g_0) \in A_0\,, \end{array}\right.$$

which implies $c_0 = A_0 \cap B_0 = \emptyset$. But this is impossible.

The obtained contradiction proves the first equality.

The proof of the second is analogous.

3. Observe that $B_t = a_t \cdot B_1$ and $a_t \in A_0$ for any $t \neq 0$. According to statement **1** of the Lemma, for any $t \neq 0$ we have

$$\begin{aligned} (A_0 \cdot c_0) \cap B_t &= (a_t \cdot a_t^{-1}) \cdot ((A_0 \cdot c_0) \cap (a_t \cdot B_1)) \\ &= a_t \cdot ((a_t^{-1} \cdot A_0 \cdot c_0) \cap B_1) \\ &= a_t \cdot ((A_0 \cdot c_0) \cap B_1) = a_t \cdot \emptyset = \emptyset, \end{aligned}$$

which gives

$$A_0 \cdot c_0 = B_0. \tag{2}$$

As we have $A_t = b_t \cdot A_0$ (where $b_1 = c_0$ and $b_t \in B_1$ for $t \neq 1$) for any t, then from (2) we obtain

$$A_t \cdot c_0 = b_t \cdot A_0 \cdot c_0 = b_t \cdot B_0 = B_{\alpha(t)}\,. \tag{3}$$

According to statement **2** of the Lemma, we have for any t

$$(A_t \cdot c_0) \cap A_t = (b_t \cdot A_0 \cdot c_0) \cap (b_t \cdot A_0) = b_t \cdot ((A_0 \cdot c_0) \cap A_0) = b_t \cdot \emptyset = \emptyset,$$

which together with (3) implies $B_{\alpha(t)} \cap A_t = \emptyset$.

It means that $B_{\alpha(t)} = B_t$ for any t. Applying (3) we obtain $A_t \cdot c_0 = B_t$, which completes the proof of the first equality. Analogously we can prove the second equality. $\square$

Corollary 1. *The following equality is true* $c_0 = c_0^{-1}$.

Proof. From the above Lemma

$$c_0^2 = c_0 \cdot c_0 = (A_1 \cap B_0) \cdot c_0 = (A_1 \cdot c_0) \cap (B_0 \cdot c_0) = B_1 \cap A_0 = e,$$

which gives $c_0 = c_0^{-1}$. □

Lemma 9. *There exists a coordinatization of plane* π *such that it satisfies all conditions mentioned above and the following equalities hold:*

$$[t, t] = b_t \cdot [0, 0] \cdot b_t^{-1} = b_t \cdot A_0 \cdot b_t^{-1}$$

for any $t \in E$.

Proof. According to Lemma 6 and Corollary 1 we have

$$\begin{aligned} b_0 \cdot A_0 \cdot b_0^{-1} &= e \cdot A_0 \cdot e = A_0 = [0, 0], \\ b_1 \cdot A_0 \cdot b_1^{-1} &= c_0 \cdot A_0 \cdot c_0^{-1} = c_0 \cdot A_0 \cdot c_0 = c_0 \cdot B_0 = B_1 = [1, 1]. \end{aligned}$$

To determine the coordinatization of the plane π, we choose the ΣG-lines $M_1 = A_0$ and $M_2 = B_1$ arbitrarily - they must only be incident to the point $(0, 1)$ and mustn't be incident to the point (∞). Let us determine the new coordinatization of π, taking instead of ΣG-line M_2 some ΣG-line M_3, which is incident to the point $(0, 1)$, but is not incident to the point (∞) and is different from ΣG-lines M_1 and M_2. This new coordinatization is determined in the same way as the coordinatization described above. Using the analogous reasonings, we obtain $M_3 = g_0 \cdot M_1 \cdot g_0^{-1}$.

But in the initial coordinatization for some $t_0 \in E$ we have

$$g_0 \in b_{t_0} \cdot A_0, \text{ i.e. } g_0 = b_{t_0} \cdot a_k \ ,$$

where $a_k \in A_0$. Thus

$$M_3 = (b_{t_0} \cdot a_k) \cdot A_0 \cdot (b_{t_0} \cdot a_k)^{-1} = b_{t_0} \cdot (a_k \cdot A_0 \cdot a_k^{-1}) \cdot b_{t_0}^{-1} = b_{t_0} \cdot A_0 \cdot b_{t_0}^{-1}.$$

By the help of renaming of points (a, a) $(a \neq 0, 1)$, which are incident to the line $[0]$, we obtain $M_3 = [t_0, t_0]$, i.e.

$$[t_0, t_0] = M_3 = b_{t_0} \cdot A_0 \cdot b_{t_0}^{-1}.$$

Using the analogous reasonings for every ΣG-line M_i, which is incident to the point $(0,1)$ and is not incident to the point (∞), we obtain $[t,t] = b_t \cdot A_0 \cdot b_t^{-1}$ for any $t \in E$. This completes our proof. $\square$

Let $\alpha_{x,y}$ be the ΣG-point $(x,y) = [0,x] \cap [1,y] = A_x \cap B_y$ $(x \neq y)$ and let $\hat{G}$ be the representation of the group G determined by the following permutations:

$$\alpha_{x,y}(t) = u \quad \stackrel{def}{\Longleftrightarrow} \quad \alpha_{x,y} \cdot A_t = A_u. \tag{4}$$

Lemma 10. *The following statements are true:*

1. *The permutation group $\hat{G}$ is isomorphic to the group G.*
2. $\alpha_{x,y}(0) = x,\ \ \alpha_{x,y}(1) = y$ *for any $x \neq y$ from E.*
3. *For any fixed elements $a, b \in E$, $(a \neq b)$, there exists an uniquely determined permutation $\alpha_{u,v}$ such that $\alpha_{u,v}(0) = a$, $\alpha_{u,v}(1) = b$.*
4. *For any fixed pairs $(a,b),(c,d) \in E \times E$, $(a \neq b,\ c \neq d)$, there exists an uniquely determined permutation $\alpha_{u,v} \in \hat{G}$ such that*
$$\alpha_{u,v}(a) = c, \quad \alpha_{u,v}(b) = d.$$
5. *For any fixed $a, b \in E$, $(a \neq b)$, there exists an uniquely determined fixed-point-free permutation $\alpha_{u,v}$ such that $\alpha_{u,v}(a) = b$.*

Proof. **1**. As we can see from (4), the representation $\hat{G}$ is a representation of the group G by left cosets with respect to the subgroup A_0. According to Theorem 5.3.2 from [3], the kernel of this representation is a subgroup H_0 of G such that $H_0 \subseteq A_0$ and $H_0 \lhd G$.

Taking $g = c_0 = A_1 \cap B_0$, we obtain

$$A_0 \supseteq H_0 = c_0 H_0 c_0^{-1} \subseteq c_0 A_0 c_0^{-1} = B_1,$$

i.e.

$$H_0 \subseteq A_0 \cap B_1 = e.$$

So representation (4) is the exact representation, and $\hat{G} \simeq G$.

2. We have

$$\alpha_{x,y}(0) = u \iff \alpha_{x,y} \cdot A_0 = A_u \implies \alpha_{x,y} \in A_u.$$

Directly from the definition of $\alpha_{x,y}$ we obtain $u = x$, i.e. $\alpha_{x,y}(0) = x$. By Lemma 6 we have

$$\alpha_{x,y}(1) = v \iff \alpha_{x,y} \cdot A_1 = A_v \iff \alpha_{x,y} \cdot A_1 \cdot c_0 = A_v \cdot c_0$$
$$\iff \alpha_{x,y} \cdot B_1 = B_v \implies \alpha_{x,y} \in B_v.$$

This gives $y = v$, i.e. $\alpha_{x,y}(1) = y$.

3. Let $a, b \in E$, $(a \neq b)$. Since $\alpha_{a,b}(0) = a$ and $\alpha_{a,b}(1) = b$, then the necessary permutation $\alpha_{u,v} \in \hat{G}$ exists and coincides with $\alpha_{a,b}$. If there exists the other permutation $\alpha_{u,v} \in \hat{G}$ such that $\alpha_{u,v}(0) = a$, $\alpha_{u,v}(1) = b$, then for the permutation $\alpha_{k,m} = \alpha_{a,b}^{-1}\alpha_{u,v}$ we have:

$$\alpha_{k,m}(0) = \alpha_{a,b}^{-1}\alpha_{u,v}(0) = \alpha_{a,b}^{-1}(a) = 0,$$
$$\alpha_{k,m}(1) = \alpha_{a,b}^{-1}\alpha_{u,v}(1) = \alpha_{a,b}^{-1}(b) = 1.$$

Moreover, applying (4) and Lemma 6, we obtain

$$\begin{cases} \alpha_{k,m} \cdot A_0 = A_0 \\ \alpha_{k,m} \cdot A_1 = A_1 \end{cases} \iff \begin{cases} \alpha_{k,m} \in A_0 \\ \alpha_{k,m} \cdot A_1 \cdot c_0 = A_1 \cdot c_0 \end{cases} \iff$$

$$\begin{cases} \alpha_{k,m} \in A_0 \\ \alpha_{k,m} \cdot B_1 = B_1 \end{cases} \iff \begin{cases} \alpha_{k,m} \in A_0 \\ \alpha_{k,m} \in B_1 \end{cases} \iff \alpha_{k,m} = A_0 \cap B_1 = e.$$

Thus $\alpha_{a,b}^{-1}\alpha_{u,v} = e$, i.e. $\alpha_{u,v} = \alpha_{a,b}$. Hence the permutation $\alpha_{u,v}$ is a uniquely determined.

4. Let $a, b, c, d \in E$, $a \neq b$, $c \neq d$ and $\alpha_{u_0,v_0} \stackrel{def}{=} \alpha_{c,d}\alpha_{a,b}^{-1}$. Then

$$\alpha_{u_0,v_0}(a) = \alpha_{c,d}\alpha_{a,b}^{-1}(a) = \alpha_{c,d}(0) = c,$$
$$\alpha_{u_0,v_0}(b) = \alpha_{c,d}\alpha_{a,b}^{-1}(b) = \alpha_{c,d}(1) = d,$$

i.e. we have proved the existence of necessary permutation $\alpha_{u,v} \in \hat{G}$.

If $\alpha_{r,s} \in \hat{G}$ and $\alpha_{r,s}(a) = c$, $\alpha_{r,s}(b) = d$, then for the permutation $\gamma = \alpha_{r,s}\alpha_{a,b}$ we have

$$\gamma(0) = \alpha_{r,s}\alpha_{a,b}(0) = \alpha_{r,s}(a) = c,$$
$$\gamma(1) = \alpha_{r,s}\alpha_{a,b}(1) = \alpha_{r,s}(b) = d.$$

By the statement **3** of the Lemma we obtain $\gamma \equiv \alpha_{c,d}$, i.e.

$$\alpha_{r,s} = \alpha_{c,d}\alpha_{a,b}^{-1} = \alpha_{u_0,v_0}.$$

This proves that the permutation $\alpha_{u,v}$ is uniquely determined.

5. Let $a, b \in E,\ a \neq b$. Since

$$G = \left(\bigcup_{k \in E} [k,k] \right) \cup [(0, \infty, 1)]$$

is the set of all ΣG-lines, which are incident to ΣG-point $(0,1)$, then we can obtain the following equivalent systems (by Lemmas 6 and 7):

$$\begin{cases} \alpha_{x,y}(a) = b \\ \alpha_{x,y}(t) \neq t \quad \forall t \in E \end{cases} \Longleftrightarrow \begin{cases} \alpha_{x,y} \cdot A_a = A_b \\ \alpha_{x,y} \cdot A_t \neq A_t \quad \forall t \in E \end{cases} \Longleftrightarrow$$

$$\begin{cases} \alpha_{x,y} \cdot b_a \cdot A_0 = b_a \cdot A_0 \\ \alpha_{x,y} \cdot b_t \cdot A_0 \neq b_t \cdot A_0 \quad \forall t \in E \end{cases} \Longleftrightarrow \begin{cases} \alpha_{x,y} \cdot b_a \in A_b \\ \alpha_{x,y} \cdot b_t \notin b_t \cdot A_0 \quad \forall t \in E \end{cases}$$

$$\Longleftrightarrow \left[\begin{array}{l} \begin{cases} a = 1, \quad b \neq 1 \\ \alpha_{x,y} \cdot c_0 \in A_b \\ \alpha_{x,y} \notin b_t A_0 b_t^{-1} \quad \forall t \in E \end{cases} \\ \begin{cases} a \neq 1, \quad b \neq a \\ \alpha_{x,y} \cdot b_a \in A_b \\ \alpha_{x,y} \notin b_t A_0 b_t^{-1} \quad \forall t \in E \end{cases} \end{array} \right. \Longleftrightarrow$$

$$\Longleftrightarrow \left[\begin{array}{l} \begin{cases} a = 1, \quad b \neq 1 \\ \alpha_{x,y} \in A_b \cdot c_0 = B_b \\ \alpha_{x,y} \notin b_t A_0 b_t^{-1} \quad \forall t \in E \end{cases} \\ \begin{cases} a \neq 1, \quad b \neq a \\ \alpha_{u,v} = \alpha_{x,y} \cdot b_a \in A_b \\ b_a^{-1} \cdot \alpha_{u,v} \notin (b_a^{-1} b_t) A_0 (b_a^{-1} b_t)^{-1} \quad \forall t \in E \end{cases} \end{array} \right. \Longleftrightarrow$$

$$\Longleftrightarrow \left[\begin{array}{l} \begin{cases} a = 1, \quad b \neq 1 \\ \alpha_{x,y} \in B_b \\ \alpha_{x,y} \notin b_t A_0 b_t^{-1} \quad \forall t \in E \end{cases} \\ \begin{cases} a \neq 1, \quad b \neq a \\ \alpha_{u,v} = \alpha_{x,y} \cdot b_a \in A_b \\ b_a^{-1} \cdot \alpha_{u,v} \notin b_{t'} A_0 b_{t'}^{-1} \quad \forall t' \in E \end{cases} \end{array} \right. \Longleftrightarrow$$

$$\Longleftrightarrow \left[\begin{array}{l} \begin{cases} a = 1, \quad b \neq 1 \\ \alpha_{x,y} \in B_b \\ \alpha_{x,y} \notin [t,t] \quad \forall t \in E \end{cases} \\ \begin{cases} a \neq 1, \quad b \neq a \\ \alpha_{u,v} = \alpha_{x,y} \cdot b_a \in A_b \\ b_a^{-1} \cdot \alpha_{u,v} \notin [t', t'] \quad \forall t' \in E \end{cases} \end{array} \right. \Longleftrightarrow$$

$$\Longleftrightarrow \left[\begin{array}{l} \left\{ \begin{array}{l} a = 1, \quad b \neq 1 \\ \alpha_{x,y} \in B_b \\ \alpha_{x,y} \in [(0, \infty, 1)] \end{array} \right. \\ \left\{ \begin{array}{l} a \neq 1, \quad b \neq a \\ \alpha_{u,v} = \alpha_{x,y} \cdot b_a \in A_b \\ b_a^{-1} \cdot \alpha_{u,v} \in [(0, \infty, 1)] \end{array} \right. \end{array} \right. \Longleftrightarrow$$

$$\Longleftrightarrow \left[\begin{array}{l} \left\{ \begin{array}{l} a = 1, \quad b \neq 1 \\ \alpha_{x,y} = [1, b] \cap [(0, \infty, 1)] \end{array} \right. \\ \left\{ \begin{array}{l} a \neq 1, \quad b \neq a \\ \alpha_{x,y} = ([0, b] \cap b_a \cdot [(0, \infty, 1)]) \cdot b_a^{-1} \end{array} \right. \end{array} \right.$$

As we can see from the last system, the existence and uniqueness of the necessary permutation $\alpha_{x,y}$ in $\hat{G}$ is obvious. □

The last lemma shows (according to Definition 2) that the group G is isomorphic to a sharply double transitive permutation group on E.

By Theorem 20.7.1 from [3] the group G is isomorphic to the group

$$H = \{\alpha_{a,b} | \alpha_{a,b}(x) = a + x \cdot (b - a), \ b \neq a\}$$

of linear transformations of some nearfield $K =< E, +, \cdot, 0, 1 >$ (a definition of a nearfield is given in [3, 9]). It is easy to see that the group operation in G can be expressed by the operations of the nearfield K in the following way:

$$\alpha_{a,b} \cdot \alpha_{c,d} = \alpha_{a+c\cdot(b-a), a+d\cdot(b-a)}. \tag{5}$$

Now we consider the following ternary operation:

$$\begin{array}{l} [x, t, y] \stackrel{def}{=} x + t \cdot (y - x), \\ [x, \infty, y] = x - y. \end{array}$$

Over D-ternar $< E, [x, t, y], 0, 1 >$ a projective plane π^* can be constructed (see Lemma 3), which is the plane dual to translation plane

[3, 8]. The incidence relation I^* on plane π^* is determined by:

$$\begin{array}{c} (a,b)\ I^*\ [c,d] \iff d = a + c\cdot(b-a) \\ (a,b)\ I^*\ [d] \iff d = a - b \\ (a)\ I^*\ [c,d] \iff a = c \\ (a)\ I^*\ [\infty], \qquad (\infty)\ I^*\ [d], \qquad (\infty)\ I^*\ [\infty] \\ (a,b)\ I^*\ [\infty] \iff (a)\ I^*\ [d] \iff (\infty)I^*[c,d] \iff false. \end{array} \tag{6}$$

Lemma 11. *The initial plane π, which has been constructed over an incidence system $\Sigma(G)$ by supplementing two lines, is isomorphic to the plane π^*.*

Proof. According to Lemma 4, ΣG-point $(0,1)$ (the unit e of group G) is incident only to ΣG-lines of π, which are subgroups from the system Σ (i.e. lines $[c,c]$, $c \in E$ and $[(0,\infty,1)]$). Since $G \backsimeq H$ then the point $(0,1)$ of the plane π^* is incident to the lines $[c,c]$, $c \in E$ and $[-1] = [[0,\infty,1]]$. Let

$M_c = \{\alpha_{a,b}{\cdot}\alpha_{u,v} | (u,v)I^*[c,c],\ c$ be an *arbitrary fixed element from* $E\}$,

$$R = \{\alpha_{a,b}\cdot\alpha_{z,w} | (z,w)I^*[-1]\},$$

where (a,b) is an arbitrary fixed point of the plane π^*, which is not incident to the lines $[0]$ and $[\infty]$. In order to prove the isomorphism of the planes π and π^*, it is sufficient to prove that the set M_c is a line $[c,d]$ on π^* (for some $d \in E$) and the set R is a line $[h]$ on π^* (for some $h \in E$). By the help of (5) and (6) we obtain

$$(k,l) \in M_c \iff \left\{ \begin{array}{c} \alpha_{k,l} = \alpha_{a,b}\cdot\alpha_{u,v} \\ (u,v)I^*[c,c] \end{array} \right. \iff \left\{ \begin{array}{c} k = a + u\cdot(b-a) \\ l = a + v\cdot(b-a) \\ c = u + c\cdot(v-u) \end{array} \right. \iff$$

$$\begin{aligned} k + c\cdot(l-k) &= a + u\cdot(b-a) + c\cdot(a + v\cdot(b-a) - a - u\cdot(b-a)) \\ &= a + u\cdot(b-a) + c\cdot(v-u)\cdot(b-a) \\ &= a + (u + c\cdot(v-u))\cdot(b-a) = a + c\cdot(b-a) = d, \end{aligned}$$

i.e. $(k,l)I^*[c,d]$.

Analogously,

$$(k,l) \in R \iff \left\{ \begin{array}{l} \alpha_{k,l} = \alpha_{a,b}\cdot\alpha_{z,w} \\ (z,w)I^*[-1] \end{array} \right. \iff \left\{ \begin{array}{l} k = a + z\cdot(b-a) \\ l = a + w\cdot(b-a) \\ -1 = z - w \end{array} \right. \iff$$

$$\begin{aligned} k - l &= a + z \cdot (b - a) - a - w \cdot (b - a) \\ &= (z - w) \cdot (b - a) = (-1) \cdot (b - a) = a - b = h, \end{aligned}$$

i.e. $(k, l)I^*[h]$, which completes our proof. □

Theorem 1 follows from the above lemmas.

6. Proof of Theorem 2

Let the assumption of Theorem 2 be satisfied.

Lemma 12. *Let $H = C_a(G)$ be the centralizer of $a \in G$, $a \neq e$. Then for any $h \in H - \{e\}$ we have $C_h(G) = H$ and H is an abelian group.*

Proof. It is evident that $e, h \in C_h(G)$ for any $h \in H - \{e\}$. If $h_1 \neq k_2$ and $h_1, h_2 \in H - \{e\}$, then

$$h_1 \in C_a(G) \iff h_1^{-1} a h_1 = a \iff a^{-1} h_1 a = h_1 \iff a \in C_{h_1}(G),$$
$$h_2 \in C_a(G) \iff h_2^{-1} a h_2 = a \iff a^{-1} h_2 a = h_2 \iff a \in C_{h_2}(G),$$

i.e. $\{e, a\} \subset C_{h_1}(G) \cap C_{h_2}(G)$.

But the centralizers $C_{h_1}(G)$ and $C_{h_2}(G)$ are lines in the plane π, so either they coincide or they have no more than one common point. Then we obtain

$$C_{h_1}(G) \equiv C_{h_2}(G) \equiv C_h(G)$$

for any $h \in H - \{e\}$. Since $a \in C_a(G) = H$ then for any $h \in H - \{e\}$ we have $H = C_h(G)$. So for any $h_1, h_2 \in H$

$$h_1 \in H = C_{h_2}(G) \iff h_1^{-1} h_2 h_1 = h_2 \iff h_2 h_1 = h_1 h_2,$$

i.e. H is an abelian group. □

This means that all ΣG-lines $[c, c]$ of the plane π are abelian groups. According to (5) we have for any $a, b \in E - \{0\}$:

$$\alpha_{0,a} \cdot \alpha_{0,b} = \alpha_{0,b \cdot a}, \qquad (0, a)I^*[0, 0].$$

So multiplication of the nearfield $K =< E, +, \cdot, 0, 1 >$ is commutative, i.e. K is a field. Then the group G is isomorphic to the group G_K

of linear transformations of the field K. The plane π is isomorphic to the plane π^*, which is constructed by the natural way over the field K, i.e. it is desarguesian [3, 8].

The proof of Theorem 2 is complete.

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