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Invertible elements in associates and semigroups. 1

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Abstract

Some invertibility criteria of an element in associates, in particular in n-ary semigroups, are given. As a corollary, axiomatics for polyagroups and n-ary groups are obtained.

Invertible elements play a special role in the theory of n-ary groupoids. For example, the structure of operations in an associate without invertible elements is still open. However, in the associate of the type (r, s, n) the structure of its operation is determined by Theorem 4 from [2] as soon as there exists at least one r-multiple invertible element in it. In particular, this theorem reduces the study of the groupoid to the study of associate of the type $(1, s, n)$ with invertible elements. Since, as was shown in [3], a binary semigroup with an invertible element is exactly a monoid, so we will take the characteristic to introduce a notion of multiary monoid.

1. Necessary informations

Let $(Q; f)$ be an $(n+1)$-ary groupoid. The operation f and the groupoid $(Q; f)$ are called (i, j)-*associative,* if the identity

$$f(x_0, \dots, x_{i-1}, f(x_i, \dots, x_{i+n}), x_{i+n+1}, \dots, x_{2n})$$
$$= f(x_0, \dots, x_{j-1}, f(x_j, \dots, x_{j+n}), x_{j+n+1}, \dots, x_{2n}).$$

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holds in $(G;f)$.

Definition 1. A groupoid $(Q;f)$ of the arity $n+1$ is said to be an *associate of the type* (r,s,n), where r divides s, s divides n, and $n>s$, if it is (i,j)-associative for all (i,j) such, that $i\equiv j\equiv 0 \pmod r$, and $i\equiv j \pmod s$. In an associate of the type (s,n), that is of the type $(1,s,n)$, the number s will be called a *degree of associativity*, and the associative operation f will be called *s-associative*. The least of the associativity degrees will be called a *period of associativity*.

The following theorem is proved in [4].

Theorem 1. *Let $(Q;f)$ be an associate of a type (r,s,n). If the words w_1 and w_2 differ from each other by bracketting only; the coordinate of every f's occurrence in the words w_1 and w_2 is divisible by r and there exists an one-to-one correspondence between f's occurrences in the word w_1 and those in the word w_2 such that the corresponding coordinates are congruent modulo s, then the formula $w_1=w_2$ is an identity in $(Q;f)$.*

Here the *coordinate of the i-th occurrence of the symbol f in a word w* is called a number of all individual variables and constants, appearing in the word w from the beginning of w to the i-th occurrence of the operation symbol f.

To define an invertible element we need the notion of a shift.

Let $(Q;f)$ be an $(n+1)$-ary groupoid. The notation $\overset{i}{a}$ denotes a sequence $a,\dots,a$ (i times).

A transformation $\lambda_{i,a}$ of the set Q, which is determined by the equality

$$\lambda_{i,a}(x)=f(\overset{i}{a},x,\overset{n-i}{a}), \tag{1}$$

is said to be an *i-th shift of the groupoid $(Q;f)$, induced by an element a*. Hence, the i-th shift is a partial case of the translation (see [1]). If an i-th shift is a substitution of the set Q, then the element a is called *i-invertible*. If an element a is i-invertible for all i multiple of r, then it is called *r-multiple invertible*, when $r=1$ it is called *invertible*. The unit is always invertible, since, it determines a shift being an identity transformation.

The notion of an invertible element for binary and n-ary groupoids coincides with a well known one. Namely, if $(Q;\cdot)$ is a semigroup, and a is its arbitrary invertible element, that is the shifts $\lambda_{0,a}$ and $\lambda_{1,a}$ are substitutions of the set Q, then it is easy to prove (see [3]), that the elements $\lambda_{0,a}^{-1}(a)$, $\lambda_{1,a}^{-1}(a)$ are right and left identity elements in the semigroup. Therefore, $\lambda_{0,a}^{-1}(a) = \lambda_{1,a}^{-1}(a)$ is an identity element, left and right inverse elements of the element a are $\lambda_{1,a}^{-2}(a)$ and $\lambda_{0,a}^{-2}(a)$ respectively. Thus, $a^{-1} := \lambda_{1,a}^{-2}(a) = \lambda_{0,a}^{-2}(a)$ is an inverse element of a.

If an element a of a multiary groupoid is i-invertible, then the element $\lambda_{i,a}^{-1}(a)$ coincides with the i*-th skew* element of a, which is denoted by $\bar{a}^i$, where $\bar{a} = \bar{a}^0$, and it is determined by the equality

$$f(\overset{i}{a}, \bar{a}^i, \overset{n-i}{a}) = a.$$

The following two lemmas are proved in [2]

Lemma 2. *If in an associate of the type (r,s,n) an element a is s-multiple invertible and $i \equiv 0 \pmod{s}$, then there exists a unique i-th skew of the element a, and, in addition, the equality*

$$\bar{a} = \bar{a}^i \tag{2}$$

holds.

Lemma 3. *In every associate of the type (r,s,n) for every s-multiple invertible element of the element a and for all $i \equiv 0 \pmod{s}$ the following identities are true*

$$f(\overset{i}{a}, \bar{a}, \overset{n-i-1}{a}, x) = x, \qquad f(x, \overset{n-i-1}{a}, \bar{a}, \overset{i}{a}) = x. \tag{3}$$

2. Criteria of invertibility of elements

One of the main results of this article is the following.

Theorem 4. *An element $a \in Q$ is r-multiple invertible in an associate $(Q;f)$ of the type (r,s,n) iff there exists an element $\bar{a} \in Q$ such that*

$$f(\bar{a}, a, \dots a, x) = x, \qquad f(x, a, \dots a, \bar{a}) = x \tag{4}$$

holds for all $x \in Q$.

Proof. If an element a is r-multiple invertible in $(Q;f)$, then the relation (4) follows from (3) when $i=0$.

Let the relationship (4) hold. To establish the invertibility of the element a, we have to prove the existence of an inverse transformation for every of the shifts induced by the element a. This follows from the following lemma.

Lemma 5. *Let $(Q;f)$ be an associate of the type (r,s,n). If for $a\in Q$ there exists an element $\bar a$ satisfying (4), then every i-th shift, induced by a, has an inverse transformation, which can be found by the formulae*

$$\begin{aligned}
&\lambda_{0,a}^{-1}(x)=f\left(x,\overset{n-s-1}{a},\bar a,\overset{s-1}{a},\bar a\right),\\
&\lambda_{n,a}^{-1}(x)=f\left(\bar a,\overset{s-1}{a},\bar a,\overset{n-s-1}{a},x\right),\\
&\lambda_{i,a}^{-1}(x)=f\left(\overset{n-s-i}{a},\bar a,\overset{s-1}{a},x,\overset{i-1}{a},\bar a\right),\quad \text{when } 0<i\leqslant n-s,\\
&\lambda_{i,a}^{-1}(x)=f\left(\bar a,\overset{n-i-1}{a},x,\overset{s-1}{a},\bar a,\overset{i-s}{a}\right),\quad \text{when } s\leqslant i<n.
\end{aligned}\tag{5}$$

Proof of Lemma. If i in (3) is a multiple of s, then

$$\begin{aligned}
x&\overset{(4)}{=}f(\bar a,\overset{n-1}{a},x)\overset{(4)}{=}f(\bar a,\overset{i-1}{a},f(\overset{n}{a},\bar a),\overset{n-i-1}{a},x)\\
&\overset{Th1}{=}f(f(\bar a,\overset{n}{a}),\overset{i-1}{a},\bar a,\overset{n-i-1}{a},x)\overset{(4)}{=}f(\overset{i}{a},\bar a,\overset{n-i-1}{a},x).
\end{aligned}$$

The other relationships from (3) are proved by the same way:

$$\begin{aligned}
x&\overset{(4)}{=}f(x,\overset{n-1}{a},\bar a)\overset{(4)}{=}f(x,\overset{n-i-1}{a},f(\bar a,\overset{n}{a}),\overset{i-1}{a},\bar a)\\
&\overset{Th1}{=}f(x,\overset{n-i-1}{a},\bar a,\overset{i-1}{a},f(\overset{n}{a},\bar a))\overset{(4)}{=}f(x,\overset{n-i-1}{a},\bar a,\overset{i}{a}).
\end{aligned}$$

Let us prove that the transformation $\lambda_{0,a}^{-1}$, which is determined by the equality (5) is inverse to $\lambda_{0,a}$.

$$\begin{aligned}
\lambda_{0,a}^{-1}\lambda_{0,a}(x)&\overset{(5)}{=}f(\lambda_{n,a}(x),\overset{n-s-1}{a},\bar a,\overset{s-1}{a},\bar a)\overset{(1)}{=}f(f(x,\overset{n}{a}),\overset{n-s-1}{a},\bar a,\overset{s-1}{a},\bar a)\\
&\overset{Th1}{=}f(x,\overset{n-s-1}{a},f(\overset{n}{a},\bar a),\overset{s-1}{a},\bar a)\overset{(3)}{=}f(x,\overset{n-1}{a},\bar a)\overset{(3)}{=}x\,,
\end{aligned}$$

$$\begin{aligned}
\lambda_{0,a}\lambda_{0,a}^{-1}(x)&\overset{(5)}{=}\lambda_{0,a}f(x,\overset{n-s-1}{a},\bar a,\overset{s-1}{a},\bar a)\overset{(1)}{=}f(f(x,\overset{n-s-1}{a},\bar a,\overset{s-1}{a},\bar a),\overset{n}{a})\\
&\overset{Th1}{=}f(x,\overset{n-s-1}{a},\bar a,\overset{s-1}{a},f(\bar a,\overset{n}{a}))\overset{(3)}{=}f(x,\overset{n-s-1}{a},\bar a,\overset{s}{a})\overset{(3)}{=}x.
\end{aligned}$$

Hence $\lambda_{0,a}\lambda_{0,a}^{-1}=\lambda_{0,a}^{-1}\lambda_{0,a}=\varepsilon$, where ε is the identity mapping. Thus, the transformation $\lambda_{0,a}^{-1}$, determined by the equality (5), is inverse to the shift $\lambda_{0,a}$. Analogously one can prove the other equalities from (5).

$$\begin{aligned}\lambda_{n,a}^{-1}\lambda_{n,a}(x) &\overset{(5)}{=} f(\bar{a},\overset{s-1}{a},\bar{a},\overset{n-s-1}{a},\lambda_{n,a}(x)) \overset{(1)}{=} f(\bar{a},\overset{s-1}{a},\bar{a},\overset{n-s-1}{a},f(\overset{n}{a},x))\\ &\overset{Th1}{=} f(\bar{a},\overset{s-1}{a},f(\bar{a},\overset{n}{a}),\overset{n-s-1}{a},x) \overset{(3)}{=} f(\bar{a},\overset{n-1}{a},x) \overset{(3)}{=} x\,,\end{aligned}$$

$$\begin{aligned}\lambda_{n,a}\lambda_{n,a}^{-1}(x) &\overset{(5)}{=} \lambda_{n,a}f(\bar{a},\overset{s-1}{a},\bar{a},\overset{n-s-1}{a},x) \overset{(1)}{=} f(\overset{n}{a},f(\bar{a},\overset{s-1}{a},\bar{a},\overset{n-s-1}{a},x))\\ &\overset{Th1}{=} f(f(\overset{n}{a},\bar{a}),\overset{s-1}{a},\bar{a},\overset{n-s-1}{a},x) \overset{(3)}{=} f(\overset{s}{a},\bar{a},\overset{n-s-1}{a},x) \overset{(3)}{=} x.\end{aligned}$$

Let number $i\leqslant n-3$ be a multiple of r. Then

$$\begin{aligned}\lambda_{i,a}\lambda_{i,a}^{-1}(x) &\overset{(5)}{=} f(\overset{i}{a},f(\overset{n-s-i}{a},\bar{a},\overset{s-1}{a},x,\overset{i-1}{a},\bar{a}),\overset{n-i}{a})\\ &\overset{(3)}{=} f(\overset{i}{a},f(\overset{n-s-i}{a},\bar{a},\overset{s-1}{a},x,\overset{i-1}{a},\bar{a}),\overset{n-i-1}{a},f(\overset{n}{a},\bar{a}))\\ &\overset{Th1}{=} f(f(\overset{n-s}{a},\bar{a},\overset{s-1}{a},x),\overset{i-1}{a},f(\bar{a},\overset{n}{a}),\overset{n-i-1}{a},\bar{a})=\\ &\overset{(3)}{=} f(f(\overset{n-s}{a},\bar{a},\overset{s-1}{a},x),\overset{n-1}{a},\bar{a}) \overset{(3)}{=} f(x,\overset{n-1}{a},\bar{a}) \overset{(3)}{=} x.\end{aligned}$$

If $i=n-s$, then the equality (5) defines the transformation

$$\lambda_{n-s,a}^{-1}(x)=f(\bar{a},\overset{s-1}{a},x,\overset{n-s-1}{a},\bar{a})\,,$$

which implies

$$\begin{aligned}\lambda_{n-s,a}^{-1}\lambda_{n-s,a}(x) &\overset{(1)}{=} f(\bar{a},\overset{s-1}{a},f(\overset{n-s}{a},x,\overset{s}{a}),\overset{n-s-1}{a},\bar{a})\\ &\overset{Th1}{=} f(f(\bar{a},\overset{n-1}{a},x),\overset{n-1}{a},\bar{a}) \overset{(3)}{=} f(x,\overset{n-1}{a},\bar{a}) \overset{(3)}{=} x.\end{aligned}$$

If $i<n-s$, then

$$\begin{aligned}\lambda_{i,a}^{-1}\lambda_{i,a}(x) &\overset{(5)}{=} f(\overset{n-s-i}{a},\bar{a},\overset{s-1}{a},f(\overset{i}{a},x,\overset{n-i}{a}),\overset{i-1}{a},\bar{a})\\ &\overset{(3)}{=} f(f(\bar{a},\overset{n}{a}),\overset{n-s-i-1}{a},\bar{a},\overset{s-1}{a},f(\overset{i}{a},x,\overset{n-i}{a}),\overset{i-1}{a},\bar{a})\\ &\overset{Th1}{=} f(f(\bar{a},\overset{n-i-1}{a},f(\overset{n-s}{a},\bar{a},\overset{s}{a}),\overset{i-1}{a},x),\overset{n-1}{a},\bar{a}) \overset{(3)}{=} x.\end{aligned}$$

If $i \geqslant s$, then

$$\lambda_{i,a}\lambda_{i,a}^{-1}(x) \overset{(5)}{=} f(\overset{i}{a}, f(\bar{a}, \overset{n-i-1}{a}, x, \overset{s-1}{a}, \bar{a}, \overset{i-s}{a}), \overset{n-i}{a})$$
$$\overset{(3)}{=} f(f(\bar{a}, \overset{n}{a}), \overset{i-1}{a}, f(\bar{a}, \overset{n-i-1}{a}, x, \overset{s-1}{a}, \bar{a}, \overset{i-s}{a}), \overset{n-i}{a})$$
$$\overset{Th1}{=} f(f(\bar{a}, \overset{i-1}{a}, f(\overset{n}{a}, \bar{a}), \overset{n-i-1}{a}, x), \overset{s-1}{a}, \bar{a}, \overset{n-s}{a}) \overset{(3)}{=} x.$$

To prove $\lambda_{i,a}^{-1}\lambda_{i,a}(x) = \varepsilon$, we consider two cases: $i = s$ and $i > s$. If $i = s$, then (5) can be rewritten as $\lambda_{s,a}^{-1}(x) = f(\bar{a}, \overset{n-s-1}{a}, x, \overset{s-1}{a}, \bar{a})$. Therefore we get

$$\lambda_{s,a}^{-1}\lambda_{s,a}(x) \overset{(1)}{=} f(\bar{a}, \overset{n-s-1}{a}, f(\overset{s}{a}, x, \overset{n-s}{a}), \overset{s-1}{a}, \bar{a})$$
$$\overset{Th1}{=} f(f(\bar{a}, \overset{n-1}{a}, x), \overset{n-1}{a}, \bar{a}) \overset{(3)}{=} f(x, \overset{n-1}{a}, \bar{a}) \overset{(3)}{=} x.$$

If $i > s$, then

$$\lambda_{i,a}^{-1}\lambda_{i,a}(x) \overset{(5)}{=} f(\bar{a}, \overset{n-i-1}{a}, f(\overset{i}{a}, x, \overset{n-i}{a}), \overset{s-1}{a}, \bar{a}, \overset{i-s}{a})$$
$$\overset{(3)}{=} f(\bar{a}, \overset{n-i-1}{a}, f(\overset{i}{a}, x, \overset{n-i}{a}), \overset{s-1}{a}, \bar{a}, \overset{i-s-1}{a}, f(\overset{n}{a}, \bar{a}))$$
$$\overset{Th1}{=} f(f(\bar{a}, \overset{n-1}{a}, x), \overset{n-i-1}{a}, f(\overset{s}{a}, \bar{a}, \overset{n-s}{a}), \overset{i-1}{a}, \bar{a}) \overset{(3)}{=} x.$$

The lemma and the theorem has been proved. □

Since for $r = s = 1$ we obtain an $(n+1)$-ary semigroup, then the following corollary is true.

Corollary 1. *An element $a \in Q$ is invertible in an $(n+1)$-ary semigroup $(Q; f)$ iff there exists an element $\bar{a} \in Q$ such that* (4) *holds for all $x \in Q$.*

3. Monoids and invertible elements

In the associate of the type (r, s, n) the structure of its operation is determined by Theorem 4 from [2] as soon as there exists at least one r-multiple invertible element in it. In particular, this theorem implies (see Corollary 11 in [2]) that the study of the groupoid reduces to the study of an associate of the type $(1, s, n)$ with invertible elements, that is why we will consider the last ones only. Since, as it was shown above, a binary semigroup with an invertible element is exactly a monoid, so we will use this characteristic to introduce its generalization and we will call it *invert* (a multiary monoid was called a semigroup with an identity element).

Since every invertible element of an invert determines some decomposition monoid, natural questions on the relations between the algebraic notions for a monoid and decomposition monoids as well as about relations between different decompositions of the same monoid arise. Here we will consider this relation between the sets of invertible elements.

Definition 2. An associate of the type $(1, s, n)$ containing at least one invertible element will be called an *invert of the type* (s, n).

When $s = 1$, then an invert is an $(n+1)$-ary semigroup containing at least one invertible element. So, every $(n + 1)$-ary monoid is an invert.

If an invert has at least one neutral element e, then, as follows from the results given below, the automorphism of its e-decomposition is identical, therefore its associativity period is equal to one, that is, such invert is a monoid. Every $(n + 1)$-ary group is an invert, since every its element is invertible.

The next statement, which follows from Theorem 4 in [2], gives a decomposition of the operation of an invert.

Theorem 6. *Let $(Q; f)$ be an $(n + 1)$-ary invert of the associativity period s. Then for every its invertible element 0 there exists a unique triple of operations $(+, \varphi, a)$ such, that $(Q; +)$ is a semigroup with a neutral element 0, an automorphism φ and an invertible element a, which satisfies the following relations:*

$$\varphi^n(x) + a = a + x, \quad \varphi^s(a) = a, \tag{6}$$

$$f(x_0, x_1, \ldots, x_n) = x_0 + \varphi(x_1) + \varphi^2(x_2) + \cdots + \varphi^n(x_n) + a. \tag{7}$$

And conversely, if an endomorphism φ and an element a of an semigroup $(Q; +)$ are connected by the relations (6)*, then the groupoid*

$(G;f)$ determined by the equality (7) *is an* $(n+1)$*-ary associate of the associativity degree* s*.*

We will use the following terminology: $(Q;+)$ is called a *monoid of the* 0*-decomposition*; φ is said to be an *automorphism of the* 0*-decomposition*; a is called a *free member of the* 0*-decomposition*; $+$, φ, a are called *components of the* 0*-decomposition*; and $(Q;+,\varphi,a)$ is said to be an *algebra of the* 0*-decomposition* of the invert $(Q;f)$.

Lemma 7. *Let k be a nonnegative integer, which is not greater than n and is a multiple of s, 0 is an arbitrary invertible element of the invert $(Q;f)$, then the components of its* 0*-decomposition are uniquely determined by the following equalities*

$$\begin{aligned} &x+y=f(x,\overset{k-1}{0},\bar{0},\overset{n-k-1}{0},y);\\ &a=f(0,0,\dots,0);\quad -a=\bar{0};\\ &\varphi^i(x)=\lambda_{0,0}^{-1}\lambda_{i,0}(x)=f(\overset{i}{0},x,\overset{n-i-1}{0},\bar{0});\\ &\varphi^{-i}(x)=\lambda_{n,0}^{-1}\lambda_{n-i,0}(x)=f(\bar{0},\overset{n-i-1}{0},x,\overset{i}{0}) \end{aligned} \tag{8}$$

for all $i=1,\dots,n-1$.

Proof. In [2] the first three of the equalities were proved. Since n divides s, then (6) implies $\varphi^n(a)=a$, therefore

$$\varphi^n(\bar{0})=\varphi^n(-a)=-\varphi^n(a)=-a=\bar{0}.$$

The transformation φ is an automorphism of the semigroup $(Q;f)$, therefore $\varphi(0)=0$ and

$$\begin{aligned} \varphi^i(x)=\varphi^i(x)-a+a=0+\varphi(0)+\dots+\varphi^{i-1}(0)+\varphi^i(x)+\\ +\varphi^{i+1}(0)+\dots+\varphi^{n-1}(0)+\varphi^n(\bar{0})+a\overset{(6)}{=}f(\overset{i}{0},x,\overset{n-i-1}{0},\bar{0}). \end{aligned}$$

Let us now make use of the relationships (5):

$$\begin{aligned} \lambda_{0,0}^{-1}\lambda_{i,0}(x)&\overset{(5)}{=}f(f(\overset{i}{0},x,\overset{n-i}{0}),\overset{n-s-1}{0},\bar{0},\overset{s-1}{0},\bar{0})\\ &\overset{Th1}{=}f(\overset{i}{0},x,\overset{n-i-1}{0},f(\overset{n-s}{0},\bar{0},\overset{s-1}{0},\bar{0}))\overset{(3)}{=}f(\overset{i}{0},x,\overset{n-i-1}{0},\bar{0})\overset{(7)}{=}\varphi^i(x). \end{aligned}$$

$$\lambda_{n,0}^{-1}\lambda_{n-i,0}(x) \overset{(5)}{=} f(\bar{0}, \overset{s-1}{0}, \bar{0}, \overset{n-s-1}{0}, f(\overset{n-i}{0}, x, \overset{i}{0})) =$$
$$\overset{Th1}{=} f(f(\bar{0}, \overset{s-1}{0}, \bar{0}, \overset{n-s}{0}), \overset{n-i-1}{0}, x, \overset{i}{0}) \overset{(3)}{=} f(\bar{0}, \overset{n-i-1}{0}, x, \overset{i}{0})$$
$$\overset{(7)}{=} \bar{0} + \varphi(0) + \cdots + \varphi^{n-i}(x) + \varphi^{n-i+1}(0) + \cdots + \varphi^{n}(0) + a$$
$$\overset{(6)}{=} -a + \varphi^{n}(\varphi^{-i}(x)) + a \overset{(6)}{=} \varphi^{-i}(x).$$

The lemma is proved. □

Corollary 2. *Under the notations of Theorem 6 the associativity period of the invert is equal to the least of the numbers s, such that $\varphi^s(a) = a$, i.e. it is equal to the length of the orbit of the element a, when we consider the action of the cyclic group $\langle\varphi\rangle$ generated by the automorphism φ.*

If an element x is invertible in an a-decomposition monoid, then its inverse element will be denoted by $-x_{\langle a\rangle}$ or by $x_{\langle a\rangle}^{-1}$ depending on additive or multiplicative notation of the a-decomposition monoid. It should be noted that the element $-x_{\langle a\rangle}$ is uniquely determined by the elements a and x.

Theorem 8. *An element of an invert will be invertible iff it is invertible in one (hence, in every) of the decomposition monoids.*

Proof. Let $(Q; f)$ be an invert of the type (s, n) with an invertible element 0 and $(Q; +)$ be a 0-decomposition monoid. Let x be invertible in $(Q; f)$ and let

$$-x_{\langle 0\rangle} := f(0, \overset{n-s-1}{x}, \bar{x}, \overset{s-1}{x}, 0). \tag{9}$$

To prove that the element $-x_{\langle 0\rangle}$ is inverse to x, we will use the equality (8) when $k = s$.

$$x + (-x_{\langle 0\rangle}) \overset{(8)}{=} f(x, \overset{s-1}{0}, \bar{0}, \overset{n-s-1}{0}, -x_{\langle 0\rangle})$$
$$\overset{(9)}{=} f(x, \overset{s-1}{0}, \bar{0}, \overset{n-s-1}{0}, f(0, \overset{n-s-1}{x}, \bar{x}, \overset{s-1}{x}, 0))$$
$$\overset{Th1}{=} f(f(x, \overset{s-1}{0}, \bar{0}, \overset{n-s}{0}), \overset{n-s-1}{x}, \bar{x}, \overset{s-1}{x}, 0) \overset{(3)}{=} f(\overset{n-s}{x}, \bar{x}, \overset{s-1}{x}, 0) \overset{(3)}{=} 0.$$

$$
\begin{aligned}
-x_{\langle 0\rangle}+x &\overset{(8)}{=} f(-x_{\langle 0\rangle},\overset{s-1}{0},\bar{0},\overset{n-s-1}{0},x)\\
&\overset{(9)}{=} f(f(0,\overset{n-s-1}{x},\bar{x},\overset{s-1}{x},0),\overset{s-1}{0},\bar{0},\overset{n-s-1}{0},x)\\
&\overset{Th1}{=} f(0,\overset{n-s-1}{x},\bar{x},\overset{s-1}{x},f(\overset{s}{0},\bar{0},\overset{n-s-1}{0},x)) \overset{(3)}{=} f(0,\overset{n-s-1}{x},\bar{x},\overset{s}{x}) \overset{(3)}{=} 0.
\end{aligned}
$$

Hence, the element $-x_{\langle 0\rangle}$ is inverse to x in $(Q;+)$.

Conversely, let the element x be invertible in the 0-decomposition monoid $(Q;+)$. Then the element

$$
f(0,x,\dots,x,0) \overset{(7)}{=} \varphi x+\varphi^2 x+\dots+\varphi^{n-1}x+a
$$

is invertible in $(Q;+)$ too. Let us define the element $\bar{x}$ by

$$
\bar{x}=-f(0,x,\dots x,0)_{\langle 0\rangle}. \tag{10}
$$

In particular, this means that

$$
\bar{x}+f(0,x,\dots x,0)_{\langle 0\rangle}=0.
$$

Then for any element y of Q we get the following relations:

$$
\begin{aligned}
y&=0+y=\bar{x}+f(0,\overset{n-1}{x},0)+y\\
&\overset{(8)}{=} f(f(\bar{x},\overset{s-1}{0},\bar{0},\overset{n-s-1}{0},f(0,\overset{n-1}{x},0)),\overset{s-1}{0},\bar{0},\overset{n-s-1}{0},y)\\
&\overset{Th1}{=} f(f(\bar{x},\overset{s-1}{0},\bar{0},\overset{n-s}{0}),\overset{n-1}{x},f(\overset{s}{0},\bar{0},\overset{n-s-1}{0},y)) \overset{(3)}{=} f(\bar{x},\overset{n-1}{x},y)\,,
\end{aligned}
$$

$$
\begin{aligned}
y&=y+0 \overset{(10)}{=} y+f(0,\overset{n-1}{x},0)+\bar{x}\\
&\overset{(8)}{=} f\Big(f(y,\overset{s-1}{0},\bar{0},\overset{n-s-1}{0},f(0,\overset{n-2}{x},0)),\overset{s-1}{0},\bar{0},\overset{n-s-1}{0},\bar{x}\Big)\\
&\overset{Th1}{=} f\Big(f(y,\overset{s-1}{0},\bar{0},\overset{n-s-1}{0}),\overset{n-2}{x},f(\overset{s}{0},\bar{0},\overset{n-s-1}{0},\bar{x})\Big) \overset{(3)}{=} f(y,\overset{n-2}{x},\bar{x}).
\end{aligned}
$$

From Theorem 4 we get the invertibility of the element x in the invert $(Q;f)$. □

Corollary 3. *The sets of all invertible elements of multiary monoid and decomposition monoids are pairwise equal.*

Corollary 4. *Let* 0 *be an invertible element of a monoid* $(Q;f)$ *of the type* (s,n) *and let* k *be multiple of* s*. Then the element* x *will be invertible in* $(Q;f)$ *iff there exists an element* $-x_{\langle 0\rangle}$ *such that*

$$f(x,\overset{s-1}{0},\bar{0},\overset{n-s-1}{0},-x_{\langle 0\rangle}) = f(-x_{\langle 0\rangle},\overset{s-1}{0},\bar{0},\overset{n-s-1}{0},x) = 0 \tag{11}$$

hold.

Proof. The equality (11) according to the equalities (8) means the truth of the relations $x+(-x_{\langle 0\rangle}) = -x_{\langle 0\rangle}+x = 0$, where $(Q;+)$ is the 0-decomposition monoid, that is the element x is invertible in $(Q;+)$. Hence, by Theorem 8 it will be invertible in the associate $(Q;f)$. □

Lemma 9. *Let* $(Q;f)$ *be an invert of the type* (s,n)*,* $(+,\varphi,a)$ *be its* 0*-decomposition. A triple* $(\cdot,\psi,b)$ *of operations defined on* Q *will be a decomposition of* $(Q;f)$ *iff there exists an invertible in* $(Q;+)$ *element* e *satisfying the conditions*

$$\begin{aligned} &x\cdot y = x-e+y, \quad \psi(x) = e+\varphi(x)-\varphi(e),\\ &b = e+\varphi(e)+\varphi^2(e)+\cdots+\varphi^n(e)+a. \end{aligned} \tag{12}$$

The algebra $(Q;\cdot,\psi,b)$ *in this case will be* e*-decomposition of the invert* $(Q;f)$*.*

Proof. Let $(\cdot,\psi,b)$ be e-decomposition of the invert $(Q;f)$, then

$$\begin{aligned} x\cdot y &\overset{(8)}{=} f(x,\overset{s-1}{e},\bar{e},\overset{n-s-1}{e},y) \overset{(7)}{=} x+\varphi e+\cdots+\varphi^{s-1}(e)+\\ &\quad+\varphi^s(\bar{e})+\varphi^{s+1}(e)+\cdots+\varphi^{n-1}(e)+\varphi^n(y)+a\\ &\overset{(6)}{=} x+(\varphi(e)+\cdots+\varphi^{s-1}(e)+\varphi^3(\bar{e})+\\ &\quad+\varphi^{s+1}(e)+\cdots+\varphi^{n-1}(e)+a)+y\,. \end{aligned}$$

Hence $x\cdot y = x+c+y$ for some $c\in Q$ and all $x,y\in Q$. In particular, when $x=e, y=0$ and $x=0, y=e$ we get the invertibility of the element c in $(Q;+)$, and the relation $c=-e$. Next,

$$\begin{aligned} \psi(x) &\overset{(8)}{=} f(e,x,\overset{n-2}{e},\bar{e}) \overset{(7)}{=} e+\varphi(x)+\varphi^2(e)+\cdots+\\ &\quad+\varphi^{n-1}(e)+\varphi^n(\bar{e})+a = e+\varphi(x)+d \end{aligned}$$

for some $d \in Q$. But $e = \psi(e) = e + \varphi(e) + d$, therefore $d = -\varphi(e)$.

On the other hand, let an element e be invertible in $(Q;+)$ and determine a triple of operations $(\cdot, \psi, b)$ on Q by the equalities (12). The invertibility of the element e in the invert $(Q; f)$ is ensured by Theorem 8. If the component of the e-decomposition of the invert $(Q; f)$ are denoted by $(\circ, \chi, c)$, then just proved assertion gives

$$x \circ y = x - e + y, \quad \chi(x) = e + \varphi(x) - \varphi(e),$$
$$c = e + \varphi(e) + \varphi^2(e) + \cdots + \varphi^n(e) + a\,.$$

Therefore $(\cdot, \psi, b) = (\circ, \chi, c)$. This means, that $(\cdot, \psi, b)$ will be a decomposition of $(Q; f)$. □

We say that the monoids $(Q; \cdot)$ and $(Q;+)$ *differ from each other by a unit*, if the equality $x \cdot y = x - e + y$ holds for some invertible in $(Q;+)$ element e, because they coincide once their units coincide. This relationship between monoids is stronger than isomorphism since the translations L_e^{-1} and R_e^{-1} are isomorphic mappings from one to the other. Therefore the following statement is obvious.

Corollary 5. *Any two decomposition monoids of the same invert differ from each other by a unit.*

Theorem 10. *The set of all invertible elements of an invert is its subquasigroup and coincides with the group of all invertible elements of any of its decomposition monoids.*

Proof. Let $(Q; f)$ be an invert of the type (s, n) and let $(+, \varphi, a)$ be its 0-decomposition. Theorem 8 implies that the sets of all invertible elements of groupoids $(Q; f)$ and $(Q;+)$ coincide. Denote this set by G. Inasmuch as G is a subgroup of the monoid $(Q;+)$ and $\varphi G = G$, $a \in G$, so for any elements $c_0, c_1, \ldots, c_n \in G$ the element

$$f(c_0, c_1, \ldots, c_n) \overset{(7)}{=} c_0 + \varphi(c_1) + \varphi^2(c_2) + \cdots + \varphi^n(c_n) + a$$

is in G also. Furthermore for any number $i = 0, 1, \ldots, n$ the solution of $f(c_0, \ldots, c_{i-1}, x, c_{i+1}, \ldots, c_n) = c$, where $c \in G$, is unique and coincides with the element

$$x \overset{(6)}{=} \varphi^{-i}\Big(-\varphi^{i-1}(c_{i-1}) - \cdots \quad -\varphi(c_1) - c_0 + c - a - \\ -\varphi^n(c_n) - \cdots - \varphi^{i+1}(c_{i+1})\Big). \tag{13}$$

which is in G too. Hence, $(G;f)$ is a subquasigroup of $(Q;f)$. □

Theorem 11. *The period of associativity of the invert determined by (θ, a) coincides with the number of different skew elements of an invertible element and with the length of the orbit $<\theta>(a)$, where $<\theta>$ is the automorphism group generated by θ.*

Proof. Let number s be the period of associativity of the $(n+1)$-ary invert $(Q;f)$ and let x be any of its invertible elements. Denote by $(*, \psi, b)$ the x-decomposition of the invert $(Q;f)$. Since

$$f(\overset{i}{x}, \psi^{n-i}(\bar{x}), \overset{n-i}{x}) \overset{(8)}{=} f(\overset{i}{x}, f(\overset{n-i}{x}, \bar{x}, \overset{i-1}{x}, \bar{x}), \overset{n-i-1}{x}, f(\overset{n}{x}, \bar{x}))$$
$$\overset{Th1}{=} f(f(\overset{n}{x}, \bar{x}), \overset{i-1}{x}, f(\bar{x}, \overset{n}{x}), \overset{n-i-1}{x}, \bar{x}) \overset{(3)}{=} f(\overset{n}{x}, \bar{x}) = x,$$

then the i-th skew $\bar{x}^i$ of the element x is determined by the equality

$$\bar{x}^i = \psi^{n-i}(\bar{x}) \overset{(8)}{=} f(\overset{n-i}{x}, \bar{x}, \overset{i-1}{x}, \bar{x}), \quad i = 0, 1, \ldots, n. \tag{14}$$

Inasmuch as, in accordance with the equality (8),

$$\psi^s(\bar{x}) = \psi^s(b^{-1}) = (\psi^s(b))^{-1} = b^{-1} = \bar{x},$$

there are at most s different skew elements of the element x: Namely $\bar{x}$, $\bar{x}^1$, ..., $\bar{x}^{s-1}$.

Suppose, for some numbers i, j with $i < j < s$, the i-th and j-th skew elements of x coincide. The results obtained imply the equality $\psi^{n-i}(\bar{x}) = \psi^{n-j}(\bar{x})$, so that $\psi^{j-i}(\bar{x}) = \bar{x}$. The last equality together with equality $\psi^s(\bar{x}) = \bar{x}$ give the relation $\psi^d(\bar{x}) = \bar{x}$, where $d =$ g.c.d.$(s, j-i)$. In view of (8) this implies $\psi^d(b) = b$. It follows from Theorem 6 that the pair (d, n) will be a type of the invert $(Q;f)$. At the same time $d < s$. A contradiction to the definition of the associativity period.

Thus, the element x has exactly s skew elements. They are determined by the relation (14) and by any full collection of pairwise noncongruent indices modulo s.

Corollary 6. *If one of skew elements of an invertible element x of a monoid coincides with x, then all skew elements of x are equal and this invert is a semigroup.*

Proof. Let $\bar{0}^i = 0$. The relation (14) implies $\varphi^{n-i}(\bar{0}) = 0$, where φ denotes an automorphism of the 0-decomposition. Apply to the last equality φ^i we obtain $\varphi^n(\bar{0}) = \varphi^i(0)$.

Since $\varphi^n(\bar{0}) = \varphi^n(-a) = -\varphi^n(a) = -a = \bar{0}$, then accounting to (8) we obtain $f(\overset{i}{0}, 0, \overset{n-i-1}{0}, \bar{0}) = \bar{0}$, that is $\bar{0} = 0$. Thus $a = f(0, \ldots, 0) = 0$ and $\varphi(a) = \varphi(0) = 0 = a$. Hence, by Theorem 11 the associativity period of the invert is equal to 1, i.e. the invert is a semigroup. □

Corollary 7. *An invert of associativity period s has at least $s + 1$ different invertible elements.*

Corollary 8. *An invert having at most two invertible elements is associative i.e. is a semigroup.*

Proof. If an invert has exactly one invertible element, then it will be associative by Corollary 6, since its skews coincide with it. If the invert has exactly two invertible elements a and b, then $\bar{a}^0 = a$ or $\bar{a}^0 = b$. If $\bar{a}^0 = a$, then according to Corollary 6 the invert is a semigroup. If $\bar{a}^0 = b$, then Theorem 11 implies $\bar{a}^1 = a$. And Theorem 11 implies that the invert is a quasigroup. □

Axiomatics of polyagroups

Both for binary and for n-ary cases an associative quasigroup is called a *group*. Therefore, retaining this regularity we will introduce the notion of a polyagroup.

Definition 3. When $s < n$ the s-associative $(n+1)$-ary quasigroup we will call a *nonsingular polyagroup of the type* (s,n).

It is easy to see, that $s = 1$ means the polyagroup is an $(n+1)$-ary group. Theorem 6 implies an analogue of Gluskin-Hosszú theorem.

Proposition 12. *Any polyagroup of the type (s,n) is (i,j)-associative for all i,j with $i \equiv j \pmod{s}$. When s is its associativity period, then no other (i,j)-associativity identity holds.*

Theorem 13. *Let $(Q;f)$ be an associate of the type (s,n) and $s < n$, $n > 1$. Then the following statements are equivalent*

1) *$(Q;f)$ is a polyagroup,*
2) *every element of the associate is invertible,*
3) *for every $x \in Q$ there exists $\bar{x} \in Q$ such that*

$$f(\bar{x}, x, \ldots, x, y) = f(y, x, \ldots, x, \bar{x}) = y \tag{15}$$

holds for all $y \in Q$,

4) *$(Q;f)$ has an invertible element 0 and for every $x \in Q$ there exists $y \in Q$ such that*

$$f(x, \overset{s-1}{0}, \bar{0}, \overset{n-s-1}{0}, y) = 0, \quad f(y, \overset{s-1}{0}, \bar{0}, \overset{n-s-1}{0}, x) = 0 \tag{16}$$

holds.

Proof. 1)⇔2) follows from Theorem 10; 2)⇔3) from Corollary 1; 2)⇔4) from Corollary 4. □

When $s = 1$ we get a criterion for n-ary groups.

Corollary 9. *Let $(Q;f)$ be $(n+1)$-ary a semigroup. Then the following statements are equivalent*

1) *$(Q;f)$ is an $(n+1)$-ary group,*
2) *every element of the semigroup is invertible,*
3) *for every $x \in Q$ there exists $\bar{x} \in Q$ such that* (15) *holds for every $y \in Q$,*
4) *$(Q;f)$ has an invertible element 0 and for every $x \in Q$ there exists $y \in Q$ such that* (16) *hold.*

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