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On n-modules with chain conditions

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Abstract

We show that the maximal n-submodules of an n-module are determined by the maximal n-subgroups of the n-group of its zero-idempotents and by the maximal n-submodules of its maximal n-submodule with zero. We state some results concerning R-n-modules with chain conditions analogous to the Jordan–Hölder Theorem, to Fitting's Lemma, to Krull–Remack–Schmidt Theorem.

1. Introduction

R-n-modules are defined as a natural generalization of the usual binary notion. In [5] and [6] we restart the study of n-modules by dropping the restriction imposed by N. Celakoski in [1], namely that the commutative n-group involved has a *unique* neutral element. In this paper we continue our investigation on R-n-modules by studying the maximal n-submodules of an n-module in terms of its canonical presentation and by retrieving some of the results on modules with chain conditions for the n-ary case.

In the sequel, we use the same conventional notations as in [5] and [6]: the sequence $a_i, \ldots, a_j$ of $j-i+1$ terms of an n-ary sum is denoted by a_i^j and if $a_i = a_{i+1} = \ldots = a_j = a$ then the sequence is denoted by $\overset{(j-i+1)}{a}$; if $i > j$, then a_i^j denotes an empty sequence. Denote by $a^{\langle k\rangle}$ the k-th power of a, which is defined by:

$$a^{\langle 0\rangle} = a \quad \text{and} \quad a^{\langle k\rangle} = [a^{\langle k-1\rangle}, \overset{(n-1)}{a}]_+, \quad k \in \mathbb{Z}$$

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In particular, $a^{\langle -1\rangle} = \overline{a}$, where $\overline{a}$ denotes the querelement of a.

The purpose of this introductory section is to recall some of the definitions and results in [5] and [6], which will be used in the sections to follow.

Throughout this paper R denotes an associative ring with unity $1 \neq 0$. For reasons similar to the ones employed in the binary case, we deal only with left n-modules and so by R-n-module we will always understand left R-n-module.

Definition 1.1. We call *left R-n-module* a commutative n-group $(M, [\,]_+)$ together with an external operation $\mu\colon R \times M \to M$ which satisfies the axioms:

A1) $\mu(r, [x_1^n]_+) = \big[\mu(r, x_1), \ldots, \mu(r, x_n)\big]_+$,

A2) $\mu\big((r_1 + \cdots + r_n), x\big) = \big[\mu(r_1, x), \ldots, \mu(r_n, x)\big]_+$,

A3) $\mu(r \cdot r', x) = \mu\big(r, \mu(r', x)\big)$,

A4) $\mu(1, x) = x$

for all $x, x_1, \ldots, x_n \in M$ and all $r, r', r_1, \ldots, r_n \in R$.

Denote the element $\mu(r, x)$ by rx and as immediate consequences of the axioms, note:

$$(r_1{+}r_2)x = [r_1x, r_2x, \overset{(n-2)}{0x}]_+\,, \qquad (-r)x = [0x, 0x, \overset{(n-3)}{rx}, r\overline{x}]_+\,,$$
$$\overline{rx} = r\overline{x}\,, \qquad \overline{x} = (-n{+}2)x = \big((-1){+}\cdots{+}(-1)\big)x.$$

The empty n-group may be regarded as an R-n-module for any ring R. If M is a non-empty R-n-module, then it necessarily has at least one neutral element; indeed, for every $x \in M$, the element $0x$ is a neuter in $(M, [\,]_+)$ (or an idempotent, since the two notions coincide in commutative n-groups). Note that $0x^{\langle k\rangle} = 0x$, $\forall x \in M$, $\forall k \in \mathbb{Z}$ (in particular $0x = 0\overline{x}$).

n-Submodules, congruences and homomorphisms are defined in the obvious way. If S is a non-empty n-submodule of an R-n-module M, then the relation ρ_S defined by $x\rho_S y \Leftrightarrow \exists s_2^n \in S : y = [x, s_2^n]_+$ is a congruence on M. This correspondence is not a bijection, still it allows us to define the factor module $M/S = M/\rho_S$.

The set of all neuters of the n-group $(M, [\,]_+)$ is denoted by $\mathcal{N}_M$ (or simply by $\mathcal{N}$) and the set of all neuters of the form $0x$, for some $x \in M$, is denoted by $\mathcal{N}_{0M}$ (or sometimes just $\mathcal{N}_0$). $\mathcal{N}_0$ is a n-submodule of $\mathcal{N}$ and they are both n-submodules of M. The elements of $\mathcal{N}_0$ are called *zero-idempotents* and they are characterized by:

$$e \in \mathcal{N}_0 \Longleftrightarrow re = e, \quad \forall r \in R,$$

which shows that the n-submodules of $\mathcal{N}_0$ coincide with the n-subgroups of $\mathcal{N}_0$. If $\mathcal{N}_0$ consists of exactly one element, then this element is called a *zero* of the n-module and it is denoted by 0.

If $f\colon M_1 \to M_2$ is a homomorphism of R-n-modules, then:

1) $f(\mathcal{N}_1) \subseteq \mathcal{N}_2$ and $f(\mathcal{N}_{01}) \subseteq \mathcal{N}_{02}$,
2) $f(\overline{x}) = \overline{f(x)}$, $\forall x \in M_1$,
3) the set $\operatorname{Ker} f = \{x \in M_1 \mid f(x) \in \mathcal{N}_{02}\}$ is an n-submodule of M_1 and $\mathcal{N}_{01} \subseteq \operatorname{Ker} f$.

The set $\operatorname{Hom}_R(M_1, M_2)$ is a commutative n-group with respect to the operation:

$$[f_1, \ldots, f_n]_+(x) = \big[f_1(x), \ldots, f_n(x)\big]_+ .$$

Any homomorphism α with $\alpha(M_1) \subseteq \mathcal{N}_{02}$ is called *nullary homomorphism* and it is a neutral element of this n-group. For each $e \in \mathcal{N}_{02}$, denote by θ_e the homomorphism given by $\theta_e(x) = e$, $\forall x \in M_1$. The set $\operatorname{End}_R M$ is an $(n,2)$-ring with respect to the above addition and to the usual multiplication of maps. An endomorphism f of M is called *nilpotent* if there exists an integer $k \geq 1$ such that f^k is a nullary endomorphism.

We have introduced in [5] a class of n-submodules and a class of automorphisms of an R-n-module which play an important role in the study of n-modules. Let M be an R-n-module. For each $e \in \mathcal{N}_0$, the set $M_e = \{x \in M \mid 0x = e\}$ is an n-submodule with zero (the element e) of M. The n-submodules M_e are all isomorphic and they form a partition of M. The maps $\varphi_{e,f}\colon M \to M$, $\varphi_{e,f}(x) = [x, \overset{(n-2)}{e}, f]_+$ are all automorphisms, for each pair of zero-idempotents $e, f \in \mathcal{N}_0$, and $\varphi_{e,f}(M_e) = M_f$. Note that $M/\mathcal{N}_0 \simeq M_e$. In fact, the whole structure of an R-n-module is determined by: the structure of an R-n-module with zero (M_e) and the structure of an idempotent commutative n-group ($\mathcal{N}_0$). This is called the canonical presentation of the R-n-

module M (see [6]).

Injective and surjective homomorphisms are characterized in [6] in terms of the data of the canonical presentation.

Proposition 1.2. *Let $f\colon M_1 \to M_2$ be a homomorphism of R-n-modules. Then f is*

(1) *injective iff* $\operatorname{Ker} f = \mathcal{N}_{01}$ *and the restriction $f|_{\mathcal{N}_{01}}$ is injective,*

(2) *surjective iff for each $e' \in \mathcal{N}_{02}$ there exists $e \in \mathcal{N}_{01}$ such that $M_{2e'} = f(M_{1e})$.*

2. Maximal submodules of an n-module

We study in this section the maximal submodules of an R-n-module, in terms of the canonical presentation of the R-n-module considered.

Theorem 2.1. *Let M be an R-n-module. Then:*

(1) *If N is a maximal n-subgroup of $\mathcal{N}_0$, then there exists a unique maximal n-submodule S of M such that $\mathcal{N}_{0S} = N$.*

(2) *If S is a maximal n-submodule of M, which does not contain $\mathcal{N}_0$, then $\mathcal{N}_{0S}$ is a maximal n-subgroup of $\mathcal{N}_0$.*

Proof. (1) It is easy to check that the set $S = \bigcup\limits_{e\in N} M_e$ is an n-submodule of M, with $\mathcal{N}_{0S} = N$.

Take now an n-submodule T of M with $S \subset T \subseteq M$ and let $x \in T \setminus S$. Then $e = 0x \in T$ and $e \notin S$ (since $e \in S$ implies $x \in S$). This shows that $\mathcal{N}_{0T} \supset \mathcal{N}_{0S} = N$, hence $\mathcal{N}_{0T} = \mathcal{N}_0$.

For any $y \in M$ one of the following holds: (a) $f = 0y \in N$ (and so $y \in S \subset T$) or (b) $f \in \mathcal{N}_0 \setminus N$ (and so $y \notin S$). We show that even in the latter case, we still have $y \in T$. Indeed, $\forall s \in S\, \exists!\, t \in \mathcal{N}_{0S} \subset S$ such that: $y = [f, \overset{(n-2)}{s}, t]_+$. Since $f \in T$, $s, t \in S \subset T$ it follows that $y \in T$. Hence $T = M$ and so S is maximal.

Let V be a maximal n-submodule of M, with $\mathcal{N}_{0V} = N = \mathcal{N}_{0S}$. Then $V \subseteq S$ (indeed, if $x \in V$ then $0x \in \mathcal{N}_{0V} = \mathcal{N}_{0S} = N$, so $x \in S$) which, together with maximality of V, implies $V = S$.

(2) Let S be a maximal n-submodule of M with $\mathcal{N}_0 \setminus S \neq \emptyset$, i.e. $\mathcal{N}_{0S} \subset \mathcal{N}_0$. Consider an n-submodule A of $\mathcal{N}_0$ such that $\mathcal{N}_{0S} \subset A \subseteq$

$\mathcal{N}_0$ and let $e \in A \setminus \mathcal{N}_{0S}$. Then $\langle S \cup \{e\} \rangle = M$ and $\forall a \in \mathcal{N}_0 \, \exists \, k \in \mathbb{N}$ and $s_{k+1}^n \in S$ such that $a = [\overset{(k)}{e}, s_{k+1}^n]_+$. By multiplying with zero, we obtain: $a = 0a = [\overset{(k)}{e}, e_{k+1}^n]_+$, with $e_i = 0s_i$, $i = 1, \ldots, n$ and $e \in M$, $e_i \in \mathcal{N}_{0S} \subset A$, $i = k+1, \ldots, n$. Now, since A is an n-submodule, we deduce that $a \in A$ and so $A = \mathcal{N}_0$. □

The above theorem shows that there exists a bijective correspondence between the set of maximal n-submodules of $\mathcal{N}_0$ and the set of maximal n-submodules of M which do not contain $\mathcal{N}_0$. A natural question arises: what can one say about the maximal n-submodules of M which *do* contain $\mathcal{N}_0$?

Theorem 2.2. *Let M be an R-n-module with the canonical presentation: $B \simeq M_e$, $A \simeq \mathcal{N}_0$. Then:*

(1) *If B has a maximal n-submodule, then M has a maximal n-submodule which contains $\mathcal{N}_0$.*

(2) *If M has a maximal n-submodule which contains $\mathcal{N}_0$, then B has a maximal n-submodule.*

Proof. (1) Let V be a maximal n-submodule of B and take an arbitrary zero-idempotent $e \in \mathcal{N}_0$. Since $B \simeq M_e$, it follows that M_e has a maximal n-submodule S_e which is isomorphic to V. Then for every $f \in \mathcal{N}_0$, the set $S_f = \varphi_{e,f}(S_e)$ is a maximal n-submodule of M_f. Define the subset S of M by: $S = \bigcup\limits_{f \in \mathcal{N}_0} S_f$. We will show that S is a maximal n-submodule of M which contains $\mathcal{N}_0$. Clearly $\mathcal{N}_0 \subseteq S$ (since $f \in S_f$, $\forall f \in \mathcal{N}_0$); equality holds when $V = \{0\}$.

Let $x \in S$; then $\exists f \in \mathcal{N}_0$ such that $x \in S_f$. Since S_f is an n-submodule it follows that $rx \in S_f$, $\forall r \in R$ and so $rx \in S$, $\forall r \in R$.

Let $x_1, \ldots, x_n \in S$; then $\exists f_i \in \mathcal{N}_0$ such that $x_i \in S_{f_i}$ and, consequently, $\exists y_i \in S_e$ such that $x_i = [y_i, \overset{(n-2)}{e}, f_i]_+$. Now we have

$$\begin{aligned}[x_1^n]_+ &= [y_1, \overset{(n-2)}{e}, f_1, \ldots, y_n, \overset{(n-2)}{e}, f_n]_+ \\ &= [[y_1^n]_+, \overset{(n-2)}{e}, [f_1^n]_+]_+ \in \varphi_{e,[f_1^n]_+}(S_e) = S_{[f_1^n]_+} \subseteq S\end{aligned}$$

and so S is an n-submodule of A.

Let T be an n-submodule of M, $S \subset T \subseteq M$ and take $x \in T \setminus S$. Define $u = 0x$ and we have $x \in M_u \setminus S_u$. Then $\tilde{x} = \varphi_{u,e}(x) \in M_e \setminus S_e$ (if $\tilde{x} \in S_e$ then $\varphi_{e,u}(\tilde{x}) = (\varphi_{e,u} \circ \varphi_{u,e})(x) = x \in S_u$, contradiction) and $\tilde{x}_f = \varphi_{e,f}(\tilde{x}) \in M_f \setminus S_f$, $\forall f \in \mathcal{N}_0$ (if $\tilde{x}_f \in S_f$ then $\exists z \in S_e$ such that $\tilde{x}_f = \varphi_{e,f}(z)$, or $\varphi_{e,f}(\tilde{x}) = \varphi_{e,f}(z)$ which implies $\tilde{x} = z \in S_e$, contradiction). Hence T contains at least one such element $\tilde{x}_f$ for each set $M_f \setminus S_f$, $f \in \mathcal{N}_0$ and so $M_f = \langle S_f \cup \{\tilde{x}_f\}\rangle$, $\forall f \in \mathcal{N}_0$. Now $\forall y \in M$ $\exists f \in \mathcal{N}_0$ such that $y \in M_f$; then there exists $k \in \mathbb{N}$ and $s_{k+1}, \ldots, s_n \in S_f$ such that: $y = [\overset{(k)}{\tilde{x}_f}, s_{k+1}^n]_+$. Since $\tilde{x}_f \in T$ and $s_{k+1}, \ldots, s_n \in S_f \subseteq S \subset T$, it follows that $y \in T$ and this shows that $T = M$.

(2) Let $S \subset M$ be a maximal n-submodule of M which contains $\mathcal{N}_0$. For each $e \in \mathcal{N}_0$ define the subset S_e of S by: $S_e = \{x \in S \mid 0x = e\}$. Clearly, $S_e = S \cap M_e$ and so S_e is an n-submodule of M_e (and of S). Moreover, $S = \bigcup\limits_{e \in \mathcal{N}_0} S_e$.

We show that, for any $e \in \mathcal{N}_0$, the n-submodule S_e is maximal in M_e. For this, let T be an n-submodul of M_e, $S_e \subset T \subseteq M_e$ and take $x \in T \setminus S_e$. Then $x \notin S$ and so $\langle S \cup \{x\}\rangle = M$. It follows that $\forall y \in M_e$ $\exists k \in \mathbb{N}$ and $s_{k+1}, \ldots, s_n \in S$ such that

$$y = [\overset{(k)}{x}, s_{k+1}^n]_+ = [\overset{(k)}{x}, \overset{(n-k-1)}{e}, [\overset{(k)}{e}, s_{k+1}^n]_+]_+.$$

By multiplying with 0 we obtain that the element $[\overset{(k)}{e}, s_{k+1}^n]_+ \in S$ belongs to M_e, which means that $[\overset{(k)}{e}, s_{k+1}^n]_+ \in S_e$. Since $x \in T$ and $e, [\overset{(k)}{e}, s_{k+1}^n]_+ \in S_e \subset T$, then $y \in T$. Hence $T = M_e$. $\square$

The above theorem shows that an n-module M has maximal n-submodules which contain $\mathcal{N}_0$ if and only if the n-submodules M_e have maximal n-submodules.

Definition 2.3. An R-n-module M is *simple* if its only congruences are the equality and the universal relation.

Remark 2.4. 1) M is simple iff its only non–void n-submodules are: $\{e\}$, with $e \in \mathcal{N}_0$ and M itself.
2) M is simple iff it has one of this canonical presentations:

(a) a simple R-n-module with zero and $\mathcal{N}_0 = \{0\}$,
(b) the R-n-module with zero is $B = \{0\}$ and $\mathcal{N}_0$ is a simple idempotent commutative n-group.

Theorem 2.5. *Let M be an R-n-module and $S \subset M$ be a non-void n-submodule. S is maximal iff M/S is simple.*

Proof. Suppose M/S is simple and let T be an n-submodule of M, with $S \subseteq T \subseteq M$. Then T/S is an n-submodule of M/S and so T/S either consists of exactly one coset (which is obviously S, since $T \supseteq S$), or $T/S = M/S$. Now $T/S = M/S$ implies that $\forall x \in M$, $\exists t \in T$, $s_1^{n-1} \in S \subseteq T$ such that $x = [t, s_1^{n-1}]_+$, i.e. $x \in T$. This shows that either $T = S$ or $T = M$.

Suppose S is maximal and consider two cases: $\mathcal{N}_0 \subseteq S$ or $\mathcal{N}_0 \setminus S \neq \emptyset$. If $\mathcal{N}_0 \subseteq S$ then M/S is an n-module with zero. Let now T be an n-submodule of M/S. Then $p^{-1}(T)$ is an n-submodule of M which contains S, so we have either $p^{-1}(T) = S$ or $p^{-1}(T) = M$. This shows that T is either the zero n-submodule or $T = M/S$.

If $\mathcal{N}_0 \setminus S \neq \emptyset$, then M/S does not have a zero element; we prove first that each coset $\hat{x} \in M/S$ contains at least one idempotent $e \in \mathcal{N}_0$ or, equivalently, that each coset is an n-submodule of M. Take now a coset $\hat{y} \in M/S$, $\hat{y} \neq S$ and a zero-idempotent $e \in \mathcal{N}_0 \setminus S$. Then $S \subset \langle S \cup \{e\} \rangle$ and so $\langle S \cup \{e\} = M$, hence y can be expressed as $y = [\overset{(k)}{e}, s_{k+1}^n]_+$, with $k \geq 1, s_{k+1}^n \in S$, and further

$$y = \big[[\overset{(k)}{e}, \overset{(n-k)}{f}]_+, \overset{(k-1)}{f}, s_{k+1}^n\big]_+ = [e', \overset{(k-1)}{f}, s_{k+1}^n]_+,$$

for any $f \in \mathcal{N}_0 \cap S$. This shows that $e' \in \hat{y}$.

Thus we have proved that each coset $\hat{x} \in M/S$ is an n-submodule of M. If $\hat{e} \in M/S$ and $f \in \mathcal{N}_0 \cap S$, then $\varphi_{f,e}(S)$ is a maximal n-submodule of M, which is contained in $\hat{e}$, hence $\varphi_{f,e}(S) = \hat{e}$. Take now an n-submodule T of M/S. If T consists of more than one element, say $\hat{e}, \hat{f} \in T$, then we have $\hat{e} \subset p^{-1}(T) \subseteq M$. This implies, since $\hat{e}$ – as n-submodule of M – is maximal, that $p^{-1}(T) = M$, and so $T = M/S$. □

Proposition 2.6. *If M is a simple R-n-module, then every endomorphism of M is either of type θ_e or an automorphism.*

Proof. If M is simple, then by Remark 2.4 it follows that either M

has a zero element and exactly two n-submodules: $\{0\}$ and M, or $M = \mathcal{N}_{0M}$ and its submodules are: $\{x\}$, $\forall x \in M$ and M. In the first case, if $f \in \mathrm{End}_R(M)$ then either $\mathrm{Ker}\, f = \{0\}$ or $\mathrm{Ker}\, f = M$, i.e. f is either injective or the zero endomorphism. If f is injective, then $\mathrm{Im} f = M$.

In the second case, either $\mathrm{Im} f = M$ or $\mathrm{Im} f = \{e\}$, $e \in M$, i.e. either f is surjective or $f = \theta_e$. If f is surjective, let $e \in M$. Then $f^{-1}(e)$ is a non–void n-submodule of M, so it is either a one–element set or the whole of M. Since f is surjective, it follows that $\forall e \in M$, the set $f^{-1}(e)$ consists of one element only. $\square$

3. Artinian and Noetherian n-modules

Definition 3.1. An R-n-module M is called *Artinian* if the set of its n-submodules satisfies the DCC (Descending Chain Condition), and it is called *Noetherian* if the set of its n-submodules satisfies the ACC (Ascending Chain Condition).

Note that every n-submodule of an Artinian (Noetherian) n-module is Artinian (Noetherian) too.

As in the binary case, the following characterization of a Noetherian n-module holds:

Proposition 3.2. *An R-n-module is Noetherian iff any n-submodule of M is finitely generated.*

Proof. Similar to the one for the binary case (see [8]). If M is Noetherian and S is an n-submodule of M, it follows that the set of all finitely generated n-submodules of S contains a maximal element A. Since A is finitely generated, it follows that $\forall x \in S$, the n-submodule $\big[\overset{(n-1)}{A}, Rx\big]_+$ of S is finitely generated which, together with the maximality of A, implies $\big[\overset{(n-1)}{A}, Rx\big]_+ = A$, and so $x \in A$. This proves that $S = A$. For the converse, see the proof for the binary case. $\square$

Proposition 3.3. *If $A \xrightarrow{f} B \xrightarrow{g} C \to 0$, is an exact sequence of R-n-modules and the homomorphism f is injective, then:*

1) *B is Artinian iff A and C are Artinian,*

2) *B is Noetherian iff A and C are Noetherian.*

Proof. 1) Suppose B is Artinian. Since f is injective, it follows that A is isomorphic to the n-submodule $f(A)$ of B, and hence it is Artinian. Let $C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$ be a descending chain of n-submodules of C. Then $g^{-1}(C_1) \supseteq g^{-1}(C_2) \supseteq g^{-1}(C_3) \supseteq \dots$ is a descending chain of n-submodules of B (with $g^{-1}(C_k) \neq \emptyset$, if $C_k \neq \emptyset$). Since B is Artinian, it follows that there exists $k > 0$ such that $g^{-1}(C_m) = g^{-1}(C_k)$, for $m > k$. But this implies – since g is surjective – that $C_m = C_k$, for $m > k$.

Conversely, assume A and C are Artinian and let

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots \tag{dc}$$

be a descending chain of n-submodules of B. By intersecting the terms of the chain *(dc)* with $f(A)$, we obtain a descending chain of n-submodules of $f(A)$:

$$B_1 \cap f(A) \supseteq B_2 \cap f(A) \supseteq B_3 \cap f(A) \supseteq \dots.$$

Since $f(A)$ is Artinian, it follows that there exists $k > 0$ such that $B_m \cap f(A) = B_k \cap f(A)$, for $m > k$. By applying g to the terms of the chain *(dc)* we obtain the descending chain of n-submodules of C:

$$g(B_1) \supseteq g(B_2) \supseteq g(B_3) \supseteq \dots,$$

so there exists $l > 0$ such that $g(B_m) = g(B_l)$, for $m > l$. Define $t = \max\{k, l\}$; we show that $B_m = B_t$, for $m > t$. Note that if $g(B_l) = \emptyset$, then $B_l = \emptyset$, hence $B_m = B_l = \emptyset$, for $m > l$; similarly, if $B_k \cap f(A) = \emptyset$, then $B_k \cap \mathcal{N}_{0B} = \emptyset$ (because $f(A) = \operatorname{Ker} g \supseteq \mathcal{N}_{0B}$), hence $B_k = \emptyset$, i.e. $B_m = B_k = \emptyset$, for $m > k$. We may therefore assume that $B_k \cap f(A) \neq \emptyset$ and $g(B_l) \neq \emptyset$. Let $b \in B_t$; $g(B_t) = g(B_m)$ implies that $\exists b' \in B_m$ such that $g(b) = g(b')$. For $e \in B_m \cap \mathcal{N}_{0B}$ (such an element exists, since $B_m \neq \emptyset$) we have:

$$\big[g(b), \overset{(n-3)}{g(b')}, g(\overline{b'}), g(e)\big]_+ = g(e) \in \mathcal{N}_{0C}$$

and hence $[b, \overset{(n-3)}{b'}, \overline{b'}, e]_+ \in \operatorname{Ker} g$. Since $m > t$, we have $B_m \subseteq B_t$ and

$$[b, \overset{(n-3)}{b'}, \overline{b'}, e]_+ \in B_t \cap \operatorname{Ker} g = B_t \cap f(A) = B_m \cap f(A).$$

Now $[b, \overset{(n-3)}{b'}, \overline{b'}, e]_+ \in B_m$, $b', e \in B_m$ implies $b \in B_m$. This shows that $B_t \subseteq B_m$.

2) The fact that if B is Noetherian then A and C are Noetherian is proved by a similar argument as above.

For the converse, we make the same constructions and use the same notations (of course by using an ascendant chain this time). We will show that $B_m = B_t$, for $m > t$. Let $b \in B_m$; $g(B_t) = g(B_m)$ implies that $\exists b' \in B_t$ such that $g(b) = g(b')$. For $e \in B_t \cap \mathcal{N}_{0B}$ we have $\big[g(b), \overset{(n-3)}{g(b')}, g(\overline{b'}), g(e)\big]_+ = g(e) \in \mathcal{N}_{0C}$ and hence $[b, \overset{(n-3)}{b'}, \overline{b'}, e]_+ \in \operatorname{Ker} g$. Since $m > t$, we have $B_t \subseteq B_m$ and

$$[b, \overset{(n-3)}{b'}, \overline{b'}, e]_+ \in B_m \cap \operatorname{Ker} g = B_m \cap f(A) = B_t \cap f(A).$$

Now $[b, \overset{(n-3)}{b'}, \overline{b'}, e]_+, b', e \in B_t$ implies $b \in B_t$ and this shows that $B_m \subseteq B_t$. □

Corollary 3.4.

1) *If S is an n-submodule of the R-n-module A, then A is Artinian (Noetherian) iff S and A/S are Artinian (Noetherian).*
2) *Let $A_1, \ldots, A_m$ be R-n-modules with zero. The R-n-module $A_1 \times \cdots \times A_m$ is Artinian (Noetherian) iff $A_1, \ldots, A_m$ are all Artinian (Noetherian).*

Proof. 1) The sequence $S \xrightarrow{i} A \xrightarrow{p} A/S \to 0$, where i is the inclusion and p is the natural homomorphism, satisfies the hypotheses of the preceding proposition.

2) The sequence $A_1 \times \cdots \times A_{n-1} \xrightarrow{f} A_1 \times \cdots \times A_n \xrightarrow{p_n} A_n \to 0$ is exact and the homomorphism f defined by

$$f\big((a_1, \ldots, a_{n-1})\big) = (a_1, \ldots, a_{n-1}, 0)$$

is injective. □

Lemma 3.5. *Let B_1, B, C_1, C be n-submodules of the R-n-module M, with $B_1 \subseteq B \subseteq M$, $C_1 \subseteq C \subseteq M$, $B_1 \cap C_1 \neq \emptyset$. Then*

$$\langle B_1 \cup (B \cap C)\rangle / \langle B_1 \cup (B \cap C_1)\rangle \simeq \langle C_1 \cup (B \cap C)\rangle / \langle C_1 \cup (B_1 \cap C)\rangle.$$

Proof. Identical to the one for the binary case (see [4]); we can apply the isomorphism theorems because $B_1 \cap C_1 \neq \emptyset$. □

Lemma 3.6. (*Schreier*) *Let* $M = S_0 \supseteq S_1 \supseteq \ldots \supseteq S_r = e$ *and* $M = T_0 \supseteq T_1 \supseteq \ldots \supseteq T_s = e$ *be two chains of* n*-submodules of the* R-n*-module* M*, where* $e \in \mathcal{N}_0$*. Define* $S_{ij} = \langle S_i \cup (S_{i-1} \cap T_j)\rangle$ *and* $T_{ij} = \langle T_j \cup (T_{j-1} \cap S_i)\rangle$*, for all* $0 \leqslant i \leqslant r$*,* $0 \leqslant j \leqslant s$*, and we obtain isomorphic refinements of the two chains:*

$$
\begin{aligned}
&S_{i-1} = S_{i0} \supseteq S_{i1} \supseteq \ldots \supseteq S_{is} = S_i, \quad 0 \leqslant i \leqslant r \\
&T_{j-1} = T_{0j} \supseteq T_{1j} \supseteq \ldots \supseteq T_{rj} = T_j, \quad 0 \leqslant j \leqslant s \\
&\qquad S_{i,j-1}/S_{ij} \simeq T_{i-1,j}/T_{ij}\,.
\end{aligned}
$$

Proof. Identical to the one for the binary case (see [4]); the preceding lemma is applicable because the zero-idempotent e belongs to each term of the two chains. □

The definition of a composition series of an R-n-module is naturally transferred from R-modules, namely: a *composition series* of an R-n-module M is a finite, strictly decreasing series of n-submodules of M,

$$M = S_0 \supset S_1 \supset \ldots \supset S_m = \{e\}, \quad e \in \mathcal{N}_0 \tag{c}$$

which does not admit strictly decreasing refinements. The series (c) is a composition series of M iff each S_i, $i = \{1, \ldots, m\}$ is a maximal n-submodule of S_{i-1}, i.e. iff the factor n-modules S_{i-1}/S_i are simple. One can easily check the validity of the Jordan-Hölder Theorem, with just one additional comment: if

$$M = S_0 \supset S_1 \supset \ldots \supset S_m = \{e\} \tag{c_1}$$
$$M = T_0 \supset T_1 \supset \ldots \supset T_r = \{f\} \tag{c_2}$$

are two composition series of M, then in order to use Schreier's Lemma one needs that the series (c_1) and (c_2) have the same last term. For this purpose, we apply to each term of the series (c_2) the automorphism $\varphi_{f,e}$ and we obtain the series:

$$\varphi_{f,e}(M) = M \supset \varphi_{f,e}(T_1) \supset \ldots \supset \varphi_{f,e}(T_r) = \{e\} \tag{c_3}$$

which is still a composition series. Schreier's Lemma may now be applied. So, if an R-n-module M has a composition series, then all its composition series have the same length, and this will be called *the length of M* (and we say that M has finite length). If M does not have composition series, then we say it has infinite length.

As in the binary case, the following hold:

1) *If S is an n-submodule of M, then* $l(M) = l(S) + l(M/S)$.
2) *If S_1, S_2 are n-submodules of M, then*
$$l(S_1) + l(S_2) = l(\langle S_1 \cup S_2 \rangle) + l(S_1 \cap S_2).$$
3) *If the sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact and the homomorphism f is injective, then* $l(B) = l(A) + l(C)$.

By using a similar argument to the one employed for usual R-modules (see [8]), one proves the following

Theorem 3.7. *An R-n-module M has composition series (i.e. M has finite length) iff M is Artinian and Noetherian.*

Proposition 3.8. *Let $f\colon M \to M$ be an endomorphism of the R-n-module M.*

1) *If M is Artinian, then f is an automorphism iff f is injective.*
2) *If M is Noetherian, then f is an automorphism iff f is surjective.*

Proof. 1) Assume f is injective; then $M \supseteq f(M) \supseteq f^2(M) \supseteq \ldots$, hence there exists m such that $f^m(M) = f^{m+1}(M) = \ldots$. This implies that $\forall y \in M \exists x \in M$ such that $f^m(y) = f^{m+1}(x)$, so $y = f(x)$.

2) Assume f is surjective; then $\mathcal{N}_0 \subseteq f^{-1}(\mathcal{N}_0) \subseteq f^{-2}(\mathcal{N}_0) \subseteq \ldots$, hence there exists m such that $f^{-m}(\mathcal{N}_0) = f^{-(m+1)}(\mathcal{N}_0) = \ldots$. Now take $x \in \operatorname{Ker} f$, that is, $f(x) \in \mathcal{N}_0$. Since f^m is surjective, $\exists x' \in M$ such that $x = f^m(x')$, whence $f^{m+1}(x') = f(x) \in \mathcal{N}_0$, or $x' \in f^{-(m+1)}(\mathcal{N}_0) = f^{-m}(\mathcal{N}_0)$. So $f^m(x') \in \mathcal{N}_0$ and $x \in \mathcal{N}_0$. This proves that $\operatorname{Ker} f = \mathcal{N}_0$ and, since f is surjective, that $f(\mathcal{N}_0) = \mathcal{N}_0$. We may then define the surjective endomorphism

$$f_1\colon \mathcal{N}_0 \to \mathcal{N}_0,\ f_1(x) = f(x),\ \forall x \in \mathcal{N}_0.$$

Being Noetherian, M is finitely generated, which in turn implies that $\mathcal{N}_0$ is finite (see [6], Theorem 3.3) and so f_1 is injective too. This shows (by 1.2) that f is also injective. $\square$

Corollary 3.9. *If $f : M \to M$ is an endomorphism of an R-n-module of finite length, then the following are equivalent:*

1) *f is an automorphism,*
2) *f is injective,*
3) *f is surjective.*

Definition 3.10. Let M be an R-n-module and let $\{M_i\}_{i\in I}$ be a family of n-submodules of M. We say that M is the *(internal) direct sum* of the family $\{M_i\}_{i\in I}$ if

(1) $M = \langle \bigcup_{i\in I} M_i \rangle$

(2) there exists an n-submodule N of $\mathcal{N}_0$ such that for every $j \in I$ we have $M_j \cap \langle \bigcup_{i\neq j} M_i \rangle = N$.

In this case, we say that M is the N*-direct sum* of the family $\{M_i\}_{i\in I}$; in particular, for $N = \emptyset$ or $N = \{e\}$ we call it *0-direct sum* or *1-direct sum*, respectively.

Remark 3.11. 1) Every n-submodule $\emptyset \neq N \subseteq \mathcal{N}_0$ determines an N-decomposition of M, namely: $M = \bigcup_{e\in N} M_e \oplus \mathcal{N}_0$. In particular, for each zero-idempotent $e \in \mathcal{N}_0$ we have a decomposition of M into a 1-direct sum:

$$M = M_e \oplus \mathcal{N}_0 \tag{D}$$

2) For each zero-idempotent $e \in \mathcal{N}_0$ we have a class of decompositions of M into 0-direct sums:

$$M = M_e \oplus \left(\oplus_{f\neq e} T_f\right) \tag{D'}$$

where each T_f is equal either to M_f or to $\{f\}$.

Definition 3.12. An n-module B with zero is called *decomposable* if B can be expressed as a direct sum $B = B_1 \oplus B_2$, with $B_1 \neq \{0\}$ and $B_2 \neq \{0\}$. Otherwise, B is called *indecomposable*.

An n-module M is called *indecomposable* if M_e is indecomposable and $\mathcal{N}_0$ is simple.

Remark 3.13. 1) Simple n-modules are indecomposable.

2) An n-submodule N of $\mathcal{N}_0$ is indecomposable iff it is simple.
3) If the n-module M is indecomposable, then its only decompositions in which M itself does not appear as a summand, are those of the forms (D) and (D').

Definition 3.14. A decomposition of an n-module into a direct sum of n-submodules is called a *canonical decomposition* if

(1) it is obtained from (D) by further decomposition of the two summands,
(2) the direct sum employed is a 1-direct sum,
(3) it does not contain summands which are one-element sets or the empty set.

In a canonical decomposition the summands are either n-modules with zero or n-submodules (n-subgroups) of $\mathcal{N}_0$.

Theorem 3.15. (*Fitting's lemma*) *If M is an R-n-module of finite length and $f\colon M \to M$ is an endomorphism, then there exists an integer $m \geqslant 1$ such that $M = f^m(M) \oplus \operatorname{Ker} f^m$.*

Proof. Similar to the one for the binary case (see [7] or [8]). Since M is Artinian, it follows – as in the proof of the preceding theorem – that there exists $m \geqslant 1$ such that $f^m(M) = f^{m+1}(M) = \ldots$, whence $f^m(M) = f^{2\cdot m}(M)$. Define the map $g\colon f^m(M) \to f^m(M)$, $g(x) = f^m(x)$ and note that g is a surjective endomorphism. Now $f^m(M)$ is Noetherian, being an n-submodule of M, so g is an automorphism. Therefore, we have

$$f^m(M) \cap \operatorname{Ker} f^m = \operatorname{Ker} g = \mathcal{N}_{0 f^m(M)} \subseteq \mathcal{N}_0.$$

In addition to that, for any $x \in M$ there exists $y \in M$ such that $f^m(x) = g\big(f^m(y)\big)$ and so

$$\big[f^m(x), \overset{(n-3)}{f^m(f^m(y))}, f^m(f^m(\overline{y})), f^m(e)\big]_+ = f^m(e),$$

$\forall e \in \mathcal{N}_0$. It follows that the element $u = \big[x, \overset{(n-3)}{f^m(y)}, f^m(\overline{y}), e\big]_+$ belongs to $\operatorname{Ker} f^m$ and: $x = [f^m(y), u, \overset{(n-2)}{e}]_+$.

This shows that $M = \big\langle f^m(M) \cup \operatorname{Ker} f^m \big\rangle$. $\square$

Corollary 3.16. *Assume that M is an indecomposable R-n-module* *of finite length.*

1) *If f is an endomorphism of M, then:*
   a) *f is an automorphism or*
   b) $\operatorname{Ker} f = \mathcal{N}_0$, $\exists e \in \mathcal{N}_0:\ f(M) = M_e$ *and the map* $g\colon M_e \to M_e$, $g(x) = f(x)$ *is an automorphism or*
   c) *f is nilpotent in the $(n,2)$-ring* $\operatorname{End}_R M$.
2) *If M is with zero, then any endomorphism of M is either nilpotent or an automorphism.*
3) *If M is with zero, and $f_i \in \operatorname{End}_R M$, $i \in \{1, 2, \ldots, m\}$, $m \equiv r(\operatorname{mod} n-1)$, while $f = [f_1, \ldots, f_m, \overset{(n-r)}{\theta}]_+$ is an automorphism, then there exists $i_0 \in \{1, \ldots, m\}$ such that f_{i_0} is an automorphism.*

Proof. 1) It follows from the preceding theorem that there exists $m \geqslant 1$ such that $M = f^m(M) \oplus \operatorname{Ker} f^m$. Since M is indecomposable, we have either $f^m(M) = \mathcal{N}_0$ or $\operatorname{Ker} f^m = \mathcal{N}_0$. In the first case, f^m is a nullary endomorphism and so f is nilpotent; in the second case we have either $f^m(M) = M$ or $f^m(M) = M_e$, for a certain $e \in \mathcal{N}_0$. If $f^m(M) = M$, then $f(M) = M$, so f is a surjective homomorphism and from Corollary 3.9 it follows that f is an automorphism. If $f^m(M) = M_e$, then (as in the proof of the preceding theorem) $M_e = f^m(M) = f^{m+1}(M) = f(M_e)$ and therefore the endomorphism $g\colon M_e \to M_e$ is surjective, so (by Corollary 3.9) it is an automorphism.

Now $\operatorname{Ker} f^m = \mathcal{N}_0$ implies that $\operatorname{Ker} f = \mathcal{N}_0$, while the fact that $\mathcal{N}_0$ is simple implies that $f(\mathcal{N}_0)$ is either a one-element set or the whole of $\mathcal{N}_0$. If $f(\mathcal{N}_0) = \mathcal{N}_0$, then the map $h\colon \mathcal{N}_0 \to \mathcal{N}_0$ is a surjective endomorphism, so an automorphism. But this fact, together with $\operatorname{Ker} f = \mathcal{N}_0$, implies that f is injective, hence f is an automorphism, which contradicts $f^m(M) = M_e$. Therefore there exists $u \in \mathcal{N}_0$ such that $f(\mathcal{N}_0) = \{u\}$; now $f(M_e) = M_e$ implies that $u = e$. Take now $y \in f(M)$ and $x \in M$ cu $y = f(x)$. If $x \in M_e$, then $y = f(x) \in M_e$; if $x \in M_v$, $v \neq e$, then let x' be the uniquely determined element of M_e such that $x = [x', \overset{(n-2)}{e}, v]_+$. Now we have

$$y = f(x) = [f(x'), \overset{(n-2)}{f(e)}, f(v)]_+ = [f(x'), \overset{(n-1)}{e}]_+ = f(x') \in M_e$$

which proves that $f(M) \subseteq M_e$.

2) Direct consequence of 1).

3) The proof is by induction on m.

If $m = 1$, then $f = [f_1, \overset{(n-1)}{\theta}]_+ = f_1$, so f_1 is an automorphism. Let now $m \geqslant 2$ and assume that the statement is true for $m-1$. The equation $f = [f_1, \ldots, f_m, \overset{(n-r)}{\theta}]_+$ implies, by right multiplication with f^{-1}, the following:

$$\mathrm{id}_M = [g_1, \ldots, g_m, \overset{(n-r)}{\theta}]_+ ,$$

where $g_i = f_i \circ f^{-1}$. If g_1 is an automorphism, then f_1 is an automorphism and $i_0 = 1$; otherwise, it follows from 2) that g_1 is nilpotent, i.e. $\exists k \geqslant 1$ such that $g_1^k = \theta$. It follows now

$$\begin{aligned}[\mathrm{id}_M, \overset{(n-3)}{g_1}, \overline{g_1}, \theta]_+ &\circ [\mathrm{id}_M, g_1, \ldots, g_1^{k-1}, \overset{(n-t)}{\theta}]_+ \\ &= \mathrm{id}_M = [\mathrm{id}_M, g_1, \ldots, g_1^{k-1}, \overset{(n-t)}{\theta}]_+ \circ [\mathrm{id}_M, \overset{(n-3)}{g_1}, \overline{g_1}, \theta]_+\end{aligned}$$

and so the map

$$[\mathrm{id}_m, \overset{(n-3)}{g_1}, \overline{g_1}, \theta]_+ = [g_2, \ldots, g_m, \overset{(n-r+1)}{\theta}]_+$$

is an automorphism for which we can apply the induction hypothesis. This completes the proof. □

Using arguments identical to those employed in the binary case ([7], [8]), one can prove the following

Theorem 3.17. *If A is an R-n-module with zero, Artinian or Noetherian, then M can be decomposed as a finite direct sum of indecomposable n-submodules.*

Also the Krull–Remack–Schmidt Theorem can be immediately transferred to the case of R-n-modules with zero: Let $B \neq \{0\}$ be an R-n-module with zero which is both Artinian and Noetherian. Then B is a finite direct sum of indecomposable n-submodules. Up to a permutation, the indecomposable components in such a direct sum are uniquely determined up to isomorphism.

Remark 3.18. Let us return now to the general case of R-n-modules (not necessarily with zero): it follows that the problem of decomposing an R-n-module M of finite length into a finite direct sum of indecomposables can be reduced to the decomposition of $\mathcal{N}_{0M}$ (since $M = M_e \oplus \mathcal{N}_{0M}$ and M_e is an n-module with zero). Recall that if M is Noetherian, then the idempotent abelian n-group $\mathcal{N}_{0M}$ is finite and $|\mathcal{N}_{0M}|$ divides $(n-1)^{k-1}$, where k is the cardinal of the generating set. Also recall that, by Remark 3.13, an n-submodule of $\mathcal{N}_0$ is indecomposable if and only if it is simple. Take $e \in \mathcal{N}_{0M}$ and let $G = \operatorname{red}_e \mathcal{N}_{0M}$ be the binary reduce of $\mathcal{N}_{0M}$ with respect to the element e (i.e. $x + y = [x, \overset{(n-2)}{e}, y]_+$); G is a (bi)group of exponent $n-1$. Note that $x_1 + \cdots + x_n = [x_1^n]_+$, which shows that $\mathcal{N}_{0M} = \operatorname{ext}^n G$. Take the decomposition (unique up to isomorphism) of G into a direct sum of indecomposable subgroups of the form $\mathbb{Z}_{p^r}$, with p prime:

$$G = G_1 \oplus \cdots \oplus G_t \tag{d_1}$$

and immediately obtain the following decomposition for $\mathcal{N}_{0M}$:

$$\mathcal{N}_{0M} = \operatorname{ext}^n G = \operatorname{ext}^n G_1 \oplus \cdots \oplus \operatorname{ext}^n G_t \tag{d_2}$$

We still did not solve the problem, since not all these summands are simple: in fact, $\operatorname{ext}^n G_i$ is simple iff G_i is of the form $\mathbb{Z}_p$, p prime. So, it remains to describe the possible decompositions of $\operatorname{ext}^n \mathbb{Z}_{p^r}$, $r > 1$, where $p^r \mid n-1$. Unfortunately, for this case one cannot prove the uniqueness of decomposition, as the following example shows.

Example 3.19. Take $n = 9$ and $A = \operatorname{ext}^9 \mathbb{Z}_8$. The 9-group A has four 9-subgroups of order 2, namely: $A_1 = \{1, 5\}, A_2 = \{2, 6\}, A_3 = \{3, 7\}, A_4 = \{0, 4\}$ and the following decompositions into direct sums:

$$\begin{aligned} A =& A_1 \oplus A_2 = A_1 \oplus A_4 = A_3 \oplus A_2 = A_3 \oplus A_4 \\ =& A_i \oplus A_j \oplus A_k = A_1 \oplus A_2 \oplus A_3 \oplus A_4 \end{aligned}$$

where i, j, k are distinct numbers in $\{1, 2, 3, 4\}$. Note that the four 9-subgroups of order 2 are mutually disjoint, which means that any decomposition of A into direct sum of indecomposables is necessarily a 0-direct sum; it is easy to check that in fact this statement is true for any n-group of the form $\operatorname{ext}^n \mathbb{Z}_{p^r}$, with $r > 1$ and $p^r \mid n-1$. Also note that $A_1 \oplus A_3 = \{1, 3, 5, 7\} \simeq \operatorname{ext}^9 \mathbb{Z}_4$, which shows that 0-direct sums with respectively isomorphic summands can give non-isomorphic results.

Summarizing, if M is a Noetherian R-n-module, then one of the following situations occurs:

- $\mathcal{N}_{0M}$ is simple. This is precisely the case when its order is a prime number p (with $p \mid n-1$);
- $\mathcal{N}_{0M}$ is not simple and it has a unique (up to isomorphism) decomposition into a finite 1-direct sum of indecomposable n-submodules. This is precisely the case when every binary reduce has in its decomposition (d_1) only summands of the form $\mathbb{Z}_{p_i}$, with p_i prime numbers.
- $\mathcal{N}_{0M}$ is not simple and it can be decomposed into finite 0-direct sums of indecomposables only. This is precisely the case when every binary reduce has at least one summand of the form $\mathbb{Z}_{p^r}$, p prime and $r > 1$, in the decomposition (d_1).

The above discussion leads us to a weaker version of the Krull–Remack–Schmidt theorem for n-modules, in the special case when $n-1 = p_1 \dots p_k$ (the prime factorization of $n-1$ is multiplicity-free).

Theorem 3.20. *Let $n > 2$ be an integer such that $n-1 = p_1 \dots p_k$ and let M be an R-n-module which is both Artinian and Noetherian. Then M has a finite canonical decomposition into indecomposable n-modules. Up to a permutation, the indecomposable components are uniquely determined up to isomorphism.*

The above theorem allows us to reduce the problem of decomposing an R-n-module into a direct sum of indecomposable n-submodules to the problem of decomposing an R-n-module with zero and an abelian n-group. Both these decompositions can be done by using the binary reduces of the two structures and then their n-ary extensions. To be more precise, if B is an R-n-module with zero, then its *binary reduce* with respect to an element $b \in B$ is the module B with the operations:

$$x + y = [x, \overset{(n-3)}{b}, \overline{b}, y]_+, \qquad r \bullet x = [rx, \overset{(n-3)}{rb}, r\overline{b}, b]_+ ,$$

for our purpose (decomposition), it is useful to consider the binary reduce with respect to the zero element. The *n-ary extension* with respect to an element a of an R-module A is the R-n-module A, with the following operations:

$$[x_1^n]_+ = x_1 + \cdots + x_n - (n-1)a\,, \qquad r \star x = rx - ra + a\,,$$

and a is the zero element in the n-ary extension. Furthermore, one can easily check that for any $a, b \in B$ we have $\mathrm{ext}_b^n(\mathrm{red}_a\, M) \simeq M$; in particular, $\mathrm{ext}_0^n(\mathrm{red}_0\, M) = M$. Note that we can talk about unique decomposition only if it is canonical, as the following example shows.

Example 3.21. Let $(\mathbb{Z}_{30}, +, \cdot)$ be the ring of integers *modulo* 30. We define on the set $M = \mathbb{Z}_{30}$ a structure of $\mathbb{Z}$-7-module by:

$$[x_1^7]_+ = x_1 + \cdots + x_7 \quad \text{and} \quad k \bullet x = (6k{+}25) \cdot x\,.$$

Then we have

$$\mathcal{N}_M = \mathcal{N}_{0M} = \{0, 5, 10, 15, 20, 25\},\ M_0 = \{0, 6, 12, 18, 24\}$$

and the following canonical decomposition of M:

$$M = \{0, 6, 12, 18, 24\} \oplus \{0, 15\} \oplus \{0, 10, 20\}$$

which is unique up to isomorphism.

However, we can give two different (non–canonical) decompositions of M into 1-direct sums of indecomposable n-submodules, namely:

$$\begin{aligned} M =& \{0, 3, 6, 9, 12, 15, 18, 21, 24, 27\} \oplus \{0, 10, 20\} \\ =& \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28\} \oplus \{0, 15\}\,. \end{aligned}$$

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