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On the classes of algebras reciprocally closed under direct products

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Abstract

The class K of algebras with the property that two algebras belongs to K iff their direct product belongs to K is studied.

The class K of algebras with the property that two algebras belongs to K iff their direct product belongs to K is called *reciprocally closed under direct products*. The formula Φ is *reciprocally preserved under direct products* if the class of algebras satisfying Φ is reciprocally closed under direct products (cf. [1]).

Three following assertions are evident.

Proposition 1. *A class of algebras closed under direct products and homomorphisms is reciprocally closed under direct products.*

Proposition 2. *A class of idempotent algebras closed under direct products and subalgebras is reciprocally closed under direct products.*

Proposition 3. *The conjunction of formulas of a fixed signature, which are reciprocally preserved under direct products, is reciprocally preserved under direct products. Similarly, the intersection of classes of algebras reciprocally closed under direct products is a class of algebras reciprocally closed under direct products.*

In this paper by a *groupoid* we mean an algebra (Q, f) with one (binary or n-ary) operation f. A groupoid (Q, f) in which for all

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$1 \leqslant i \leqslant n$ and $a_i \in Q$ the equation

$$f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n) = a_i$$

has a unique solution $x_i \in Q$ (denoted by $f^i(a_1, \ldots, a_n)$) is called a *quasigroup*. A *loop* is a quasigroup with a neutral element; a *semigroup* - an associative groupoid; a *group* - an associative quasigroup.

A formula Φ of the signature Ω is called *conjunctively-positive* iff its record has no predicate letter, except the symbols of equality, no logical connective, except the symbols of conjunction, and no term, except terms of the signature Ω.

A formula Φ is *prenex almost conjunctively-positive formula of a signature* Ω, iff all quantifiers and symbols "$\exists!$" in its shortened record, obtained only by reductions to the symbols "$\exists!$", precede the quantifier-free part, and the shortened record has no predicate letter, except the symbols of equality, no logical connective, except the symbols of conjunction, and no term, except terms of the signature Ω. Obviously, prenex normal form of a conjunctively-positive formula of a signature Ω is a prenex almost conjunctively-positive formula of the signature Ω.

Lemma 4. *Every prenex almost conjunctively-positive formula is reciprocally preserved under direct products.*

Proof. The given formula is equivalent to a closed formula of the form

$$(Q_1 x_1) \ldots (Q_k x_k)(w_1 = w_2 \,\&\, \ldots \,\&\, w_{2m-1} = w_{2m}), \tag{1}$$

where $Q_1, ..., Q_k$ are quantifiers $\forall$, $\exists$ and symbols "$\exists!$", and $w_1, ..., w_{2m}$ are terms of the signature of the given formula. The formula (1) has the signature of algebras of some type. Fix arbitrary algebras $\langle G, \Omega_1 \rangle$ and $\langle H, \Omega_2 \rangle$ of the type. Denote the direct product of the first of them by the second of them by $\langle M, \Omega \rangle$. Validity of the formula (1) in the algebra $\langle M, \Omega \rangle$ is equivalent to the formula

$$(Q_1 \langle y_1, z_1 \rangle \in M) \ldots (Q_k \langle y_k, z_k \rangle \in M)\, P(\langle y_1, z_1 \rangle, \ldots, \langle y_k, z_k \rangle), \tag{2}$$

where $P(x_1, \ldots, x_k)$ is the quantifier-free part of the formula (1). Next, the formula (2) is equivalent to the formula

$$(Q_1 y_1 \in G, z_1 \in H) \ldots (Q_k y_k \in G, z_k \in H)\, (P'(y_1, \ldots, y_k) \,\& \\ \& \, P''(z_1, \ldots, z_k)), \tag{3}$$

where P' and P'' are formulas obtained from P by the way of the replacement of every propositional variable x_i respectively with y_i and z_i and of every functional variable f of the signature Ω with the respective functional variable (f_1 of the signature Ω_1 and f_2 of the signature Ω_2 respectively). At last, formula (3) and, therefore, formula (2), are equivalent to the formula

$$((Q_1 y_1 \in G) \dots (Q_k y_k \in G)\, P'(y_1, \dots, y_k))\,\& \\ \&\,((Q_1 z_1 \in H) \dots (Q_k z_k \in H)\, P''(z_1, \dots, z_k)),$$

that is equivalent to simultaneous validity of the formula (1) in both $\langle G, \Omega_1\rangle$ and $\langle H, \Omega_2\rangle$ algebras. □

Corollary 5. *Every conjunctively-positive formula is reciprocally preserved under direct products.*

Corollary 6. *The class of all quasigroups (of all groups, of all semigroups, of all monoids, of all loops) is reciprocally closed under direct products.*

As it is well known, the *direct product* $\rho \times \tau$ *of binary relations* ρ *and* τ is defined as the relation

$$\langle a, b\rangle(\rho \times \tau)\langle c, d\rangle \iff (a\,\rho\,c)\,\&\,(b\,\tau\,d).$$

It is clear, that for mappings f and g the relation $f \times g$ is a mapping with the domain equal to the Cartesian product of the domains of the mappings f and g and $(f \times g)(\langle x, y\rangle) = \langle f(x), g(y)\rangle$.

A groupoid (G, g) is called an *isotope of a binary semigroup* $(Q, +)$ iff there exists a collection $\langle \alpha_1, \dots, \alpha_n, \alpha\rangle$ of bijections from the set G onto the set Q satisfying the identity

$$\alpha g(x_1, \dots, x_n) = \alpha_1 x_1 + \dots + \alpha_n x_n. \tag{4}$$

An isotope of a group is called also a *group isotope.* It is easy to see that an isotope of a group is a quasigroup. A transformation α of a set Q is called a *linear transformation of a group* $(Q, +)$ if there exist an endomorphism θ and a right translation R_c of this group such that $\alpha = R_c\theta$. An isotope of a group $(Q, +)$ defined by (4) is called *i-linear* if the bijections α_i and α are linear transformations of

$(Q,+)$. An isotope is *linear* if it is i-linear for all i. Obviously, every groupoid isomorphic to a linear or i-linear group isotope is a linear or, respectively, i-linear group isotope.

Lemma 7. *The direct product of an isotope (A,g) of a semigroup $(G,+)$ by an isotope (B,h) of a semigroup $(H,\cdot)$ defined by (4) and*

$$\beta h(x_1,\ldots,x_n)=\beta_1 x_1\cdot\ldots\cdot\beta_n x_n$$

is an isotope (C,f) of the semigroup $(M,\circ)$ determined by

$$(M,\circ)=(G\times H,\circ)=(G,+)\times(H,\cdot)$$

and by

$$(\alpha\times\beta)f(x_1,\ldots,x_n)=(\alpha_1\times\beta_1)x_1\circ\ldots\circ(\alpha_n\times\beta_n)x_n,$$

where $\alpha_1,\ldots,\alpha_n$ and α are bijections from A onto G, and $\beta_1,\ldots,\beta_n$ and β are bijections from B onto H.

Proof. Indeed, let f be the operation of the given direct product of the isotopes of the semigroups. Then

$$\begin{aligned}(\alpha\times\beta)&f(\langle x_1,y_1\rangle,\ldots,\langle x_n,y_n\rangle)\\ &=(\alpha\times\beta)(\langle g(x_1,\ldots,x_n),h(y_1,\ldots,y_n)\rangle)\\ &=\langle\alpha g(x_1,\ldots,x_n),\beta h(y_1,\ldots,y_n)\rangle\\ &=\langle\alpha_1x_1+\ldots+\alpha_nx_n,\beta_1y_1\cdot\ldots\cdot\beta_ny_n\rangle\\ &=\langle\alpha_1x_1,\beta_1y_1\rangle\circ\ldots\circ\langle\alpha_nx_n,\beta_ny_n\rangle\\ &=(\alpha_1\times\beta_1)\langle x_1,y_1\rangle\circ\ldots\circ(\alpha_n\times\beta_n)\langle x_n,y_n\rangle,\end{aligned}$$

which completes the proof. □

If (Q,f) is a quasigroup of an arity $n\geqslant 2$ then $(Q,f,f^1,\ldots,f^n)$ is called the *primitive quasigroup* which corresponds to the quasigroup (Q,f). Such quasigroup maybe defined as an algebra $(Q,f,f^1,\ldots,f^n)$ with $n+1$ n-ary operations satisfying $2n$ identities:

$$f(x_1,\ldots,x_{i-1},f^i(x_1,\ldots,x_n),x_{i+1},\ldots,x_n)=x_i,$$
$$f^i(x_1,\ldots,x_{i-1},f(x_1,\ldots,x_n),x_{i+1},\ldots,x_n)=x_i.$$

A congruence on a quasigroup (Q,f) is called *normal* if it is a congruence on the corresponding primitive quasigroup.

Lemma 8. *The homomorphic image of a group isotope, where the congruence which corresponds to the respective homomorphism is normal, is a group isotope.*

Proof. Let (Q, f) be the given group isotope, φ the given homomorphism of the group isotope (Q, f) onto a groupoid (G, g), and π the respective normal congruence on (Q, f). Then π is a congruence on the primitive quasigroup $(Q, f, f^1, \dots, f^n)$. Denote the respective natural homomorphism by ψ. From [2] it follows that the class of all n-ary group isotopes is a variety of quasigroups. Therefore, the class of all primitive quasigroups which correspond to n-ary group isotopes is closed under homomorphisms, whence ψ is a homomorphism of the group isotope (Q, f) onto some group isotope $(Q/\pi, h)$. Hence (G, g) is a group isotope. □

Lemma 9. *A homomorphism of a quasigroup (Q, f) into a quasigroup (G, g) is a homomorphism of a quasigroup $(Q, f, f^1, \dots, f^n)$ into a quasigroup $(G, g, g^1, \dots, g^n)$.*

Proof. Denote the given homomorphism by φ. Let a_1, …, a_n be arbitrary elements from Q, and i be a natural number not greater than n. If $b_i = f^i(a_1, \dots, a_n)$, then

$$\varphi a_i = \varphi f(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n) \\ = g(\varphi a_1, \dots, \varphi a_{i-1}, \varphi b_i, \varphi a_{i+1}, \dots, \varphi a_n),$$

whence, it follows that

$$\varphi f^i(a_1, \dots, a_n) = \varphi b_i = g^i(\varphi a_1, \dots, \varphi a_n)$$

for all a_1, …, $a_n \in Q$ and all $1 \leqslant i \leqslant n$. Thus we have the identity

$$\varphi f^y(x_1, \dots, x_n) = g^y(\varphi x_1, \dots, \varphi x_n).$$

This completes the proof. □

Corollary 10. *The congruence which corresponds to a homomorphism of a quasigroup into a quasigroup is normal.*

Corollary 11. *If there exists a homomorphism φ of a group isotope into a quasigroup (Q, f), then the groupoid $(Im\varphi, f)$ is a group isotope.*

Proof. It is enough to add the statement of Lemma 8 to the statement of Corollary 10. □

Example 14. Let $(Q,+)$ be an arbitrary infinite group. Since the sets Q and Q^2 have the same cardinal number, then there exists a bijection f of Q^2 onto Q. Let $(Q^3,*)$ be the direct product $(Q,+)\times(Q,+)\times(Q,+)$ and let (Q^3,g) be the isotope of the group $(Q^3,*)$ defined by the identity

$$g(x_1,\ldots,x_n)=\alpha x_1*\ldots*\alpha x_n,$$

where $n\geqslant 2$ is an arbitrary fixed number and α is a substitution of Q^3 defined by the identity

$$\alpha(\langle x,y,z\rangle)=\langle f^{-1}(x),f(y,z)\rangle.$$

Let φ be a mapping $\varphi: Q^3\to Q^2$ such that

$$\varphi:\langle x,y,z\rangle\mapsto\langle x,y\rangle,$$

and let h be the operation of the arity $n\geqslant 2$ defined on Q^2 by the formula

$$h(\langle x_1,y_1\rangle,\ldots,\langle x_n,y_n\rangle)=\varphi g(\langle x_1,y_1,z_1\rangle,\ldots,\langle x_n,y_n,z_n\rangle),$$

where $z_1,\ldots,z_n\in Q$ are arbitrary.

The operation h is not dependent on that choice of $z_1,\ldots,z_n\in Q$, since for the direct product $(Q^2,\star)$ of the group $(Q,+)$ we have

$$\begin{aligned}
&\varphi g(\langle x_1,y_1,z_1\rangle,\ldots,\langle x_n,y_n,z_n\rangle)\\
&\qquad=\varphi(\alpha(\langle x_1,y_1,z_1\rangle)*\ldots*\alpha(\langle x_n,y_n,z_n\rangle))\\
&\qquad=\varphi(\langle f^{-1}(x_1),f(y_1,z_1)\rangle*\ldots*\langle f^{-1}(x_n),f(y_n,z_n)\rangle)\\
&\qquad=\varphi(\langle f^{-1}(x_1)\star\ldots\star f^{-1}(x_n),f(y_1,z_1)+\ldots+f(y_n,z_n)\rangle)\\
&\qquad=f^{-1}(x_1)\star\ldots\star f^{-1}(x_n).
\end{aligned}$$

Moreover, from these equalities it follows that the operation h is not a quasigroup one, since all divisions are multivalued. But the identity

$$h(\varphi x_1,\ldots,\varphi x_n)=\varphi g(x_1,\ldots,x_n)$$

holds. Thus, φ is a homomorphism of the group isotope (Q^3, g) onto the groupoid (Q^2, h), which is not even a quasigroup. The congruence corresponding to it by Lemma 8 is not normal. □

Theorem 13. *The class of all group isotopes is reciprocally closed under direct products.*

Proof. By Lemma 7 the direct product of two group isotopes of the same arity is a group isotope. Let the direct product (M, f) of a groupoid (G, g) by a groupoid (H, h) be a group isotope. By Corollary 6 the groupoids (G, g) and (H, h) are quasigroups. It is easy to see that the mappings φ_1 and φ_2 from the group isotope (M, f) into the quasigroups (G, g) and (H, h) respectively, for which

$$(\forall x \in G)(\forall y \in H)(\varphi_1(\langle x, y\rangle) = x \ \&\ \varphi_2(\langle x, y\rangle) = y),$$

are homomorphisms of the group isotope (M, f) onto the quasigroups (G, g) and (H, h), respectively. By Corollary 11 these two quasigroups are group isotopes. □

Theorem 14. *The class of all i-linear n-ary group isotopes, where i and n are fixed numbers, is reciprocally closed under direct products.*

Proof. By Lemma 7 the direct product of two i-linear n-ary group isotopes is an i-linear group isotope. Let the direct product (M, f) of a groupoid (G, g) by a groupoid (H, h) be i-linear n-ary group isotope. By Theorem 13 (G, g) and (H, h) are group isotopes. The repeated application of Lemma 7 gives i-linearity of these group isotopes. □

Corollary 15. *The class of all linear group isotopes is reciprocally closed under direct products.*

In spite of the collection of the above results and the results of Horn from [1] which describe the structure of the classes of algebras reciprocally closed under direct products, the question about criterion for a class of algebras to be reciprocally closed under direct products, or, at least, for a formula to be reciprocally preserved under direct products, remains open.

References

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