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Remarks on polyadic groups

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Abstract

We prove that an n-ary semigroup $(G, [\ ])$ is an n-ary group ($n \geqslant 3$) iff there exists $d \in G$ such that for every $a, b \in G$ and some fixed $i, j \in \{1, \ldots, n-1\}$ the following two equations $[\overset{(i)}{a}, \overset{(n-i-1)}{b}, x] = d$ and $[y, \overset{(n-j-1)}{b}, \overset{(j)}{a}] = b$ are solvable.

Generalizing the group result from [4], V. I. Tyutin proved in [3] that an n-ary group $(G, [\ ])$ may be defined as an n-ary semigroup $(G, [\ ])$ in which for some fixed $d \in G$ and every $a, b \in G$ the following two equations

$$[\overset{(n-1)}{a}, x] = b, \qquad [y, \overset{(n-1)}{a}] = d$$

are solvable.

On the other hand, the author proved in [2] the following characterization of n-ary group

Theorem 1. *An n-ary semigroup $(G, [\ ])$ is an n-ary group iff for every $a, b \in G$ and some fixed $i, j \in \{1, \ldots, n-1\}$ the following two equations*

$$[\overset{(i)}{a}, \overset{(n-i-1)}{b}, x] = b, \qquad [y, \overset{(n-j-1)}{b}, \overset{(j)}{a}] = b$$

are solvable. □

Note that this theorem was also obtained by W. A. Dudek as a consequence of some general results (cf. [1]).

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Theorem 2. *An n-ary semigroup $(G,[\ ])$ is an n-ary group $(n \geqslant 3)$ iff there exists $d \in G$ such that for every $a, b \in G$ and some fixed $i, j \in \{1, \ldots, n-1\}$ the following equations*

$$[\overset{(i)}{a}, \overset{(n-i-1)}{b}, x] = d, \qquad [y, \overset{(n-j-1)}{b}, \overset{(j)}{a}] = b$$

are solvable.

Proof. It is clear that in an n-ary group $(G,[\ ])$ these equations have unique solutions for every $a, b, d \in G$ and every $i, j \in \{1, \ldots, n-1\}$.

On the other hand, if $(G,[\ ])$ is an n-ary semigroup in which there exists an element d such that for some fixed $i, j \in \{1, \ldots, n-1\}$ and every $a, b \in G$ these equations are solvable, then there are elements $u, v, w, z \in G$ such that

$$[\overset{(n-1)}{d}, u] = d, \quad [v, \overset{(n-j-1)}{a}, \overset{(j)}{d}] = a, \quad [w, \overset{(n-1)}{a}] = a, \quad [\overset{(i)}{a}, \overset{(n-i-1)}{b}, z] = d.$$

For such $u, v, w, z, d \in G$ and every $a \in G$ we have also

$$[a, \overset{(n-2)}{d}, u] = a, \quad [w, \overset{(n-2)}{a}, d] = d, \quad [\overset{(n-2)}{d}, u, a] = a,$$

because

$$\begin{aligned}[a, \overset{(n-2)}{d}, u] &= [[v, \overset{(n-j-1)}{a}, \overset{(j)}{d}], \overset{(n-2)}{d}, u] = [v, \overset{(n-j-1)}{a}, \overset{(j-1)}{d}, [\overset{(n-1)}{d}, u]] \\ &= [v, \overset{(n-j-1)}{a}, \overset{(j-1)}{d}, d] = [v, \overset{(n-j-1)}{a}, \overset{(j)}{d}] = a,\end{aligned}$$

$$\begin{aligned}[w, \overset{(n-2)}{a}, d] &= [w, \overset{(n-2)}{a}, [\overset{(i)}{a}, \overset{(n-i-1)}{b}, z]] = [[w, \overset{(n-1)}{a}], \overset{(i-1)}{a}, \overset{(n-i-1)}{b}, z] \\ &= [a, \overset{(i-1)}{a}, \overset{(n-i-1)}{b}, z] = [\overset{(i)}{a}, \overset{(n-i-1)}{b}, z] = d\end{aligned}$$

and

$$\begin{aligned}[\overset{(n-2)}{d}, u, a] &= [[w, \overset{(n-2)}{a}, d], \overset{(n-3)}{d}, u, a] = [w, \overset{(n-3)}{a}, [a, \overset{(n-2)}{d}, u], a] \\ &= [w, \overset{(n-3)}{a}, a, a] = [w, \overset{(n-1)}{a}] = a.\end{aligned}$$

As a consequence we obtain

$$\begin{aligned}[\overset{(i)}{a}, \overset{(n-i-1)}{b}, [z, \overset{(n-3)}{d}, u, b]] &= [[\overset{(i)}{a}, \overset{(n-i-1)}{b}, z], \overset{(n-3)}{d}, u, b] \\ &= [d, \overset{(n-3)}{d}, u, b] = [\overset{(n-2)}{d}, u, b] = b,\end{aligned}$$

which proves that $x = [\,z, \overset{(n-3)}{d}, u, b\,]$ is the solution of the first equation from our Theorem 1.

Since the second equation in Theorem 2 is the same as the second equation in Theorem 1 and has (by the assumption) a solution, Theorem 1 shows that $(G, [\ \ ])$ is an n-ary group.

This completes the proof. $\square$

Corollary. *An n-ary semigroup $(G, [\ \ ])$ is an n-ary group $(n \geqslant 3)$ iff there exists $d \in G$ such that for every $a, b \in G$ (at least) one of the following pairs of equations is solvable:*

a) $\quad [\,a, \overset{(n-2)}{b}, x\,] = d, \qquad [\,y, \overset{(n-2)}{b}, a\,] = b;$

b) $\quad [a, \overset{(n-2)}{b}, x\,] = d, \qquad [\,y, \overset{(n-1)}{a}\,] = b;$

c) $\quad [\overset{(n-1)}{a}, x\,] = d, \qquad [\,y, \overset{(n-2)}{b}, a] = b;$

d) $\quad [\overset{(n-1)}{a}, x\,] = d, \qquad [\,y, \overset{(n-1)}{a}\,] = b.$ $\square$

In the same manner as the above Theorem 2 we can prove

Theorem 3. *An n-ary semigroup $(G, [\ \ ])$ is an n-ary group $(n \geqslant 3)$ iff there exists $d \in G$ such that for every $a, b \in G$ and some fixed $i, j \in \{1, \ldots, n-1\}$ the following equations*

$$[\,\overset{(i)}{a}, \overset{(n-i-1)}{b}, x\,] = b, \qquad [\,y, \overset{(n-j-1)}{b}, \overset{(j)}{a}\,] = d$$

are solvable. $\square$

Putting in this Theorem $i = j = n-1$, we obtain Tyutin's definition of n-ary groups mentioned at the beginning of this paper.

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