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On supermatrix operator semigroups

Steven Duplij

Abstract

One-parameter semigroups of antitriangle idempotent supermatrices and corresponding superoperator semigroups are introduced and investigated. It is shown that t-linear idempotent superoperators and exponential superoperators are mutually dual in some sense, and the first give additional to exponential different solution to the initial Cauchy problem. The corresponding functional equation and analog of resolvent are found for them. Differential and functional equations for idempotent (super)operators are derived for their general t power-type dependence.

1. Introduction

Operator semigroups [1] play an important role in mathematical physics [2, 3, 4] viewed as a general theory of evolution systems [5, 6, 7]. Its development covers many new fields [8, 9, 10, 11], but one of vital for modern theoretical physics directions — supersymmetry and related mathematical structures — was not considered before in application to operator semigroup theory. The main difference between previous considerations is the fact that among building blocks (e.g. elements of corresponding matrices) there exist noninvertible objects (divisors

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of zero and nilpotents) which by themselves can form another semigroup. Therefore, we have to take that into account and investigate it properly, which can be called a *semigroup* $\times$ *semigroup* method.

Here we study continuous supermatrix representations of idempotent operator semigroups firstly introduced in [12, 13] for bands. Usually matrix semigroups are defined over a field $\mathbb{K}$ [14] (on some nonsupersymmetric generalizations of $\mathbb{K}$-representations see [15, 16]). But after discovery of supersymmetry [17, 18] the realistic unified particle theories began to be considered in superspace [19, 20]. So all variables and functions were defined not over a field $\mathbb{K}$, but over Grassmann-Banach superalgebras over $\mathbb{K}$ [21, 22, 23], becoming in general noninvertible, and therefore they should be considered by semigroup theory, which was claimed in [24, 25], and some semigroups having nontrivial abstract properties were found [26]. Also, it was shown that supermatrices of the special (antitriangle) shape can form various strange and sandwich semigroups not known before [27, 28]. Here we consider one-parametric semigroups (for general theory see [2, 5, 29]) of antitriangle supermatrices and corresponding superoperator semigroups. The first ones continuously represent idempotent semigroups and second ones lead to new superoperator semigroups with nontrivial properties.

Let Λ be a commutative Banach $\mathbb{Z}_2$-graded superalgebra [30, 31] over a field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}_p$) with a decomposition into the direct sum: $\Lambda = \Lambda_0 \oplus \Lambda_1$. The elements a from Λ_0 and Λ_1 are homogeneous and have the fixed even and odd parity defined as $|a| \stackrel{def}{=} \{i \in \{0,1\} = \mathbb{Z}_2 | \, a \in \Lambda_i\}$. The even homomorphism $\mathfrak{r}_{body} : \Lambda \to \mathbb{B}$ is called a body map and the odd homomorphism $\mathfrak{r}_{soul} : \Lambda \to \mathbb{S}$ is called a soul map [32], where $\mathbb{B}$ and $\mathbb{S}$ are purely even and odd algebras over $\mathbb{K}$ and $\Lambda = \mathbb{B} \oplus \mathbb{S}$. It can be thought that, if we have the Grassmann algebra Λ with generators $\xi_i, \ldots, \xi_n$ $\xi_i \xi_j + \xi_j \xi_i = 0$, $1 \le i, \, j \le n$, in particular $\xi_i^2 = 0$ (n can be infinite, and only this case is nontrivial and interesting), then any even x and odd $\varkappa$ elements have the expansions

(which can be infinite)

$$x = x_{body} + x_{soul} = x_{body} + x_{12}\xi_1\xi_2 + x_{13}\xi_1\xi_3 + \ldots = \\ x_{body} + \sum_{1\leq r\leq n}\ \sum_{1<i_1<\ldots<i_{2r}\leq n} x_{i_1\ldots i_{2r}}\xi_{i_1}\ldots\xi_{i_{2r}} \tag{1}$$

$$\varkappa = \varkappa_{soul} = \varkappa_1\xi_1 + \varkappa_2\xi_2 + \ldots + x_{123}\xi_1\xi_2\xi_3 + \ldots = \\ \sum_{1\leq r\leq n}\ \sum_{1<i_1<\ldots<i_r\leq n} \varkappa_{i_1\ldots i_{2r-1}}\xi_{i_1}\ldots\xi_{i_{2r-1}} \tag{2}$$

From (1)-(2) it follows

Corollary 0.1. *The equations $x^2 = 0$ and $x\varkappa = 0$ can have nonzero nontrivial solutions (appearing zero divisors and even nilpotents, while odd objects are always nilpotent).*

Conjecture 0.2. *If zero divisors and nilpotents will be included in the following analysis as elements of matrices, then one can find new and unusual properties of corresponding semigroups.*

For that we should consider general properties of supermatrices [30] and introduce their additional reductions [12].

1 Supermatrices and their even-odd classification

Let us consider $(p|q)$-dimensional linear model superspace $\Lambda^{p|q}$ over Λ (in the sense of [30, 33]) as the even sector of the direct product $\Lambda^{p|q} = \Lambda_0^p \times \Lambda_1^q$ [32, 23]. The even morphisms $\mathrm{Hom}_0\left(\Lambda^{p|q}, \Lambda^{m|n}\right)$ between superlinear spaces $\Lambda^{p|q} \to \Lambda^{m|n}$ are described by means of $(m+n) \times (p+q)$-supermatrices [30, 33] (for some nontrivial properties see [34, 35]). In what follows we will treat noninvertible morphisms [36, 37] on a par with invertible ones [12].

First we consider $(1+1) \times (1+1)$-supermatrices describing the elements from $\mathrm{Hom}_0\left(\Lambda^{1|1}, \Lambda^{1|1}\right)$ in the standard $\Lambda^{1|1}$ basis [30]

$$M \equiv \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \in \mathrm{Mat}_\Lambda(1|1) \tag{3}$$

where $a, b \in \Lambda_0$, $\alpha, \beta \in \Lambda_1$, $\alpha^2 = \beta^2 = 0$ (in the following we use Latin letters for elements from Λ_0 and Greek letters for ones from Λ_1, and all odd elements are nilpotent of index 2).

The supertrace and Berezinian (superdeterminant) are defined by [30]

$$\mathrm{str} M = a - b, \tag{4}$$

$$\mathrm{Ber} M = \frac{a}{b} + \frac{\beta\alpha}{b^2}. \tag{5}$$

First term corresponds to triangle supermatrices, second term - to antitriangle ones. So we obviously have different **two dual types** of supermatrices [12].

Definition 1.1. *Even-reduced supermatrices* are elements from $\mathrm{Mat}_\Lambda (1|1)$ of the form

$$M_{even} \equiv \begin{pmatrix} a & \alpha \\ 0 & b \end{pmatrix} \in \mathrm{RMat}_\Lambda^{even} (1|1) \subset \mathrm{Mat}_\Lambda (1|1). \tag{6}$$

Odd-reduced supermatrices are elements from $\mathrm{Mat}_\Lambda (1|1)$ of the form

$$M_{odd} \equiv \begin{pmatrix} 0 & \alpha \\ \beta & b \end{pmatrix} \in \mathrm{RMat}_\Lambda^{odd} (1|1) \subset \mathrm{Mat}_\Lambda (1|1). \tag{7}$$

Conjecture 1.2. *The odd-reduced supermatrices have a nilpotent (but nonzero) Berezinian*

$$\mathrm{Ber} M_{odd} = \frac{\beta\alpha}{b^2} \neq 0, \quad (\mathrm{Ber} M_{odd})^2 = 0. \tag{8}$$

Remark 1. Indeed this property (8) prevented in the past the use of this type (odd-reduced) of supermatrices in physics. All previous applications (excluding [12, 13, 38]) were connected with triangle (even-reduced, similar to Borel) ones and first term in Berezinian $\mathrm{Ber} M = \frac{a}{b}$ (5).

The even- and odd-reduced supermatrices are *mutually dual* in the sense of the Berezinian addition formula [12]

$$\mathrm{Ber} M = \mathrm{Ber} M_{even} + \mathrm{Ber} M_{odd}. \tag{9}$$

For sets of matrices we use corresponding bold symbols, e.g. $\mathbf{M} \stackrel{def}{=} \{M \in \mathrm{Mat}_\Lambda(1|1)\}$, and the set product is standard

$$\mathbf{M} \cdot \mathbf{N} \stackrel{def}{=} \{\cup MN \mid M, N \in \mathrm{Mat}_\Lambda(1|1)\}.$$

The matrices from $\mathrm{Mat}(1|1)$ form a linear semigroup of $(1+1) \times (1+1)$-supermatrices under the standard supermatrix multiplication $\mathfrak{M}(1|1) \stackrel{def}{=} \{\mathbf{M} \mid \cdot\}$ [30]. Obviously, the even-reduced matrices $\mathbf{M}_{even}$ form a semigroup $\mathfrak{M}_{even}(1|1)$ which is a subsemigroup of $\mathfrak{M}(1|1)$, because of $\mathbf{M}_{even} \cdot \mathbf{M}_{even} \subseteq \mathbf{M}_{even}$ and the unity is in $\mathfrak{M}_{even}(1|1)$. This trivial observation leads to general structure (Borel) theory for matrices: triangle matrices form corresponding substructures (subgroups and subsemigroups). It was believed before that in case of supermatrices the situation does not changed, because supermatrix multiplication is the same [30]. But they did not take into account *zero divisors and nilpotents* appearing naturally and inevitably in supercase.

Conjecture 1.3. *Standard (lower/upper) triangle supermatrices are not the only substructures due to unusual properties of zero divisors and nilpotents appearing among elements (see (1)-(2) and* ***Corollary 0.1****).*

It means that in such consideration we have additional (to triangle) class of subsemigroups. Then we can formulate the following general

For a given n, m, p, q to describe and classify all possible substructures (subgroups and subsemigroups) of $(m+n)\times(p+q)$-supermatrices.

First example of such new substructures are Γ-matrices considered below.

Conjecture 1.4. *These new substructures lead to* ***new corresponding superoperators*** *which are represented by one-parameter substructures of supermatrices.*

Therefore we first consider possible (not triangle) subsemigroups of supermatrices.

2 Odd-reduced supermatrix semigroups

In general, the odd-reduced matrices M_{odd} do not form a semigroup, since their multiplication is not closed in general $\mathbf{M}_{odd} \cdot \mathbf{M}_{odd} \subset \mathbf{M}$. Nevertheless, some subset of $\mathbf{M}_{odd}$ *can form* a semigroup [12]. That can happen due to the existence of zero divisors in Λ, and so we have $\mathbf{M}_{odd} \cdot \mathbf{M}_{odd} \cap \mathbf{M}_{odd} = \mathbf{M}_{odd}^{smg} \neq \emptyset$.

To find $\mathbf{M}_{odd}^{smg}$ we consider a $(1+1) \times (1+1)$ example. Let $\alpha, \beta \in \Gamma_{set}$, where $\Gamma_{set} \subset \Lambda_1$. We denote $\operatorname{Ann} \alpha \stackrel{def}{=} \{\gamma \in \Lambda_1 \mid \gamma \cdot \alpha = 0\}$ and $\operatorname{Ann} \Gamma_{set} = \bigcap\limits_{\alpha \in \Gamma} \operatorname{Ann} \alpha$ (here the intersection is crucial). Then we define *left* and *right* Γ*-matrices*

$$\mathbf{M}_{odd(L)}^{\Gamma} \stackrel{def}{=} \begin{pmatrix} 0 & \Gamma_{set} \\ \operatorname{Ann} \Gamma_{set} & b \end{pmatrix}, \tag{10}$$

$$\mathbf{M}_{odd(R)}^{\Gamma} \stackrel{def}{=} \begin{pmatrix} 0 & \operatorname{Ann} \Gamma_{set} \\ \Gamma_{set} & b \end{pmatrix}. \tag{11}$$

Proposition 2.1. *The* Γ*-matrices* $\mathbf{M}_{odd(L,R)}^{\Gamma} \subset \mathbf{M}_{odd}$ *form subsemigroups of* $\mathfrak{M}(1|1)$ *under the standard supermatrix multiplication, if* $b\Gamma \subseteq \Gamma$.

Definition 2.2. Γ-semigroups $\mathfrak{M}_{odd(L,R)}^{\Gamma}(1|1)$ are subsemigroups of $\mathfrak{M}(1|1)$ formed by the Γ-matrices $\mathbf{M}_{odd(L,R)}^{\Gamma}$ under supermatrix multiplication.

Corollary 2.3. *The* Γ*-matrices are additional to triangle supermatrices substructures which form semigroups.*

Let us consider general square antitriangle $(p+q)\times(p+q)$-supermatrices (having even parity in notations of [30]) of the form

$$M_{odd}^{p|q} \stackrel{def}{=} \begin{pmatrix} 0_{p\times p} & \Gamma_{p\times q} \\ \Delta_{q\times p} & B_{q\times q} \end{pmatrix}, \tag{12}$$

where ordinary matrix $B_{q\times q}$ consists of even elements and matrices $\Gamma_{p\times q}$ and $\Delta_{q\times p}$ consist of odd elements [30, 33] (we drop their indices below). The Berezinian of $M_{odd}^{p|q}$ can be obtained from the general formula by reduction and in case of invertible B (which is implied here) is (cf. (8))

$$\operatorname{Ber} M_{odd}^{p|q} = -\frac{\det\left(\Gamma B^{-1} \Delta\right)}{\det B}. \tag{13}$$

A set of supermatrices $\mathbf{M}_{odd}^{p|q}$ form a semigroup $\mathfrak{M}_{odd}^{\Gamma}(p|q)$ of $\Gamma^{p|q}$-matrices, if $\Gamma_{set}\Delta_{set} = 0$, i.e. antidiagonal matrices are orthogonal, and $\Gamma_{set}\mathbf{B} \subset \Gamma_{set}$, $\mathbf{B}\Delta_{set} \subset \Delta_{set}$.

Proof. Consider the product

$$M_{odd_1}^{p|q} M_{odd_2}^{p|q} = \begin{pmatrix} \Gamma_1\Delta_2 & \Gamma_1 B_2 \\ B_1\Delta_2 & B_1 B_2 + \Delta_1\Gamma_2 \end{pmatrix} \tag{14}$$

and observe the condition of vanishing even-even block, which gives $\Gamma_1\Delta_2 = 0$, and other conditions follows obviously. $\square$

From (14) it follows

Corollary 2.4. *Two $\Gamma^{p|q}$-matrices satisfy the band relation $M_1 M_2 = M_1$, if $\Gamma_1 B_2 = \Gamma_1$, $B_1\Delta_2 = \Delta_2$, $B_1 B_2 + \Delta_1\Gamma_2 = B_1$.*

Definition 2.5. We call a set of $\Gamma^{p|q}$-matrices satisfying additional condition $\Delta_{set}\Gamma_{set} = 0$, a set of strong $\Gamma^{p|q}$-matrices.

Strong $\Gamma^{p|q}$-matrices have some extra nice features and all supermatrices considered below are of this class.

Corollary 2.6. *Idempotent strong $\Gamma^{p|q}$-matrices are defined by relations $\Gamma B = \Gamma$, $B\Delta = \Delta$, $B^2 = B$.*

The product of n strong $\Gamma^{p|q}$-matrices M_i has the following form

$$M_1 M_2 \ldots M_n = \begin{pmatrix} 0 & \Gamma_1 A_{n-1} B_n \\ B_1 A_{n-1}\Delta_n & B_1 A_{n-1} B_n \end{pmatrix}, \tag{15}$$

where $A_{n-1} = B_2 B_3 \ldots B_{n-1}$, and its Berezinian is

$$\operatorname{Ber}\left(M_1 M_2 \ldots M_n\right) = -\frac{\det\left(\Gamma_1 A_{n-1}\Delta_n\right)}{\det\left(B_1 A_{n-1} B_n\right)}. \tag{16}$$

3 One-even-parameter supermatrix idempotent semigroups

Here we investigate one-even-parameter subsemigroups of Γ-semigroups and as a particular example for clearness of statements consider $\mathfrak{M}_{odd}(1|1)$, where all characteristic features taking place in general $(p+q)\times(p+q)$ as well can be seen. These formulas will be applied for establishing corresponding superoperator semigroup properties.

A simplest semigroup can be constructed from antidiagonal nilpotent supermatrices of the shape

$$Y_\alpha(t) \overset{def}{=} \begin{pmatrix} 0 & \alpha t \\ \alpha & 0 \end{pmatrix}. \tag{17}$$

where $t \in \Lambda^{1|0}$ is an even parameter of the Grassmann algebra Λ which continuously "numbers" elements $Y_\alpha(t)$ and $\alpha \in \Lambda^{0|1}$ is a fixed odd element of Λ which "numbers" the sets $\mathbf{Y}_\alpha = \bigcup_t Y_\alpha(t)$.

Definition 3.1. The supermatrices $Y_\alpha(t)$ together with a null supermatrix $Z \overset{def}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ form a *continuous null semigroup* $\mathfrak{Z}_\alpha(1|1) = \{\mathbf{Y}_\alpha \cup Z; \cdot\}$ having the null multiplication

$$Y_\alpha(t)\, Y_\alpha(u) = Z. \tag{18}$$

For any fixed $t \in \Lambda^{1|0}$ the set $\{Y_\alpha(t), Z\}$ is a 0-minimal ideal in $\mathfrak{Z}_\alpha(1|1)$.

Remark 2. If we consider, for instance, a one-even-parameter odd-reduced supermatrix $R_\alpha(t) = \begin{pmatrix} 0 & \alpha \\ \alpha & t \end{pmatrix}$, then multiplication of $R_\alpha(t)$ is not closed since $R_\alpha(t)\, R_\alpha(u) = \begin{pmatrix} 0 & \alpha u \\ \alpha t & tu \end{pmatrix} \notin \mathbf{R}_\alpha = \bigcup_t R_\alpha(t)$. Any other possibility except ones considered below also do not give closure of multiplication.

Thus the only nontrivial closed systems of one-even-parameter odd-reduced (antitriangle) $(1+1)\times(1+1)$ supermatrices are $\mathbf{P}_\alpha = \bigcup_t P_\alpha(t)$ where

$$P_\alpha(t) \stackrel{def}{=} \begin{pmatrix} 0 & \alpha t \\ \alpha & 1 \end{pmatrix} \tag{19}$$

and $\mathbf{Q}_\alpha = \bigcup_t Q_\alpha(u)$ where

$$Q_\alpha(u) \stackrel{def}{=} \begin{pmatrix} 0 & \alpha \\ \alpha u & 1 \end{pmatrix}. \tag{20}$$

First, we establish multiplication properties of supermatrices $P_\alpha(t)$ and $Q_\alpha(u)$. Obviously, that they are idempotent.

Sets of idempotent supermatrices $\mathbf{P}_\alpha$ and $\mathbf{Q}_\alpha$ form left zero and right zero semigroups respectively with multiplication

$$P_\alpha(t)\,P_\alpha(u) = P_\alpha(t), \tag{21}$$

$$Q_\alpha(t)\,Q_\alpha(u) = Q_\alpha(u). \tag{22}$$

if and only if $\alpha^2 = 0$.

Proof. It simply follows from supermatrix multiplication law and general previous considerations. □

Corollary 3.2. *The sets $\mathbf{P}_\alpha$ and $\mathbf{Q}_\alpha$ are rectangular bands since*

$$P_\alpha(t)\,P_\alpha(u)\,P_\alpha(t) = P_\alpha(t), \tag{23}$$

$$P_\alpha(u)\,P_\alpha(t)\,P_\alpha(u) = P_\alpha(u) \tag{24}$$

and

$$Q_\alpha(u)\,Q_\alpha(t)\,Q_\alpha(u) = Q_\alpha(u), \tag{25}$$

$$Q_\alpha(t)\,Q_\alpha(u)\,Q_\alpha(t) = Q_\alpha(t) \tag{26}$$

with components $t = t_0 + \operatorname{Ann}\alpha$ and $u = u_0 + \operatorname{Ann}\alpha$ correspondingly.

They are orthogonal in sense of

$$Q_\alpha(t)\,P_\alpha(u) = E_\alpha, \tag{27}$$

where

$$E_\alpha \overset{def}{=} \begin{pmatrix} 0 & \alpha \\ \alpha & 1 \end{pmatrix} \tag{28}$$

is a right "unity" and left "zero" in semigroup $\mathbf{P}_\alpha$, because

$$P_\alpha(t)\,E_\alpha = P_\alpha(t)\,, \quad E_\alpha P_\alpha(t) = E_\alpha \tag{29}$$

and a left "unity" and right "zero" in semigroup $\mathbf{Q}_\alpha$, because

$$Q_\alpha(t)\,E_\alpha = E_\alpha, \quad E_\alpha Q_\alpha(t) = Q_\alpha(t)\,. \tag{30}$$

It is important to note that

$$P_\alpha(t=1) = Q_\alpha(t=1) = E_\alpha, \tag{31}$$

and so $\mathbf{P}_\alpha \cap \mathbf{Q}_\alpha = E_\alpha$. Therefore, almost all properties of $\mathbf{P}_\alpha$ and $\mathbf{Q}_\alpha$ are similar, and we will consider only one of them in what follows. For generalized Green's relations and more detail properties of odd-reduced supermatrices see [13, 27] .

4 Odd-reduced supermatrix operator semigroups

Let us consider a semigroup $\mathcal{P}$ of superoperators $\mathsf{P}(t)$ (see for general theory [2, 3, 5]) represented by the one-even-parameter semigroup $\mathbf{P}_\alpha$ of odd-reduced supermatrices $P_\alpha(t)$ (19) which act on $(1|1)$-dimensional superspace $\mathbb{R}^{1|1}$ as follows $P_\alpha(t)\,\mathtt{X}$, where $\mathtt{X} = \begin{pmatrix} x \\ \varkappa \end{pmatrix} \in \mathbb{R}^{1|1}$, where x - even coordinate, $\varkappa$ - odd coordinate ($\varkappa^2 = 0$) having expansions (1) and (2) respectively (see **Corollary 0.1**). We have a representation $\rho : \mathcal{P} \to \mathbf{P}_\alpha$ with correspondence $\mathsf{P}(t) \to P_\alpha(t)$, but (as is usually made, e.g. [5]) we identify space of superoperators with the space of corresponding matrices (nevertheless, we use here operator notations for convenience).

Definition 4.1. An odd-reduced "dynamical" system on $\mathbb{R}^{1|1}$ is defined by an odd-reduced supermatrix-valued function $\mathsf{P}(\cdot) : \mathbb{R}_+ \to \mathfrak{M}_{odd}(1|1)$ and "time evolution" of the state $\mathtt{X}(0) \in \mathbb{R}^{1|1}$given by the function $\mathtt{X}(t) : \mathbb{R}_+ \to \mathbb{R}^{1|1}$, where

$$\mathtt{X}(t) = \mathsf{P}(t)\,\mathtt{X}(0) \tag{32}$$

and can be called as orbit of $\mathtt{X}(0)$ under $\mathsf{P}(\cdot)$.

Remark 3. In general the definition, the continuity, the functional equation and most of conclusions below hold valid also for $t \in \mathbb{R}^{1|0}$ (as e.g. in [5, p. 9]) including "nilpotent time" directions (see **Corollary 0.1**).

From (21) it follows that

$$\mathsf{P}(t)\,\mathsf{P}(s) = \mathsf{P}(t), \tag{33}$$

and so superoperators $\mathsf{P}(t)$ are idempotent. Also they form a rectangular band, because of

$$\mathsf{P}(t)\,\mathsf{P}(s)\,\mathsf{P}(t) = \mathsf{P}(t), \tag{34}$$

$$\mathsf{P}(s)\,\mathsf{P}(t)\,\mathsf{P}(s) = \mathsf{P}(s). \tag{35}$$

We observe that

$$\mathsf{P}(0) = \begin{pmatrix} 0 & 0 \\ \alpha & 1 \end{pmatrix} \neq \mathsf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{36}$$

as opposite to the standard case [2]. A "generator" $\mathsf{A} = \mathsf{P}'(t)$ is

$$\mathsf{A} = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \tag{37}$$

and so the standard definition of generator [2]

$$\mathsf{A} = \lim_{t\to 0} \frac{\mathsf{P}(t) - \mathsf{P}(0)}{t}. \tag{38}$$

holds and for difference we have the standard relation

$$\mathsf{P}(t) - \mathsf{P}(s) = \mathsf{A} \cdot (t - s). \tag{39}$$

The following properties of the generator A take place

$$\mathsf{P}(t)\,\mathsf{A} = \mathsf{Z}, \tag{40}$$

$$\mathsf{A}\mathsf{P}(t) = \mathsf{A}, \tag{41}$$

where "zero operator" Z is represented by the null supermatrix, $\mathsf{A}^2 = \mathsf{Z}$, and therefore generator A is a nilpotent of second degree.

From (38) it follows that

$$\mathsf{P}(t) = \mathsf{P}(0) + \mathsf{A} \cdot t. \tag{42}$$

Definition 4.2. We call operators which can be presented as a linear supermatrix function of t a t-linear superoperators.

From (42) it follows that $\mathsf{P}(t)$ is a t-linear superoperator.

Proposition 4.3. *Superoperators $\mathsf{P}(t)$* ***cannot be presented*** *as an exponent (as for the standard superoperator semigroups $\mathsf{T}(t) = e^{\mathsf{A}\cdot t}$ [2]).*

Proof. In our case

$$\mathsf{T}(t) = e^{\mathsf{A}\cdot t} = \mathsf{I} + \mathsf{A} \cdot t = \begin{pmatrix} 1 & \alpha t \\ 0 & 1 \end{pmatrix} \notin \mathbf{P}_\alpha. \tag{43}$$

□

Remark 4. Exponential superoperator $\mathsf{T}(t) = e^{\mathsf{A}\cdot t}$ is represented by even-reduced supermatrices $\mathsf{T}(\cdot) : \mathbb{R}_+ \to \mathfrak{M}_{even}(1|1)$ [5], but idempotent superoperator $\mathsf{P}(t)$ is represented by odd-reduced supermatrices $\mathsf{P}(\cdot) : \mathbb{R}_+ \to \mathfrak{M}_{odd}(1|1)$ (see **Definition 1.1**).

Nevertheless, the superoperator $\mathsf{P}(t)$ satisfies the same linear differential equation

$$\mathsf{P}'(t) = \mathsf{A} \cdot \mathsf{P}(t) \tag{44}$$

as the standard exponential superoperator $\mathsf{T}(t)$ (the initial value problem [5])

$$\mathsf{T}'(t) = \mathsf{A} \cdot \mathsf{T}(t) . \tag{45}$$

That leads to the following

Corollary 4.4. *In case initial state does not equal unity* $\mathsf{P}(0) \neq \mathsf{I}$*, there exists an additional class of solutions of the initial value problem (44)-(45) among odd-reduced (antidiagonal) idempotent t-linear (nonexponential) superoperators.*

Let us compare behavior of superoperators $\mathsf{P}(t)$ and $\mathsf{T}(t)$. First of all, their generators coincide

$$\mathsf{P}'(0) = \mathsf{T}'(0) = \mathsf{A}. \tag{46}$$

But powers of $\mathsf{P}(t)$ and $\mathsf{T}(t)$ are different $\mathsf{P}^n(t) = \mathsf{P}(t)$ and $\mathsf{T}^n(t) = \mathsf{T}(nt)$. In their common actions the superoperator which is from the left transfers its properties to the right hand side as follows

$$\mathsf{T}^n(t)\,\mathsf{P}(t) = \mathsf{P}((n+1)\,t) , \tag{47}$$

$$\mathsf{P}^n(t)\,\mathsf{T}(t) = \mathsf{P}(t) . \tag{48}$$

Their commutator is nonvanishing

$$[\mathsf{T}(t)\,\mathsf{P}(s)] = \mathsf{P}'(0)\,t = \mathsf{T}'(0)\,t = \mathsf{A}t, \tag{49}$$

which can be compared with the pure exponential commutator (for our case) $[\mathsf{T}(t)\,\mathsf{T}(u)] = 0$ and idempotent commutator

$$[\mathsf{P}(t)\,\mathsf{P}(s)] = \mathsf{P}'(0)\,(t-s) = \mathsf{A}\,(t-s) . \tag{50}$$

All superoperators $\mathsf{P}(t)$ and $\mathsf{T}(t)$ commute in case of "nilpotent time" and

$$t \in \operatorname{Ann} \alpha. \tag{51}$$

Remark 5. The uniqueness theorem [5, p. 3] holds only for $\mathsf{T}(t)$, because the nonvanishing commutator $[\mathsf{A}, \mathsf{P}(t)] = \mathsf{A} \neq 0$.

Corollary 4.5. *The superoperator* $\mathsf{T}(t)$ *is an inner inverse for* $\mathsf{P}(t)$*, because of*

$$\mathsf{P}(t)\,\mathsf{T}(t)\,\mathsf{P}(t) = \mathsf{P}(t)\,, \tag{52}$$

but it is not an outer inverse, because

$$\mathsf{T}(t)\,\mathsf{P}(t)\,\mathsf{T}(t) = \mathsf{P}(2t)\,. \tag{53}$$

Let us try to find a (possibly noninvertible) operator U which connects exponential and idempotent superoperators $\mathsf{P}(t)$ and $\mathsf{T}(t)$.

The "semi-similarity" relation

$$\mathsf{T}(t)\,\mathsf{U} = \mathsf{U}\mathsf{P}(t) \tag{54}$$

holds if

$$\mathsf{U} = \begin{pmatrix} \sigma\alpha & \sigma \\ 0 & \rho\alpha \end{pmatrix} \tag{55}$$

which is noninvertible triangle and depends from two odd constants, and the "adjoint" relation

$$\mathsf{U}^{*}\mathsf{T}(t) = \mathsf{P}(t)\,\mathsf{U}^{*} \tag{56}$$

holds if

$$\mathsf{U}^{*} = \begin{pmatrix} 0 & \alpha v t \\ \alpha u & v \end{pmatrix} \tag{57}$$

which is also noninvertible antitriangle and depends from two even constants and "time".

Note that U is nilpotent of third degree, since $\mathsf{U}^2 = \sigma\rho\mathsf{A}$, but the "adjoint" superoperator is not nilpotent at all if v is not nilpotent.

Both A and Z behave as zeroes, but $\mathsf{Y}(t)$ (see (17)) is a two-sided zero for $\mathsf{T}(t)$ only, since

$$\mathsf{T}(t)\,\mathsf{Y}(t) = \mathsf{Y}(t)\,\mathsf{T}(t) = \mathsf{Y}(t)\,, \tag{58}$$

but

$$\mathsf{P}(t)\,\mathsf{Y}(t) = \mathsf{Y}(0)\,, \tag{59}$$

$$\mathsf{Y}(t)\,\mathsf{P}(t) = \mathsf{A}t. \tag{60}$$

If we add A and Z to superoperators $\mathsf{P}(t)$, then we obtain an extended odd-reduced noncommutative superoperator semigroup $\mathcal{P}_{odd} = \bigcup \mathsf{P}(t) \bigcup \mathsf{A} \bigcup \mathsf{Z}$ with the following Cayley table (for convenience we add $\mathsf{Y}(t)$ and $\mathsf{T}(t)$ as well)

$1 \setminus 2$	$\mathsf{P}(t)$	$\mathsf{P}(s)$	A	Z	$\mathsf{Y}(t)$	$\mathsf{T}(t)$	$\mathsf{T}(s)$
$\mathsf{P}(t)$	$\mathsf{P}(t)$	$\mathsf{P}(t)$	Z	Z	$\mathsf{P}(t)$	$\mathsf{P}(t)$	$\mathsf{P}(t)$
$\mathsf{P}(s)$	$\mathsf{P}(s)$	$\mathsf{P}(s)$	Z	Z	$\mathsf{P}(s)$	$\mathsf{P}(s)$	$\mathsf{P}(s)$
A	A	A	Z	Z	Z	A	A
Z	Z	Z	Z	Z	Z	Z	Z
$\mathsf{Y}(t)$	$\mathsf{A}t$	$\mathsf{A}s$	Z	Z	Z	$\mathsf{Y}(t)$	$\mathsf{Y}(t)$
$\mathsf{T}(t)$	$\mathsf{P}(2t)$	$\mathsf{P}(t+s)$	A	Z	$\mathsf{Y}(t)$	$\mathsf{T}(2t)$	$\mathsf{T}(t+s)$
$\mathsf{T}(s)$	$\mathsf{P}(t+s)$	$\mathsf{P}(2s)$	A	Z	$\mathsf{Y}(t)$	$\mathsf{T}(t+s)$	$\mathsf{T}(2s)$

(61a)

It is easily seen that associativity in the left upper square holds, and so the table (61a) is actually represents a semigroup of superoperators $\mathcal{P}_{odd}$ (under supermatrix multiplication).

The analogs of the "smoothing operator" $\mathsf{V}(t)$ [5] are

$$\mathsf{V}_P(t) = \int_0^t \mathsf{P}(s)\, ds = \frac{t}{2}\left(\mathsf{P}(t) + \mathsf{P}(0)\right) = \begin{pmatrix} 0 & \alpha \dfrac{t^2}{2} \\ \alpha t & t \end{pmatrix}, \tag{62}$$

$$\mathsf{V}_T(t) = \int_0^t \mathsf{T}(s)\, ds = \frac{t}{2}\left(\mathsf{T}(t) + \mathsf{T}(0)\right) = \begin{pmatrix} t & \alpha \dfrac{t^2}{2} \\ 0 & t \end{pmatrix}. \tag{63}$$

Let us consider the differential sequence of sets of superoperators $\mathsf{P}(t)$

$$\mathsf{S}_n \xrightarrow{\partial} \mathsf{S}_{n-1} \xrightarrow{\partial} \ldots \mathsf{S}_1 \xrightarrow{\partial} \mathsf{S}_0 \xrightarrow{\partial} \mathsf{A} \xrightarrow{\partial} \mathsf{Z}, \tag{64}$$

where $\partial = d/dt$ and

$$\mathsf{S}_n = \bigcup_t \frac{t^n}{n(n-1)\ldots 1} \mathsf{P}\left(\frac{t}{n+1}\right), \tag{65}$$

and by definition

$$\mathsf{S}_0 = \bigcup_t \mathsf{P}(t), \tag{66}$$

$$\mathsf{S}_1 = \bigcup_t \mathsf{V}_P(t). \tag{67}$$

Now we construct an analog of the standard operator semigroup functional equation [2, 5]

$$\mathsf{T}(t+s) = \mathsf{T}(t)\,\mathsf{T}(s). \tag{68}$$

Using the multiplication law (33) and manifest representation (19). for the idempotent superoperators $\mathsf{P}(t)$ we can formulate

Definition 4.6. The odd-reduced idempotent superoperators $\mathsf{P}(t)$ satisfy the following generalized functional equation

$$\mathsf{P}(t+s) = \mathsf{P}(t)\,\mathsf{P}(s) + \mathsf{N}(t,s), \tag{69}$$

where

$$\mathsf{N}(t,s) = \mathsf{P}'(t)\,s.$$

The presence of second term $\mathsf{N}(t,s)$ in the right hand side of the generalized functional equation (69) can be connected with nonautonomous and deterministic properties of systems describing by it [5]. Indeed, from (32) it follows that

$$\begin{aligned}\mathtt{X}(t+s) &= \mathsf{P}(t+s)\,\mathtt{X}(0) = \mathsf{P}(t)\,\mathsf{P}(s)\,\mathtt{X}(0) + \mathsf{P}'(t)\,s\mathtt{X}(0) \\ &= \mathsf{P}(t)\,\mathtt{X}(s) + \mathsf{P}'(t)\,s\mathtt{X}(0) \neq \mathsf{P}(t)\,\mathtt{X}(s)\end{aligned} \tag{70}$$

as opposite to the always implied relation for exponential superoperators $\mathsf{T}(t)$ (translational property [2, 5])

$$\mathsf{T}(t)\,\mathtt{X}(s) = \mathtt{X}(t+s), \tag{71}$$

which follows from (68). Instead of (71), using the band property (33) we obtain

$$\mathsf{P}(t)\,\mathtt{X}(s) = \mathtt{X}(t), \tag{72}$$

which can be called the "moving time" property.

Find a "dynamical system" with time evolution satisfying the "moving time" property (72) instead of the translational property (71).

For "nilpotent time" satisfying (51) the generalized functional equation (69) coincides with the standard functional equation (68), and therefore the idempotent operators $\mathsf{P}(t)$ describe autonomous and deterministic "dynamical" system and satisfy the translational property (71).

Proof. Follows from (51) and (70). □

Find all maps $\mathsf{P}(\cdot) : \mathbb{R}_{+} \to \mathfrak{M}(p|q)$ satisfying the generalized functional equation (69).

We turn to this problem later, and now consider some features of the Cauchy problem for idempotent superoperators.

5 Cauchy problem

Let us consider an action (32) of superoperator $\mathsf{P}(t)$ in superspace $\mathbb{R}^{1|1}$ as $\mathtt{X}(t) = \mathsf{P}(t)\,\mathtt{X}(0)$, where the initial components are $\mathtt{X}(0) = \begin{pmatrix} x_0 \\ \varkappa_0 \end{pmatrix}$. From (32) the evolution of the components has the form

$$\begin{pmatrix} x(t) \\ \varkappa(t) \end{pmatrix} = \begin{pmatrix} \alpha\varkappa_0 t \\ \alpha x_0 + \varkappa_0 \end{pmatrix} \tag{73}$$

which shows that superoperator $\mathsf{P}(t)$ does not lead to time dependence of odd components. Then from (73) we see that

$$\mathtt{X}'(t) = \begin{pmatrix} \alpha\varkappa_0 \\ 0 \end{pmatrix} = const. \tag{74}$$

This is in full agreement with an analog of the Cauchy problem for our case

$$\mathtt{X}'(t) = \mathsf{A}\cdot\mathtt{X}(t). \tag{75}$$

The solution of the Cauchy problem (75) is given by (32), but the idempotent superoperator $\mathsf{P}(t)$ **can not be presented** in exponential form as in the standard case [2], but only in the t-linear form $\mathsf{P}(t) = \mathsf{P}(0) + \mathsf{A} \cdot t \neq e^{\mathsf{A} \cdot t}$, as we have already shown in (42).

This allows us to formulate

Theorem 5.1. *In superspace the solution of the Cauchy initial problem with the same generator* A *is two-fold and is given by two different type of superoperators:*

1. *Exponential superoperator* $\mathsf{T}(t)$ *represented by the even-reduced supermatrices;*

2. *Idempotent t-linear superoperator* $\mathsf{P}(t)$ *represented by the odd-reduced supermatrices.*

For comparison the standard solution of the Cauchy problem (75)

$$\mathtt{X}(t) = \mathsf{T}(t)\,\mathtt{X}(0)$$

in components is

$$\begin{pmatrix} x(t) \\ \varkappa(t) \end{pmatrix} = \begin{pmatrix} x_0 + \alpha\varkappa_0 t \\ \varkappa_0 \end{pmatrix}, \tag{76}$$

which shows that the time evolution of even coordinate is also in nilpotent even direction $\alpha\varkappa_0$ as in (73), but with addition of initial (possibly nonilpotent) x_0, while odd coordinate is (another) constant as well. That leads to

"Even" and "odd" evolutions coincide if even initial coordinate vanishes $x_0 = 0$ or common starting point is pure odd $\mathtt{X}(0) = \begin{pmatrix} 0 \\ \varkappa_0 \end{pmatrix}$.

A very much important formula is the condition of commutativity [2]

$$[\mathsf{A}, \mathsf{P}(t)]\,\mathtt{X}(t) = \mathsf{A}\mathtt{X}(t) = \begin{pmatrix} \alpha\varkappa(t) \\ 0 \end{pmatrix} = 0, \tag{77}$$

which satisfies, when $\alpha \cdot \varkappa(t) = 0$, while in the standard case the commutator $[\mathsf{A}, \mathsf{T}(t)]\,\mathtt{X}(t) = 0$, i.e. vanishes without any additional conditions [2].

6 Superanalog of resolvent for exponential and idempotent superoperators

For resolvents $\mathsf{R}_P(z)$ and $\mathsf{R}_T(z)$ we use analog the standard formula from [2] in the form

$$\mathsf{R}_P(z) = \int_0^\infty e^{-zt}\mathsf{P}(t)\,dt, \tag{78}$$

$$\mathsf{R}_T(z) = \int_0^\infty e^{-zt}\mathsf{T}(t)\,dt. \tag{79}$$

Using the supermatrix representation (19) we obtain

$$\mathsf{R}_P(z) = \begin{pmatrix} 0 & \dfrac{\alpha}{z^2} \\ \dfrac{\alpha}{z} & \dfrac{1}{z} \end{pmatrix}, \tag{80}$$

$$\mathsf{R}_T(z) = \begin{pmatrix} \dfrac{1}{z} & \dfrac{\alpha}{z^2} \\ 0 & \dfrac{1}{z} \end{pmatrix}. \tag{81}$$

We observe, that $\mathsf{R}_T(z)$ satisfies the standard resolvent relation [5]

$$\mathsf{R}_T(z) - \mathsf{R}_T(w) = (w-z)\,\mathsf{R}_T(z)\,\mathsf{R}_T(w), \tag{82}$$

but its analog for $\mathsf{R}_P(z)$

$$\mathsf{R}_P(z) - \mathsf{R}_P(w) = (w-z)\,\mathsf{R}_P(z)\,\mathsf{R}_P(w) + \frac{w-z}{zw^2}\mathsf{A} \tag{83}$$

has additional term proportional to the generator A.

7 Properties of t-linear idempotent (super)operators

Here we consider properties of general t-linear (super)operators of the form

$$\mathsf{K}(t) = \mathsf{K}_0 + \mathsf{K}_1 t, \tag{84}$$

where $\mathsf{K}_0 = \mathsf{K}(0)$ and $\mathsf{K}_1 = \mathsf{K}'(0)$ are constant (super)operators represented by $(n \times n)$ matrices or $(p+q) \times .(p+q)$ supermatrices with t ("time") independent entries. Obviously, that the generator of a general t-linear (super)operator is

$$\mathsf{A}_K = \mathsf{K}'(0) = \mathsf{K}_1. \tag{85}$$

We will find system of equations for K_0 and K_1 for some special cases appeared in above consideration.

If a t-linear (super)operator $\mathsf{K}(t)$ satisfies the band equation (33)

$$\mathsf{K}(t)\,\mathsf{K}(s) = \mathsf{K}(t)\,, \tag{86}$$

then it is idempotent and the constant component (super)operators K_0 and K_1 satisfy the system of equations

$$\mathsf{K}_0^2 = \mathsf{K}_0, \tag{87}$$

$$\mathsf{K}_1^2 = \mathsf{Z}, \tag{88}$$

$$\mathsf{K}_1\mathsf{K}_0 = \mathsf{K}_1, \tag{89}$$

$$\mathsf{K}_0\mathsf{K}_1 = \mathsf{Z}, \tag{90}$$

from which it follows, that K_0 is idempotent, K_1 is nilpotent, and K_1 is right divisor of zero and left zero for K_0.

For non-supersymmetric operators we have

Corollary 7.1. *The components of t-linear operator $\mathsf{K}(t)$ have the following properties: idempotent matrix K_0 is similar to an upper triangular matrix with 1 on the main diagonal and nilpotent matrix K_1 is similar to an upper triangular matrix with 0 on the main diagonal [14, 39].*

Comparing with the previous particular super case (42) we have $\mathsf{K}_0 = \mathsf{P}(0)$ and $\mathsf{K}_1 = \mathsf{A} = \mathsf{P}'(0)$.

Remark 6. In case of $(p+q) \times (p+q)$ supermatrices the triangularization properties of **Corollary 7.1** do not hold valid due to presence divisors of zero and nilpotents among entries (see **Corollary 0.1**), and so the inner structure of the component supermatrices satisfying (87)-(90) can be much different from the standard non-supersymmetric case [14, 39].

Let us consider the structure of t-linear operator $\mathsf{K}(t)$ satisfying the generalized functional equation (69).

If a t-linear (super)operator $\mathsf{K}(t)$ satisfies the generalized functional equation

$$\mathsf{K}(t+s) = \mathsf{K}(t)\,\mathsf{K}(s) + \mathsf{K}'(t)\,s, \tag{91}$$

then its component (super)operators K_0 and K_1 satisfy the system of equations

$$\mathsf{K}_0^2 = \mathsf{K}_0, \tag{92}$$
$$\mathsf{K}_1^2 = \mathsf{Z}, \tag{93}$$
$$\mathsf{K}_1\mathsf{K}_0 = \mathsf{K}_1, \tag{94}$$
$$\mathsf{K}_0\mathsf{K}_1 = \mathsf{Z}, \tag{95}$$

We observe that the systems (87)-(90) and (92)-(95) are fully identical. It is important to observe the connection of the above properties with the differential equation for t-linear (super)operator $\mathsf{K}(t)$

$$\mathsf{K}'(t) = \mathsf{A}_K \cdot \mathsf{K}(t)\,. \tag{96}$$

Using (85) we obtain the equation for components

$$\mathsf{K}_1^2 = \mathsf{Z}, \tag{97}$$
$$\mathsf{K}_1\mathsf{K}_0 = \mathsf{K}_1. \tag{98}$$

That leads to the following

Theorem 7.2. *For any t-linear (super)operator $\mathsf{K}(t) = \mathsf{K}_0 + \mathsf{K}_1 t$ the next statements are equivalent:*

1. *$\mathsf{K}(t)$ is idempotent and satisfies the band equation (86);*
2. *$\mathsf{K}(t)$ satisfies the generalized functional equation (91);*
3. *$\mathsf{K}(t)$ satisfies the differential equation (96) and has idempotent time independent part $\mathsf{K}_0^2 = \mathsf{K}_0$ which is orthogonal to its generator $\mathsf{K}_0\mathsf{A} = \mathsf{Z}$.*

8 General t-power-type idempotent (super)operators

Let us consider idempotent (super)operators which depend from time by power-type function, and so they have the form

$$\mathsf{K}(t) = \sum_{m=0}^{n} \mathsf{K}_m t^m, \tag{99}$$

where K_m are t-independent (super)operators represented by $(n \times n)$ matrices or $(p+q) \times .(p+q)$ supermatrices. This power-type dependence of is very much important for super case, when supermatrix elements take value in Grassmann algebra, and therefore can be nilpotent (see (1)–(2) and **Corollary 0.1**).

We now start from the band property $\mathsf{K}(t)\mathsf{K}(s) = \mathsf{K}(t)$ and then find analogs of the functional equation and differential equation for them. Expanding the band property (86) in component we obtain n-dimensional analog of (87)-(90) as

$$\mathsf{K}_0^2 = \mathsf{K}_0, \tag{100}$$
$$\mathsf{K}_i^2 = \mathsf{Z},\ 1 \leq i \leq n, \tag{101}$$
$$\mathsf{K}_i\mathsf{K}_0 = \mathsf{K}_i,\ 1 \leq i \leq n, \tag{102}$$
$$\mathsf{K}_0\mathsf{K}_i = \mathsf{Z},\ 1 \leq i \leq n, \tag{103}$$
$$\mathsf{K}_i\mathsf{K}_j = \mathsf{Z},\ 1 \leq i, j \leq n, i \neq j. \tag{104}$$

Proposition 8.1. *The n-generalized functional equation for any t-power-type idempotent (super)operators (99) has the form*

$$\mathsf{K}(t+s) = \mathsf{K}(t)\mathsf{K}(s) + \mathsf{N}_n(t,s), \tag{105}$$

where

$$\mathsf{N}_n(t,s) = \sum_{m=1}^{n}\sum_{l=m}^{n} \mathsf{K}_l \frac{l(l-1)\ldots(l-m+1)}{m!} s^m t^{l-m}. \tag{106}$$

Proof. For the difference using the band property (86) we have $\mathsf{N}_n(t,s) = \mathsf{K}(t+s) - \mathsf{K}(t)\mathsf{K}(s) = \mathsf{K}(t+s) - \mathsf{K}(t)$. Then we expand in Taylor series around t and obtain $\mathsf{N}_n(t,s) = \sum\limits_{m=1}^{n} \mathsf{K}^{(m)}(t)\,\dfrac{s^m}{m!}$, where $\mathsf{K}^{(m)}(t)$ denotes n-th derivative which is a finite series for the power-type $\mathsf{K}(t)$ (99). □

The differential equation for idempotent (super)operators coincide with the standard initial value problem only for t-linear operators. In case of the power-type operators (99) we have

Proposition 8.2. *The n-generalized differential equation for any t-power-type idempotent (super)operators (99) has the form*

$$\mathsf{K}'(t) = \mathsf{A}_K \cdot \mathsf{K}(t) + \mathsf{U}_n(t), \tag{107}$$

where

$$\mathsf{U}_n(t) = \begin{cases} 0 & n = 1 \\ \sum\limits_{m=2}^{n} m\mathsf{K}_m t^{m-1} & n \geq 2 \end{cases}. \tag{108}$$

Proof. To find the difference $\mathsf{U}_n(t)$ we use the expansion (99) and the band conditions for components (100)–(104). □

9 Conclusion

In general one-parametric semigroups and corresponding superoperator semigroups represented by antitriangle idempotent supermatrices and their generalization for any dimensions p, q, m, n have many unusual and nontrivial properties [12, 13, 27, 28]. Here we considered only some of them related to their connection with functional and differential equations. It would be interesting to generalize the above constructions to higher dimensions and to study continuity properties of the introduced idempotent superoperators, to consider similar constructions in quasigroups [40] and n-groups [41]. These questions will be investigated elsewhere.

Note by the way, that in 1973 Nambu [43] proposed an interesting generalization of classical Hamiltonian mechanics. It is based on a new notion of brackets, now called *the Nambu bracket*, generalizing the usual Poisson bracket, which is a binary operation on an algebra of classical observables on a phase space, to the multiple operation of higher order $n \geqslant 3$. This new operation is skew-symmetric and satisfies the Leibnitz rule with respect to the usual multiplication of functions as well as the so-called *fundamental identity*, which is a natural generalization of the Jacobi identity.

The canonical Nambu bracket for n classical observables on the phase space R^n with coordinates $x_1, x_2, \ldots, x_n$ is defined by

$$\{f_1, f_2, \ldots, f_n\} = J(f_1, f_2, \ldots, f_n),$$

where the right-hand side stands for the Jacobian of the mapping

$$f = \{f_1, f_2, \ldots, f_n\} : R^n \to R^n.$$

The author of [45] suspects that such structures might clarify many important problems of modern mathematical physics (Yang-Baxter equation, Poisson-Lie groups, quantum groups) for higher-dimensional cases (see also [46]).

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E-mail: Steven.A.Duplij@univer.kharkov.ua
Theory Group, Nuclear physics Laboratory
Kharkov National University, Kharkov 61001, Ukraine