On B-algebras and quasigroups

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Abstract

In this paper we discuss further relations between B-algebras and quasigroups.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([2, 3]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [4, 5] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim introduced in [8] the notion of d-algebras, i.e. algebras satisfying (1) \( xx = 0 \), (5) \( 0x = 0 \), (6) \( xy = 0 \) and \( yx = 0 \) imply \( x = y \), which is another useful generalization of BCK-algebras, and then they investigated several relations between d-algebras and BCK-algebras as well as some other interesting relations between d-algebras and oriented digraphs. Recently, Y. B. Jun, E. H. Roh and H. S. Kim introduced in [6] a new notion, called an BH-algebra, determined by (1), (2) \( x0 = x \) and (6), which is a generalization of BCH/BCI/BCK-algebras. They also defined the notions of ideals and boundedness in BH-algebras, and showed that there is a maximal ideal in bounded BH-algebras. J. Neggers and H. S. Kim introduced in [9] and investigated a class of algebras which is related to several classes of algebras of interest such as BCH/BCI/BCK-algebras and which seems to have rather nice properties without being excessively complicated otherwise. In this paper we discuss further relations between B-algebras and other topics, especially quasigroups. This is a continuation of [9].

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2. Preliminaries

A \textit{B-algebra} is a non-empty set \(X\) with a constant 0 and a binary operation \(\cdot\) (denoted by juxtaposition) satisfying the following axioms:

(1) \(xx = 0\),
(2) \(x0 = x\),
(3) \((xy)z = x(z(0y))\)

for all \(x, y, z \in X\).

\textbf{Example 2.1.} It is easy to see that \(X = \{0, 1, 2, 3, 4, 5\}\) with the multiplication:

\[
\begin{array}{cccccc}
  & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 2 & 1 & 3 & 4 & 5 \\
1 & 1 & 0 & 2 & 4 & 5 & 3 \\
2 & 2 & 1 & 0 & 5 & 3 & 4 \\
3 & 3 & 4 & 5 & 0 & 2 & 1 \\
4 & 4 & 5 & 3 & 1 & 0 & 2 \\
5 & 5 & 3 & 4 & 2 & 1 & 0 \\
\end{array}
\]

is a \textit{B-algebra}.

The following result is proved in [9].

\textbf{Proposition 2.2.} If \((X; \cdot, 0)\) is a \textit{B-algebra}, then

(i) \(x(yz) = (x(0z))y\),
(ii) \((xy)(0y) = x\),
(iii) \(xz = yz\) implies \(x = y\)

for all \(x, y, z \in X\).

A \textit{B-algebra} \((X; \cdot, 0)\) is said to be \textit{0-commutative} if \(x(0y) = y(0x)\) for any \(x, y \in X\).

The \textit{B-algebra} from the above example is not 0-commutative, since we have \(3 \cdot (0 \cdot 4) = 2 \neq 1 = 4 \cdot (0 \cdot 3)\). A simple example of a 0-commutative \textit{B-algebra} is a Boolean group. It is not difficult to see that a \textit{B-algebra} is a Boolean group if it satisfies one from the following identities: \(0x = x\), \(xy = yx\), \((xy)z = x(yz)\).
3. B-algebras and quasigroups

Lemma 3.1. Let \((X;\cdot,0)\) be a \(B\)-algebra. Then for all \(x,y \in X\)

(i) \(xy = 0\) implies \(x = y\),
(ii) \(0x = 0y\) implies \(x = y\),
(iii) \(0(0x) = x\).

Proof. (i) Trivially follows from Proposition 2.2 (iii) and the fact that \(0 = yy\).
(ii) If \(0x = 0y\), then

\[
0 = xx = (xx)0 = x(0(0x)) = x(0(0y)) = (xy)0 = xy,
\]

and hence \(x = y\) by (i).
(iii) For any \(x \in X\), since \(0x = (0x)0 = 0(0(0x))\) by (ii), we have \(x = 0(0x)\).

Theorem 3.2. In any \(B\)-algebra the left cancellation law holds.

Proof. Assume that \(xy = xz\). Then \(0(xy) = 0(xz)\). By Proposition 2.2 (i), we obtain that \((0(0y))x = (0(0z))x\). By Lemma 3.1 (iii) we have \(yx = zx\). Hence \(y = z\) by Proposition 2.2 (iii).

Let \(L_a\) and \(R_a\) be the left and right translation of \(X\) (respectively), i.e. let \(L_a(x) = ax\) and \(R_a(x) = xa\) for all \(x \in X\).

Lemma 3.3. If \((X;\cdot,0)\) is a \(B\)-algebra, then

(i) \(L_0\) is a bijection,
(ii) \(R_0 = R_0^{-1} = id_X\),
(iii) \(L_a\) and \(R_a\) are injective for all \(a \in X\),
(iv) \(L_0^{-1}(0 \cdot x) = L_0^{-1}(L_0(x)) = x\) and

\[
0 \cdot (L_0^{-1}(x)) = L_0(L_0^{-1}(x)) = x \text{ for } x \in X.
\]

Proof. (i) Since \(0(0x) = x\), \(L_0^2 = id_X\) and so \(L_0\) is a bijection.
(ii) is a consequence of (2).
(iii) follows from Proposition 2.2 (iii) and Theorem 3.2.

Lemma 3.4. \(L_a\) and \(R_a\) are surjective for all \(a \in X\).
Proof. Let $c \in X$. Putting $b = (L_{0}^{-1}(c)) \cdot (0 \cdot a)$, we obtain

$$L_{a}(b) = L_{a}(L_{0}^{-1}(c) \cdot (0 \cdot a)) = a \cdot (L_{0}^{-1}(c) \cdot (0 \cdot a)) = (a \cdot a) \cdot (L_{0}^{-1}(c)) = 0 \cdot (L_{0}^{-1}(c)) = c.$$ 

Thus $L_{a}$ is surjective.

Similarly, for $b = c \cdot (L_{0}^{-1}(a))$ we have

$$R_{a}(b) = R_{a}((c \cdot (L_{0}^{-1}(a))) \cdot a = (c \cdot (L_{0}^{-1}(a))) \cdot (0 \cdot (L_{0}^{-1}(a))) = c.$$ 

by Proposition 2.2 (ii). Hence $R_{a}$ is surjective.

\[ \square \]

**Theorem 3.5.** Every $B$-algebra is a quasigroup.

Proof. By Lemma 3.3 (iii) and Lemma 3.4.

\[ \square \]

**Proposition 3.6.** A $B$-algebra $(X; \cdot, 0)$ satisfies the identity $(yx)x = y$ if and only if it is a loop and 0 is its neutral element.

Proof. If a $B$-algebra $(X; \cdot, 0)$ satisfies the identity $(yx)x = y$, then putting $y = 0$ in this identity we have $(0x)x = 0$, which by Lemma 3.1 (i) gives $0x = x$. Hence 0 is the neutral element of $(X; \cdot, 0)$. By Theorem 3.5 $(X; \cdot, 0)$ is a loop.

Conversely, if 0 is the neutral element of a $B$-algebra $(X; \cdot, 0)$, then

$$(yx)x = y(x(0x)) = y(xx) = y0 = y$$

for all $x, y \in X$. This proves the proposition.

\[ \square \]

**Theorem 3.7.** A $B$-algebra satisfies the identity $x(xy) = y$ if and only if it is 0-commutative.

Proof. If a $B$-algebra $(X; \cdot, 0)$ satisfies the identity $x(xy) = y$, then

$$(x(0y))y = x(y(0(0y))) = x(yy) = x0 = x = y(yx) = y(y(0x)) = (y(0x))y.$$ 

Hence we have $(x(0y))y = (y(0x))y$. Then, by the right cancellation law, we obtain $x(0y) = y(0x)$. 

\[ \square \]
The converse statement is proved in [9].

Remark. A $B$-algebra satisfying the identity $x(xy) = y$ is not, in general, a loop. Indeed, if $(G, +, 0)$ is an abelian group, then $G$ with the operation $x \cdot y = x - y$ is an example of a 0-commutative $B$-algebra, which satisfies this identity but it is not a loop.

References


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