Quasigroups and Related Systems **8** (2001), 87 – 96

Representations of positional algebras

Valentin S. Trokhimenko

Abstract

In the paper we consider representations of positional algebras in the sense of V. D. Belousov [1] by partial multiplace functions. We prove that any such representation has a special construction.

On the sets of multiplace functions of several arities one often considers the binary operations of superpositions, which are called positional superpositions. Such operations are used in the theory of functional equations and in the theory of n-ary quasigroups [1]. Thus the study of positional algebras and their representations by multiplace functions has the particular interest. For descriptions of such representations we use the generalization of the method of determining pairs, which B. M. Schein considered for semigroups of transformations [2].

A *positional algebra* is a partial algebra of the form

$$\mathfrak{G} = (G; \overset{1}{+}, \overset{2}{+}, \ldots, \overset{n}{+}, \ldots),$$

where $\overset{1}{+}, \overset{2}{+}, \ldots, \overset{n}{+}, \ldots$ are partial binary operations on a set G satisfying the Axioms $\mathbf{A_1} - \mathbf{A_5}$.

$\mathbf{A_1}$ $\{x\} \overset{1}{+} \{y\} \neq \emptyset$ for all $x, y \in G$.

$\mathbf{A_2}$ For every $x \in G$ there exists $n \in \mathbb{N}$ such that

$$i \leqslant n \Longleftrightarrow \{x\} \overset{n}{+} \{y\} \neq \emptyset$$

for all $i \in \mathbb{N}$ and $y \in G$.

2000 Mathematics Subject Classification: 20N15, 08N05
Keywords: vector valued function
The main result of this paper was announced on the Third International Algebraic Conference in Ukraine, Sumy, 2 – 8 July 2001.

Let α be the binary relation on $G \times \mathbb{N}$ such that $(x,n) \in \alpha$ if and only if

$$(\forall i \in \mathbb{N})\,(\forall y \in G)\ \left(i \leqslant n \Longleftrightarrow \{x\} \overset{i}{+} \{y\} \neq \emptyset \right). \tag{1}$$

Proposition 1. *The relation α is single valued.*

Proof. Let $(x,n) \in \alpha$ and $(x,m) \in \alpha$ for some $x \in G$, $n,m \in \mathbb{N}$. Assume that $n \neq m$, then we suppose, without restricting generality, that $n < m$. According to (1) we have

$$(\forall i \in \mathbb{N})\,(\forall y \in G)\ \left(i \leqslant n \Longleftrightarrow \{x\} \overset{i}{+} \{y\} \neq \emptyset \right), \tag{2}$$

$$(\forall i \in \mathbb{N})\,(\forall y \in G)\ \left(i \leqslant m \Longleftrightarrow \{x\} \overset{i}{+} \{y\} \neq \emptyset \right). \tag{3}$$

As it is not difficult to see (2) is equivalent to

$$(\forall i \in \mathbb{N})\,(\forall y \in G)\ \left(i > n \Longleftrightarrow \{x\} \overset{i}{+} \{y\} = \emptyset \right). \tag{4}$$

From (3) it follows $\{x\} \overset{m}{+} \{y\} \neq \emptyset$ for all $y \in G$. Since $m > n$, then, from (4), for each $y \in G$ we obtain $\{x\} \overset{m}{+} \{y\} = \emptyset$. The obtained contradiction proves that $n = m$. □

Further by the arity of an element $x \in G$ we mean the value $\alpha(x)$ and we denote it by $|x|$. Thus, $|x| = \alpha(x)$. From the definition of α it follows that for $x, y \in G$ and $i \in \mathbb{N}$ the result of $x \overset{i}{+} y$ is defined if and only if $i \leqslant |x|$.

A$_3$ For all $x, y \in G$, $i \in \mathbb{N}$, if $i \leqslant |x|$, then

$$|x \overset{i}{+} y| = |x| + |y| - 1.$$

A$_4$ For $x, y, z \in G$ and $n, m \in \mathbb{N}$ such that $n \leqslant |x|$, $m \leqslant |y|$, we have

$$x \overset{n}{+} (y \overset{m}{+} z) = (x \overset{n}{+} y) \overset{n+m-1}{+} z\,.$$

A$_5$ For $x, y, z \in G$ and $n, m \in \mathbb{N}$ such that $m < n \leqslant |x|$, holds

$$(x \overset{n}{+} y) \overset{m}{+} z = (x \overset{m}{+} z) \overset{n+|z|-1}{+} y.$$

Let $\mathcal{T}_n(A) = \mathcal{T}(A^n, A)$ be the set of all full multiplace functions (i.e. operations) on a set A. For all $f, g \in \mathcal{T}(A) = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n(A)$ such that $|f| = n$, $|g| = m$ we define a *positional superposition* $\overset{i}{+}$ $(i \in \mathbb{N})$ putting:

$$(f \overset{i}{+} g)(a_1^{n+m-1}) = f(a_1^{i-1}, g(a_i^{i+m-1}), a_{i+m}^{n+m-1}) \tag{5}$$

where $a_1, \dots, a_{n+m-1} \in A$ and a_i^j denotes the sequence $a_i, a_{i+1}, \dots, a_j$ if $i \leqslant j$, and the empty symbol if $i > j$.

An algebra $(\mathcal{T}(A), \overset{1}{+}, \overset{2}{+}, \dots)$ is called a *symmetrical positional algebra of operations*, its subalgebras – *positional algebras of operations*.

Let $\mathcal{F}_n(A)$ be the set of all partial n-place transformations on A and let Θ_n be an empty mapping from A^n into A. On the set

$$\mathcal{F}(A) = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n(A) \cup \{\Theta_n\}$$

we consider partial binary operations $\overset{i}{+}$ $(i \in \mathbb{N})$ defined for $f \in \mathcal{F}_n(A)$, $g \in \mathcal{F}_m(A)$ and $a_1, \dots, a_{n+m-1}, b, c \in A$ by the formula

$$(a_1^{n+m-1}, c) \in f \overset{i}{+} g \Longleftrightarrow (\exists b)\Big((a_i^{i+m-1}, b) \in g \wedge (a_1^{i-1} b a_{i+m}^{n+m-1}, c) \in f\Big). \tag{6}$$

If $f \overset{i}{+} g$ is an empty transformation, then we put $f \overset{i}{+} g = \Theta_{n+m-1}$.

We assume that $\Theta_n \overset{i}{+} \Theta_m = \Theta_{n+m-1}$ for all $n, m \in \mathbb{N}$ and $i \leqslant n$. We assume also that $f \overset{i}{+} \Theta_m = \Theta_n \overset{i}{+} g = \Theta_{n+m-1}$. It is clear that the system $(\mathcal{F}(A), \overset{1}{+}, \overset{2}{+}, \dots)$ is a positional algebra. This algebra is called a *symmetrical positional algebra of multiplace functions*, its subalgebras – *positional algebras of multiplace functions*.

Let $\mathfrak{G}_1 = (G_1, \overset{1}{+}, \overset{2}{+}, \dots)$ and $\mathfrak{G}_2 = (G_2, \overset{1}{+}, \overset{2}{+}, \dots)$ be two positional algebras. The mapping $P : G_1 \to G_2$ such that

1. $|g| = |P(g)|$ for each $g \in G_1$,

2. $P(g_1 \overset{i}{+} g_2) = P(g_1) \overset{i}{+} P(g_2)$ for all $g_1, g_2 \in G_1$ and $i \leqslant |g_1|$,

We put $n = |f|$ if and only if $f \in \mathcal{T}(A^n, A)$.

Analougously we define the operations $\overset{i}{+}$ on the set of all relations.

is called a *strong homomorphism of* $\mathfrak{G}_1$ *into* $\mathfrak{G}_2$. A strong homomorphism of a positional algebra $\mathfrak{G}$ into a symmetrical positional algebra of operations (multiplace functions) is called a *representation of* $\mathfrak{G}$ *by operations* (or *by multiplace functions*). A representation which is an isomorphism is called *faithful* (or *isomorphic*).

Let $\mathfrak{G} = (G, \overset{1}{+}, \overset{2}{+}, \ldots)$ be a positional algebra, e – an element not belonging to G, $G^* = G \cup \{e\}$. We put $|e| = 1$, $e \overset{1}{+} e = e$, $e \overset{1}{+} g = g$, $g \overset{i}{+} e = g$ for every $g \in G$ and $i \leqslant |g|$. It is not difficult to see that $\mathfrak{G}^* = (G^*, \overset{1}{+}, \overset{2}{+}, \ldots)$ is a positional algebra.

The following theorem was proved by V. D. Belousov (cf. [1]).

Theorem 1. *Every positional algebra is isomorphic to some positional algebra of operations.*

Corollary 1. *Every positional algebra is isomorphic to some positional algebra of multiplace functions and to some positional algebra of relations.*

Let $\mathfrak{G} = (G, \overset{1}{+}, \overset{2}{+}, \ldots)$ be a positional algebra, A – a non-empty set, $\Omega(A)$ – the set of all words on it. If $\omega_1, \ldots, \omega_n \in \Omega(A)$, then the word $\omega = \omega_1\omega_2 \ldots \omega_n$ is the sum of words $\omega_1, \omega_2, \ldots, \omega_n$. By $l(\omega)$ we denote the length of $\omega \in \Omega(A)$. For each word $\omega \in \Omega(A)$ of length $l(\omega) = n$ by ε^ω we denote some equivalence relation on $G_n = \{g \in G \,|\, |g| = n\}$, which corresponds to ω. So, $\varepsilon^\omega \subset G_{l(\omega)} \times G_{l(\omega)}$.

Let $\mathcal{E} = (\varepsilon^\omega)^{\omega \in B}$, where $B \subset \Omega(A)$, be a family of equivalence relations.

Definition 1. A family $\mathcal{E}$ is called *permissible* for positional algebra $\mathfrak{G}$, if for all $g, x_i, y_i \in G$, $i = 1, \ldots, n$ and $n = |g|$

$$x_1 \equiv y_1(\varepsilon^{\omega_1}) \wedge \ldots \wedge x_n \equiv y_n(\varepsilon^{\omega_n}) \implies g \underset{n}{\overset{1}{+}} x_n^1 \equiv g \underset{n}{\overset{1}{+}} y_n^1(\varepsilon^{\omega_1 \cdots \omega_n}),$$

where $g \underset{n}{\overset{1}{+}} x_n^1$ denotes $(\ldots((g \overset{n}{+} x_n) \overset{n-1}{+} x_{n-1}) \ldots) \overset{1}{+} x_1$.

Definition 2. A family $\mathcal{W} = (W^\omega)^{\omega \in B}$, where W^ω is a subset of $G_{l(\omega)}$, is an *l-ideal*, if for all $g, x_k \in G$, $k \neq i$, $k, i = 1, \ldots, n$, where $|g| = n$ and $l(\omega_1) = l(\omega) + \sum_{k=1, k\neq i}^{n} |x_k|$ the following implication is valid:

$$h \in W^\omega \implies ((g \underset{n}{\overset{i+1}{+}} x_n^{i+1}) \overset{i}{+} h) \underset{i-1}{\overset{1}{+}} x_{i-1}^1 \in W^{\omega_1}.$$

Definition 3. By a *determining pair* of a positional algebra $\mathfrak{G}$ we mean an ordered pair $(\mathcal{E},\mathcal{W})$, where $\mathcal{E}$ is a family of equivalence relations permissible for a positional algebra $\mathfrak{G}^*$, $\mathcal{W}$ is an l-ideal of a family of subsets W^ω such that W^ω is either empty or an ε^ω-class.

By $(H_a^\omega)_{a\in I_\omega}$, where $W^\omega\neq H_a^\omega$ for all $a\in I_\omega$, we denote the family of all ε^ω-classes (uniquely indexed by elements of some fixed set I_ω) such that the following implication, where $n=|g|$, holds

$$\left.\begin{array}{l}(\ldots((g\overset{n}{+}H_{a_n}^{\omega_n})\overset{n-1}{+}H_{a_{n-1}}^{\omega_{n-1}})\ldots)\overset{1}{+}H_{a_1}^{\omega_1}\subset H_b^{\omega_1\cdots\omega_n},\\ (\ldots((g\overset{n}{+}H_{a_n}^{\omega'_n})\overset{n-1}{+}H_{a_{n-1}}^{\omega'_{n-1}})\ldots)\overset{1}{+}H_{a_1}^{\omega'_1}\subset H_c^{\omega'_1\cdots\omega'_n}\end{array}\right\}\Longrightarrow b=c. \tag{7}$$

Obviously for $I_\omega\cap I_{\omega'}=\emptyset$ the condition (7) is satisfied.

For every $g\in G$, $|g|=n$, we define the partial n-place function $P_{(\mathcal{E},\mathcal{W})}(g)$, where $(\mathcal{E},\mathcal{W})$ is a determining pair of a positional algebra $\mathfrak{G}$, putting

$$(a_1^n,b)\in P_{(\mathcal{E},\mathcal{W})}(g)\Longleftrightarrow(\ldots((g\overset{n}{+}H_{a_n}^{\omega_n})\overset{n-1}{+}H_{a_{n-1}}^{\omega_{n-1}})\ldots)\overset{1}{+}H_{a_1}^{\omega_1}\subset H_b^{\omega_1\cdots\omega_n} \tag{8}$$

for some $\omega_1,\ldots,\omega_n\in\Omega(A)$.

Theorem 2. *If* $(\mathcal{E},\mathcal{W})$ *is a determining pair of a positional algebra*

$$\mathfrak{G}=(G;\overset{1}{+},\overset{2}{+},\ldots,\overset{n}{+},\ldots),$$

then the mapping $P_{(\mathcal{E},\mathcal{W})}:g\longmapsto P_{(\mathcal{E},\mathcal{W})}(g)$, *where* $g\in G$, *is its representation by multiplace functions.*

Proof. Let g_1, g_2 be arbitrary elements of G such that $|g_1|=n$, $|g_2|=m$. Assume that $(a_1^{n+m-1},c)\in P_{(\mathcal{E},\mathcal{W})}(g_1\overset{i}{+}g_2)$ for $i\leqslant n$. Then, by (8), we obtain

$$(\ldots((g_1\overset{i}{+}g_2)\overset{n+m-1}{+}H_{a_{n+m-1}}^{\omega_{n+m-1}})\overset{n-1}{+}\ldots)\overset{1}{+}H_{a_1}^{\omega_1}\subset H_c^{\omega_1\cdots\omega_{n+m-1}}.$$

If $x_i\in H_{a_i}^{\omega_i}$, $i=1,\ldots,n+m-1$, then

$$(g_1\overset{i}{+}g_2)\underset{n+m-1}{\overset{1}{+}}x_{n+m-1}^1\in H_c^{\omega_1\ldots\omega_{n+m-1}},$$

which, by the axioms of a positional algebra, gives

$$(g_1\overset{i}{+}g_2)\underset{n+m-1}{\overset{1}{+}}x_{n+m-1}^1=\Big(\Big(g_1\underset{n}{\overset{i+1}{+}}x_{n+m-1}^{i+m}\Big)\overset{i}{+}\Big(g_2\underset{m}{\overset{1}{+}}x_{i+m-1}^i\Big)\Big)\underset{i-1}{\overset{1}{+}}x_{i-1}^1\,.$$

Therefore

$$\left(\left(g_1 \overset{i+1}{\underset{n}{+}} x_{n+m-1}^{i+m}\right) \overset{i}{+} \left(g_2 \overset{1}{\underset{m}{+}} x_{i+m-1}^{i}\right)\right) \underset{i-1}{\overset{1}{+}} x_{i-1}^{1} \in H_c^{\omega_1\ldots\omega_{n+m-1}} . \tag{9}$$

Hence

$$\left(\left(g_1 \overset{i+1}{\underset{n}{+}} x_{n+m-1}^{i+m}\right) \overset{i}{+} \left(g_2 \overset{1}{\underset{m}{+}} x_{i+m-1}^{i}\right)\right) \underset{i-1}{\overset{1}{+}} x_{i-1}^{1} \notin W^{\omega_1\ldots\omega_{n+m-1}} .$$

Since the family $\mathcal{W}$ is an l-ideal, then from the last condition follows that $g_2 \overset{1}{\underset{m}{+}} x_{i+m-1}^{i} \notin W^{\omega_i\ldots\omega_{i+m-1}}$.

Suppose that

$$g_2 \overset{1}{\underset{m}{+}} x_{i+m-1}^{i} \in H_b^{\omega_i\ldots\omega_{i+m-1}} . \tag{10}$$

This, by the permissibility of $\mathcal{E}$, gives

$$\left(\ldots\left(g_2 \overset{m}{+} H_{a_{i+m-1}}^{\omega_{i+m-1}}\right) \overset{m-1}{+} \ldots\right) \overset{1}{+} H_{a_i}^{\omega_i} \subset H_b^{\omega_i\ldots\omega_{i+m-1}} ,$$

which implies

$$(a_i^{i+m-1}, b) \in P_{(\mathcal{E},\mathcal{W})}(g_2) . \tag{11}$$

Thus (9) together with (10) proves that

$$g_1 \overset{n}{+} H_{a_{n+m-1}}^{\omega_{n+m-1}} \overset{n-1}{+} \cdots \overset{i+1}{+} H_{a_{i+m}}^{\omega_{i+m}} \overset{i}{+} H_b^{\omega_i\ldots\omega_{i+m-1}} \overset{i-1}{+} H_{a_{i-1}}^{\omega_{i-1}} \overset{i-2}{+}\overset{i-2}{+} \cdots \overset{1}{+} H_{a_1}^{\omega_1}$$

is contained in $H_c^{\omega_1\ldots\omega_{n+m-1}}$. Hence

$$(a_1^{i-1} b\, a_{i+m}^{n+m-1}, c) \in P_{(\mathcal{E},\mathcal{W})}(g_1) . \tag{12}$$

Now, comparing (11) with (12) we obtain

$$(a_1^{n+m-1}, c) \in P_{(\mathcal{E},\mathcal{W})}(g_1) \overset{i}{+} P_{(\mathcal{E},\mathcal{W})}(g_2) .$$

So, we have proved that

$$P_{(\mathcal{E},\mathcal{W})}(g_1 \overset{i}{+} g_2) \subset P_{(\mathcal{E},\mathcal{W})}(g_1) \overset{i}{+} P_{(\mathcal{E},\mathcal{W})}(g_2) .$$

The converse inclusion can be proved in the similar way. Thus

$$P_{(\mathcal{E},\mathcal{W})}(g_1 \overset{i}{+} g_2) = P_{(\mathcal{E},\mathcal{W})}(g_1) \overset{i}{+} P_{(\mathcal{E},\mathcal{W})}(g_2)$$

for all $g_1, g_2 \in G$ and $i \leqslant n$. Hence $P_{(\mathcal{E},\mathcal{W})}$ is a representation of the positional algebra $\mathfrak{G}$.

The fact that $P_{(\mathcal{E},\mathcal{W})}(g)$ is a function is a consequence of the permissibility of $\mathcal{E}$. □

We say that a *representation* P of a given positional algebra *is generated by a determining pair* if there exists a determining pair $(\mathcal{E},\mathcal{W})$ of this algebra such that $P = P_{(\mathcal{E},\mathcal{W})}$.

Theorem 3. *Every representation of a positional algebra by multiplace functions is generated by some of its determining pair.*

Proof. Let P be a representation of a positional algebra $\mathfrak{G} = (G; \overset{1}{+}, \overset{2}{+}, \ldots)$ by multiplace functions on a set A. For each vector $a_1^n = (a_1, \ldots, a_n) \in A^n$ we define the binary relation $\varepsilon^{a_1^n} \subset G_n \times G_n$ and the subset $W^{a_1^n} \subset G_n$ putting (for $g, g_1, g_2 \in G_n$)

$$
\begin{aligned}
g_1 \equiv g_2(\varepsilon^{a_1^n}) &\Longleftrightarrow P(g_1)\langle a_1^n\rangle = P(g_2)\langle a_1^n\rangle\,,\\
g \in W^{a_1^n} &\Longleftrightarrow P(g)\langle a_1^n\rangle = \emptyset.
\end{aligned}
$$

Moreover, let

$$
\begin{aligned}
I(G) &= \{n \in \mathbb{N} \mid (\exists g \in G)\, n = |g|\}\,,\\
\mathcal{E}_P &= \{\varepsilon^{a_1^n} \mid a_1^n \in A^n, n \in I(G)\}\,,\\
\mathcal{W}_P &= \{W^{a_1^n} \mid a_1^n \in A^n, n \in I(G)\}\,.
\end{aligned}
$$

We prove that $(\mathcal{E}_P, \mathcal{W}_P)$ is a determining pair of the positional algebra $\mathfrak{G}$ such that $P = P_{(\mathcal{E}_P,\mathcal{W}_P)}$.

It is clear that $\varepsilon^{a_1^n}$ is an equivalence relation on G_n. To prove that $\mathcal{E}_P$ is permissible for the positional algebra $\mathfrak{G}^*$, let $g \in G$, $|g| = n$ and

$$
x_1 \equiv y_1(\varepsilon^{a_1^{m_1}}),\ x_2 \equiv y_2(\varepsilon^{b_1^{m_2}})\,, \ldots,\, x_n \equiv y_n(\varepsilon^{c_1^{m_n}})\,.
$$

This, by the definition, implies

$$
\begin{aligned}
P(x_1)\langle a_1^{m_1}\rangle &= P(y_1)\langle a_1^{m_1}\rangle,\\
P(x_2)\langle b_1^{m_2}\rangle &= P(y_2)\langle b_1^{m_2}\rangle,\\
&\ldots\ldots\ldots\ldots\ldots\\
P(x_n)\langle c_1^{m_n}\rangle &= P(y_n)\langle c_1^{m_n}\rangle,
\end{aligned}
$$

which gives

$$P(g)\Big(P(x_1)\langle a_1^{m_1}\rangle, P(x_2)\langle b_1^{m_2}\rangle,\ldots,P(x_n)\langle c_1^{m_n}\rangle\Big)$$
$$= P(g)\Big(P(y_1)\langle a_1^{m_1}\rangle, P(y_2)\langle b_1^{m_2}\rangle,\ldots,P(y_n)\langle c_1^{m_n}\rangle\Big).$$

That is equivalent to

$$\Big(P(g)\overset{n}{+}P(x_n)\overset{n-1}{+}\cdots\overset{1}{+}P(x_1)\Big)\langle a_1^{m_1}b_1^{m_2}\ldots c_1^{m_n}\rangle$$
$$=\Big(P(g)\overset{n}{+}P(y_n)\overset{n-1}{+}\cdots\overset{1}{+}P(y_1)\Big)\langle a_1^{m_1}b_1^{m_2}\ldots c_1^{m_n}\rangle.$$

Since P is a homomorphism, we have

$$P(g\underset{n}{\overset{1}{+}}x_n^1)\langle a_1^{m_1}\ldots c_1^{m_n}\rangle = P(g\underset{n}{\overset{1}{+}}y_n^1)\langle a_1^{m_1}\ldots c_1^{m_n}\rangle,$$

i. e.

$$g\underset{n}{\overset{1}{+}}x_n^1\equiv g\underset{n}{\overset{1}{+}}y_n^1(\varepsilon^{a_1^{m_1}\ldots c_1^{m_n}}).$$

So, $\mathcal{E}_P$ is permissible for the positional algebra $\mathfrak{G}^*$.

To prove that $\mathcal{W}_P$ is an l-ideal, consider $g,x_i\in G$, $|g|=n$, $|x_i|=m_i$, $i=1,\ldots,n$. By the definition $|g\underset{n}{\overset{1}{+}}x_n^1|=\sum\limits_{i=1}^{n}m_i=m$.

If $g\underset{n}{\overset{1}{+}}x_n^1\notin W^{a_1^{n_1}}$, then $P(g\underset{n}{\overset{1}{+}}x_n^1)\langle a_1^m\rangle\neq\emptyset$, whence

$$\Big(P(g)\overset{n}{+}P(x_n)\overset{n-1}{+}\cdots\overset{i}{+}P(x_i)\overset{i-1}{+}\cdots\overset{1}{+}P(x_1)\Big)\langle a_1^m\rangle\neq\emptyset,$$

therefore

$$P(g)\Big(P(x_1)\langle a_1^{m_1}\rangle,\ldots,P(x_i)\langle a_{s_{i-1}+1}^{s_{i-1}+m_i}\rangle,\ldots,P(x_n)\langle a_{s_{n-1}}^{m}\rangle\Big)\neq\emptyset,$$

where $s_{i-1}=\sum\limits_{k=1}^{i-1}m_k$. Hence $P(x_i)\langle a_{s_{i-1}+1}^{s_{i-1}+m_i}\rangle\neq\emptyset$, i.e. $x_i\notin W^{\omega_i^{i+m_i}}$ for each $i=1,\ldots,n$. So, $\mathcal{W}_P$ is an l-ideal and, in the consequence, $(\mathcal{E}_P,\mathcal{W}_P)$ is a determining pair of the positional algebra $\mathfrak{G}$.

To prove that $P=P_{(\mathcal{E}_P,\mathcal{W}_P)}$, let $a_1^n\in A^n$, $b\in A$ and

$$H_b^{a_1^n}=\{g\in G_n\,|\,P(g)\langle a_1^n\rangle=\{b\}\},$$

i. e.

$$g\in H_b^{a_1^n}\Longleftrightarrow(a_1^n,b)\in P(g)\,.$$

It is clear that $e\in H_a^a$ for any $a\in A$. Also it is not difficult to see that $\{H_b^{a_1^n}\mid b\in A\}$ is the set of $\varepsilon^{a_1^n}$-classes, which are disjoint with $W^{a_1^n}$.

The set of all such classes satisfies (7). Indeed, if

$$g\overset{n}{+}H_{b_n}^{c_1^{m_n}}\overset{n-1}{+}\cdots\overset{1}{+}H_{b_1}^{a_1^{m_1}}\subset H_c^{a_1^{m_1}\ldots c_1^{m_n}}$$

and

$$g\overset{n}{+}H_{b_n}^{{c'}_1^{m'_n}}\overset{n-1}{+}\cdots\overset{1}{+}H_{b_1}^{{a'}_1^{m'_1}}\subset H_d^{{a'}_1^{m'_1}\ldots {c'}_1^{m'_n}},$$

then

$$g\underset{n}{\overset{1}{+}}x_n^1\in H_c^{a_1^{m_1}\ldots c_1^{m_n}}\quad\text{and}\quad g\underset{n}{\overset{1}{+}}y_n^1\in H_c^{{a'}_1^{m'_1}\ldots {c'}_1^{m'_n}},$$

where $x_i\in H_{b_i}^{d_1^{m_i}}$, $y_i\in H_{b_i}^{{d'}_1^{m'_i}}$, $i=1,\ldots,n$, $d\in\{a,\ldots,c\}$. Since P is a homomorphism

$$\begin{aligned}c&=P(g)\Big(P(x_1)(a_1^{m_1}),\ldots,P(x_n)(c_1^{m_n})\Big)=P(g)(b_1,\ldots,b_n)\\&=P(g)\Big(P(y_1)({a'}_1^{m'_1}),\ldots,P(y_n)({c'}_1^{m'_n})\Big)=d,\end{aligned}$$

i.e. $c=d$. So, the condition (7) is satisfied.

Now let $(b_1^n,c)\in P(g)$, where $|g|=n$. Therefore $g\in H_c^{b_1^n}$. But $e\in H_{b_i}^{b_i}$, $i=1,\ldots,n$, and $g=g\underset{n}{\overset{1}{+}}e$ imply

$$g\overset{n}{+}H_{b_n}^{b_n}\overset{n-1}{+}H_{b_{n-1}}^{b_{n-1}}\overset{n-1}{+}\cdots\overset{1}{+}H_{b_1}^{b_1}\subset H_c^{b_1^n},$$

which gives

$$(b_1^n,c)\in P_{(\mathcal{E}_P,\mathcal{W}_P)}(g). \tag{13}$$

Conversely, if (13) holds, then for some $a_1^{m_1},\ldots,c_1^{m_n}$ we have

$$g\overset{n}{+}H_{b_n}^{c_1^{m_n}}\overset{n-1}{+}\cdots\overset{1}{+}H_{b_1}^{a_1^{m_1}}\subset H_c^{a_1^{m_1}\ldots c_1^{m_n}}.$$

This means that $g\underset{n}{\overset{1}{+}}x_n^1\in H_c^{a_1^{m_1}\ldots c_1^{m_n}}$ for $x_1\in H_{b_1}^{a_1^{m_1}}$, …, $x_n\in H_{b_n}^{c_1^{m_n}}$. Thus $P(x_1)(a_1^{m_1})=b_1$,…, $P(x_n)(c_1^{m_n})=b_n$ and $(a_1^{m_1}\ldots c_1^{m_n},c)\in P(g\underset{n}{\overset{1}{+}}x_n^1)$.

But P is a homomorphism, hence

$$(a_1^{m_1}\ldots c_1^{m_n},c)\in P(g)\overset{n}{+}P(x_n)\overset{n-1}{+}\cdots\overset{1}{+}P(x_1).$$

Therefore

$$c = P(g)\Big(P(x_1)(a_1^{m_1}),\ldots,P(x_n)(c_1^{m_n})\Big) = P(g)(b_1,\ldots,b_n) = P(g)(b_1^n),$$

whence $(b_1^n, c) \in P(g)$.

So, $P(g) = P_{(\mathcal{E}_P,\mathcal{W}_P)}(g)$ for all $g \in G$, which proves $P = P_{(\mathcal{E}_P,\mathcal{W}_P)}$. $\square$

Problems

1. *Describe all representations of positional algebras by n-ary relations.*

2. *Find an abstract characterization of symmetrical positional algebras of operations (multiplace functions, n-ary relations).*

3. *Find an abstract characteristic of the class of all positional algebras of multiplace functions ordered by the relation of the set-theoretical inclusion.*
(For n-ary relations this problem was solved by F. M. Sokhatsky in [3].)

4. *Describe all automorphisms of the symmetrical positional algebra of operations (multiplace functions, n-ary relations).*

References

[1] **V. D. Belousov**: *n–ary quasigroups*, (Russian), Ştiinţa, Kishinev 1972.

[2] **B. M. Schein**: *Lectures on transformation semigroups*, (Russian), Special course. Izdat. Saratov. Univ., Saratov 1970. (English translation: *Lectures on semigroups of transformations*, Amer. Math. Soc. Translations, II ser. **113** (1979), 123 – 181.)

[3] **F. M. Sokhatsky**: *On positional algebras*, (Russian), Mat. Issled. **71**, (1983), 104 – 117.

Department of Mathematics
Pedagogical University
21100 Vinnitsa
Ukraine
e-mail: vtrokhim@sovauma.com

Received September 29, 2001