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The topological quasigroups with multiple identities

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Abstract

In this article we describe the topological quasigroups with (n, m)-identities, which are obtained by using isotopies of topological groups. Such quasigroups are called the (n, m)-homogeneous quasigroups. Our main goal is to extend some affirmations of the theory of topological groups on the class of topological (n, m)-homogeneous quasigroups.

1. General notes

A non-empty set G is said to be a *groupoid* relative to a binary operation denoted by $\cdot$ or by juxtaposition, if for every ordered pair a, b of elements of G, is defined a unique element $ab \in G$.

If the groupoid G is a topological space and the multiplication operation $(a, b) \to a \cdot b$ is continuous, then G is called a *topological groupoid.*

A groupoid G is called a *groupoid with division*, if for every $a, b \in G$ the equations $ax = b$ and $ya = b$ have solutions, not necessarily unique.

A groupoid G is called *reducible* or *cancellative*, if for each equality $xy = uv$ the equality $x = u$ is equivalent to the equality $y = v$.

A groupoid G is called a *primitive groupoid with the divisions*, if there exist two binary operations $l : G \times G \to G$, $r : G \times G \to G$ such that $l\,(a, b) \cdot a = b$, $a \cdot r\,(a, b) = b$ for all $a, b \in G$. Thus a primitive groupoid with divisions is a universal algebra with three binary operations.

If in a topological groupoid G the primitive divisions l and r are continuous, then we can say that G is a *topological primitive groupoid with continuous divisions.*

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A primitive groupoid G with divisions is called a *quasigroup* if every of the equations $ax = b$ and $ya = b$ has unique solution. In the quasigroup G the divisions l, r are uniques.

An element $e \in G$ is called an *identity* if $ex = xe = x$ for every $x \in X$. A quasigroup with an identity is called a *loop*.

If a multiplication operation in a quasigroup $(G, \cdot)$ with a topology is continuous, then G is called a *semitopological quasigroup*.

If in a semitopological quasigroup G the divisions l and r are continuous, then G is called a *topological quasigroup*.

A quasigroup G is called *medial* if it satisfies the law $xy \cdot zt = xz \cdot yt$ for all $x, y, z, t \in G$.

If a medial quasigroup G contains an element e such that $e \cdot x = x$ $(x \cdot e = x)$ for all x in G, then e is called a *left* (*right*) *identity element* of G and G is called a *left* (*right*) *medial loop*.

Let $N = \{1, 2, ...\}$ and $Z = \{..., -2, -1, 0, 1, 2, ...\}$. We shall use the terminology from [3, 5].

2. Multiple identities

We consider a groupoid $(G, +)$. For every two elements a, b from $(G, +)$ we denote

$$\begin{aligned} 1\,(a, b, +) &= (a, b, +)\,1 = a + b\,, \\ n\,(a, b, +) &= a + (n-1)\,(a, b, +)\,, \\ (a, b, +)\,n &= (a, b, +)\,(n-1) + b \end{aligned}$$

for all $n \geqslant 2$.

If a binary operation $(+)$ is given on a set G, then we shall use the symbols $n(a, b)$ and $(a, b)n$ instead of $n\,(a, b, +)$ and $(a, b, +)\,n$.

Definition 1. Let $(G, +)$ be a groupoid, $n \geqslant 1$ and $m \geqslant 1$. The element e of a groupoid $(G, +)$ is called an (n, m)*-zero* of G if $e + e = e$ and $n\,(e, x) = (x, e)\,m = x$ for every $x \in G$. If $e + e = e$ and $n\,(e, x) = x$ for every $x \in G$, then e is called an (n, ∞)*-zero*. If $e + e = e$ and $(x, e)\,m = x$ for every $x \in G$, then e is called an (∞, m)*-zero*. It is clear that $e \in G$ is an (n, m)-zero, if it is an (n, ∞)-zero and an (∞, m)-zero.

Remark 1. In the multiplicative groupoid $(G, \cdot)$ the element e is called an (n, m)*-identity*. The notion of the (n, m)-identity was introduced in [4].

Theorem 1. *Let $(G, \cdot)$ be a multiplicative groupoid, $e \in G$ and the following conditions hold:*

1. $ex = x$ *for every* $x \in G$;
2. $x^2 = x \cdot x = e$ *for every* $x \in G$;
3. $x \cdot yz = y \cdot xz$ *for all* $x, y, z \in G$;
4. *For every* $a, b \in G$ *there exists a unique point* $y \in G$ *such that* $ay = b$.

Then e *is a* $(1,2)$*-identity in* G.

Proof. Fix $x \in G$. Pick $y \in G$ such that $xe \cdot y = x$. By virtue of the condition 2 we have $x \cdot (xe \cdot y) = x \cdot x = e$, i.e. $x \cdot (xe \cdot y) = e$. From the condition 3 it follows that $xe \cdot xy = e$. It is clear that $xe \cdot xe = e$. Thus $xe \cdot xy = xe \cdot xe$, $xy = xe$ and $y = e$. Therefore $(x \cdot e) \cdot e = (x \cdot e) \cdot y = x$ and e is a $(1,2)$-identity. The proof is complete. □

Example 1. Let $(G, +)$ be a commutative additive group with a zero 0. Consider a new binary operation $x \cdot y = y - x$. Then $(G, \cdot)$ is a medial quasigroup with a $(1,2)$-identity 0. If $x + x \neq 0$ for some $x \in G$, then 0 is not an identity in $(G, \cdot)$.

Theorem 2. *Let $(G, \cdot)$ be a multiplicative groupoid, $e \in G$ and the following conditions hold:*

1. $ex = x$ *for every* $x \in G$;
2. $x \cdot x = e$ *for every* $x \in G$;
3. $xy \cdot uv = xu \cdot yv$ *for all* $x, y, u, v \in G$;
4. *If* $xa = ya$, *then* $x = y$.

Then G *is a medial quasigroup with a* $(1,2)$*-identity* e.

Proof. If $x \in G$, then $xe \cdot e = xe \cdot xx = xx \cdot ex = e \cdot ex = x$. Thus e is a $(1,2)$-identity.

Consider the equation $xa = b$. Then $xa \cdot e = b \cdot e$, $xa \cdot ee = be$ and $xe \cdot ae = be$. Thus $(xe \cdot ae) \cdot (be) = e$, $(xe \cdot b) \cdot (ae \cdot e) = e$, $(xe \cdot b)\, a = e$, $(xe \cdot b) \cdot (ea) = e$, $(xe \cdot e) \cdot (ba) = e$ and $x \cdot ba = e$. Therefore $x \cdot ba = ba \cdot ba$ and $x = ba$. Since $ba \cdot a = ba \cdot ea = be \cdot aa = be \cdot e = b$, the element $x = ba$ is a unique solution of the equation $xa = e$. Now we consider the equation $ay = b$. In this case $be = ay \cdot e = ay \cdot aa = aa \cdot ya = e \cdot ya = ya$. Thus $y = be \cdot a$ is a unique solution of the equation $ay = b$. The proof is complete. □

Corollary 1. *Let $(G, \cdot)$ be a left medial loop, $e \in G$ and $x^2 = e$ for every $x \in G$. Then e is a $(1,2)$-identity.*

3. Homogeneous isotopes

Definition 2. Let $(G,+)$ be a topological groupoid. A groupoid $(G, \cdot)$ is called a *homogeneous isotope* of the topological groupoid $(G,+)$ if there exist two topological automorphisms $\varphi, \psi : (G,+) \to (G,+)$ such that $x \cdot y = \varphi(x) + \psi(y)$ for all $x, y \in G$.

If $h : X \to X$ is a mapping, then $h^1(x) = h(x)$ and $h^n(x) = h\left(h^{n-1}(x)\right)$ for all $x \in X$ and $n \geqslant 2$.

Definition 3. Let $n, m \leqslant \infty$. A groupoid $(G, \cdot)$ is called an *(n,m)-homogeneous isotope* of a topological groupoid $(G,+)$ if there exist two topological automorphisms $\varphi, \psi : (G,+) \to (G,+)$ such that:

1. $x \cdot y = \varphi(x) + \psi(y)$ for all $x, y \in G$;
2. $\varphi\psi = \psi\varphi$;
3. If $n < +\infty$, then $\varphi^n(x) = x$ for every $x \in G$.
4. If $m < +\infty$, then $\psi^m(x) = x$ for every $x \in G$.

Definition 4. A groupoid $(G, \cdot)$ is called an *isotope* of a topological groupoid $(G,+)$, if there exist two homeomorphisms $\varphi, \psi : (G,+) \to (G,+)$ such that $x \cdot y = \varphi(x) + \psi(y)$ for all $x, y \in G$.

Under the conditions of Definition 4 we shall say that the isotope $(G, \cdot)$ is generated by the homeomorphisms φ, ψ of the topological groupoids $(G,+)$ and denote $(G, \cdot) = g(G, +, \varphi, \psi)$.

Theorem 3. *Let $(G,+)$ be a topological groupoid, $\varphi, \psi : G \to G$ be homeomorphisms and $(G, \cdot) = g(G, +, \varphi, \psi)$. Then:*

1. $(G,+) = \left(G, \cdot, \varphi^{-1}, \psi^{-1}\right)$;
2. *$(G, \cdot)$ is a topological groupoid;*
3. *If $(G,+)$ is a reducible groupoid, then $(G, \cdot)$ is a reducible groupoid too;*
4. *If $(G,+)$ is a groupoid with a division, then $(G, \cdot)$ is a groupoid with a division too;*
5. *If $(G,+)$ is a topological primitive groupoid with a division, then $(G, \cdot)$*

is a topological primitive groupoid with a division too;

6. *If* $(G,+)$ *is a topological quasigroup, then* $(G,\cdot)$ *is a topological quasigroup too;*
7. *If* $n,m,p,k \in N$ *and* $(G,\cdot)$ *is an* (n,m)*-homogeneous isotop of the groupoid* $(G,+)$ *and* e *is a* (k,p)*-zero in* $(G,+)$*, then* e *is an* (mk,np)*-identity in* $(G,\cdot)$.

Proof. We have $x \cdot y = \varphi(x) + \psi(y)$. Therefore

$$\varphi^{-1}(x) \cdot \psi^{-1}(y) = \varphi\left(\varphi^{-1}(x)\right) + \psi\left(\psi^{-1}(y)\right) = x + y$$

and $(G,+) = g\left(G,\cdot,\varphi^{-1},\psi^{-1}\right)$. The assertion 1 is proved. The assertion 2 and 3 are obvious.

Let $(G,+,r,l)$ be a topological primitive groupoid with the divisions, where $l : G \times G \to G$ and $r : G \times G \to G$ be continuous primitive divisions. Then the mappings $l_1(a,b) = \varphi^{-1}\left(l\left(\psi(a),b\right)\right)$ and $r_1(a,b) = \psi^{-1}\left(r\left(\varphi(a),b\right)\right)$ are the divisions of the groupoid $(G,\cdot)$. The divisions l_1, r_1 are continuous if and only if the divisions l, r are continuous. The assertions 4, 5 and 6 are proved.

Let $(G,\cdot)$ be an (n,m)-homogeneous isotope of the groupoid $(G,+)$ and e be a (k,p)-zero in $(G,+)$. We mention that $\varphi^q(e) = \psi^q(e) = e$ for every $q \in N$. If $k < +\infty$, then in $(G,+)$ we have $qk\,(e,x,+) = x$ for each $x \in G$ and for every $q \in N$.

Let $m < +\infty$ and $\psi^m(x) = x$ for all $x \in G$.

Then $1\,(e,x,\cdot) = 1\,(e,\psi(x),+)$ and $q\,(e,x,\cdot) = q\,(e,\psi^q(x),+)$ for every $q \geqslant 1$. Therefore

$$mk\,(e,x,\cdot) = mk\left(e,\psi^{mk}(x),+\right) = mk\,(e,x,+) = x.$$

Analogously we obtain that

$$(e,x,\cdot)\,np = (e,\varphi^{np}(x),+)\,np = (e,x,+)\,np = x.$$

Hence e is an (mk,np)-identity in $(G,\cdot)$. The statement 7 is proved. The proof of Theorem 3 is complete. □

Remark 2. Let $(G,+)$ be a topological quasigroup, $a,b \in G$ and φ,ψ be two automorphisms of $(G,+)$. If $x \cdot y = (a + \varphi(x)) + (\psi(y) + b)$, then we denote $(G,\cdot) = g\,(G,+,\varphi,\psi,a,b)$. It is clear that $(G,\cdot)$ is a topological quasigroup too. If $\varphi_1(x) = a + \varphi(x)$ and $\psi_1(x) = \psi(x) + b$, then φ_1, ψ_1 are homeomorphism of $(G,+)$ and $(G,+,\varphi,\psi,a,b) = (G,+,\varphi_1,\psi_1)$.

4. The homogeneous isotopes and congruences

We consider a topological groupoid $(G,+)$. If α is a relation on G, then $\alpha(x)=\{y\in G : x\alpha y\}$ for every $x\in G$.

An equivalence relation α on G is called a *congruence* on $(G,+)$ if from $x\alpha u$ and $y\alpha v$ it follows $(x+y)\,\alpha\,(u+v)$. If $(G,+)$ is a primitive groupoid with divisions l and r, then we consider that $l\,(x,y)\,\alpha l\,(u,v)$ and $r\,(x,y)\,\alpha r\,(u,v)$ provided $x\alpha u$ and $y\alpha v$.

Two congruences α and β on G are called *conjugate* if there exists a topological automorphism $\varphi : G\to G$ such that the relation $x\alpha y$ is equivalent to the relation $\varphi(x)\beta\varphi(y)$.

Let α, β be two conjugate congruences on G and φ be the topological automorphism for which the relation $x\alpha y$ is equivalent to the relation $\varphi(x)\beta\varphi(y)$. Let $\alpha\,(x)=\{y\in G : x\alpha y\}$. Then $\varphi\,(\alpha(x))=\beta\,(\varphi(x))$. If $\{\beta_\mu : \mu\in M\}$ is a family of congruences on $(G,+)$, then there exists the intersection $\beta=\cap\{\beta_\mu : \mu\in M\}$, where $\beta\,(x)=\cap\{\beta_\mu\,(x) : \mu\in M\}$. The relation $x\beta y$ is hold, if and only if $x\beta_\mu y$ is hold for every $\mu\in M$.

Theorem 4. *Let $(G,\cdot)=g\,(G,+,\varphi,\psi)$ be an isotope of the topological primitive groupoid $(G,+)$ with the divisions $\{r,l\}$, φ,ψ be topological automorphisms of $(G,+)$, and α be a congruence on the groupoid $(G,+,l,r)$. Then:*

1. *If $(G,\cdot)$ is a homogeneous isotope, then there exists a countable set of congruences $\{\beta_n : n\in N\}$ of the groupoid $(G,+)$, conjugate to α, such that $\alpha\in\{\beta_n : n\in N\}$ and $\beta=\cap\{\beta_n : n\in N\}$ is a common congruence of the groupoids $(G,+)$ and $(G,\cdot)$.*
2. *If $(G,\cdot)$ is an (n,m)-homogeneous isotope of the groupoid $(G,+)$, and $n,m<+\infty$, then there exists a finite set of congruences $\{\beta_i : i\leqslant nm\}$ of the groupoid $(G,+)$, conjugate to α, such that $\beta=\cap\{\beta_i : i\leqslant nm\}$ is a common congruence of the groupoids $(G,+)$ and $(G,\cdot)$.*

Proof. Let Z be the set of all integer numbers. If $n=0$, then $\varphi^0(x)=x$ for all $x\in G$. If $n\in Z$ and $n<0$, then $\varphi^n=\left(\varphi^{-1}\right)^{-n}$. Denote by $\{h_n : n\in Z\}$ the set of the all automorphisms

$$\left\{\varphi^{k_1}\circ\psi^{m_1}\circ\varphi^{k_2}\circ\psi^{m_2}\circ...\circ\varphi^{k_n}\circ\psi^{m_n}\ :\ n\in N,\ \ k_1,m_1,...,k_n,m_n\in Z\right\}.$$

If $\varphi\psi=\psi\varphi$, then

$$\{h_n : n\in Z\}=\left\{\varphi^k\circ\psi^m : \ k,m\in Z\right\}.$$

For each $n \in N$ we define the congruence $\beta_n(x) = h_n(\alpha(x))$ for all $x \in G$.

Denote $\beta = \cap\{\beta_k : k \in N\}$. Then $\varphi(\beta(x)) = \psi(\beta(x)) = \beta(x)$ for each $x \in G$. Hence β is a common congruence of groupoids $(G,+)$ and $(G,\cdot)$. Suppose that automorphisms φ and ψ satisfy the Definition 3 and $(G,\cdot)$ is an (n,m)-isotope of groupoid $(G,+)$. In this case we have

$$\varphi^{k_1} \cdot \psi^{q_1} \cdot \varphi^{k_2} \cdot \psi^{q_2} \cdot \ldots \cdot \varphi^{k_n} \cdot \psi^{q_n} = \left(\varphi^{k_1+\ldots+k_n}\right) \cdot \left(\psi^{q_1+\ldots+q_n}\right)$$

Therefore

$$\{h_k : k \in N\} = \left\{\varphi^i \cdot \psi^j \,:\, i = 1,\ldots,n,\ j = 1,\ldots,m\right\} = \{h_k \,:\, k \leqslant nm\}$$

and the set $\{\beta_n : n \in N\}$ is finite and contains no more than nm distinct elements. The proof is complete. □

Remark 3. Let α and β be two conjugate congruences on a topological groupoid G. Then:

1. The sets $\alpha(x)$ are G_δ-sets iff the sets $\beta(x)$ are G_δ-sets in G.
2. The sets $\alpha(x)$ are closed in G iff the sets $\beta(x)$ are closed in G.
3. The sets $\alpha(x)$ are open in G iff the sets $\beta(x)$ are open in G.

Remark 4. Let $\{\beta_n : n \in N' \subset N\}$ be a family of congruences on a topological goupoid G and $\beta = \cap\{\beta_n : n \in N\}$. Then:

1. If the sets $\beta_n(x)$ are G_δ-sets in G, then the sets $\beta(x)$ are G_δ-sets in G too.
2. If the set N' is finite and the sets $\beta_n(x)$ are open, then the sets $\beta(x)$ are open in G.

5. General properties of medial quasigroups

Let $(G,\cdot)$ be a topological medial quasigroup. By virtue of Toyoda's Theorem [7] there exist a binary operation $(+)$ on G, two elements $0, c \in G$ and two topological automorphisms $\varphi, \psi : (G,+) \to (G,+)$ such that $(G,+)$ is a topological commutative group, 0 is the zero of $(G,+)$ and $(G,\cdot) = g(G,+,\varphi,\psi,0,c)$ is a homogeneous isotope of $(G,+)$. In particular, $\varphi\psi = \psi\varphi$.

In [2] G.B. Beleavskaya has proved a generalization of Toyoda's Theorem.

Theorem 5. *Let $(G,+)$ be a topological quasigroup, $0 \in G, 0+0=0, \varphi, \psi$ be two automorphisms of $(G,+)$ and $(G,\cdot)=(G,+,\varphi,\psi)$. Then:*

1. *$\{0\}$ is a subquasigroup of the quasigroups $(G,+)$ and $(G,\cdot)$.*
2. *If $n<+\infty$, then 0 is an (n,∞)-identity of $(G,\cdot)$ iff $\varphi^n(x)=x$ for every $x \in G$.*
3. *If $m<+\infty$, then 0 is an (∞,m)-identity of $(G,\cdot)$ iff $\psi^m(x)=x$ for every $x \in G$.*
4. *If $n,m<+\infty$, then 0 is an (n,m)-identity of $(G,\cdot)$ iff $\varphi^n(x)=\psi^m(x)=x$ for every $x \in G$.*

Proof. Let n $<+\infty$. If $\varphi^n(x)=x$ for every $x \in G$, then from Theorem 3 it follows that 0 is an $(n,+\infty)$-identity in $(G,\cdot)$.

Let 0 be an (n,∞)-identity in $(G,\cdot)$. By construction, $\varphi(0)=\psi(0)=0$ and $x \cdot y=\varphi(x)+\psi(y)$. Then $(x,0)\,k=\varphi^k(x)$ and $(0,x)\,k=\varphi^k(x)$ for every $k \in N$. Since $(x,0)\,n=x$ we obtain that $\varphi^n(x)=x$. The proof is complete. □

Consider on G some equivalence relation α. Denote by G/α the collection of classes of equivalence $\alpha(x)$ and $\pi_\alpha : G \to G/\alpha$ is the natural projection. On G/α we consider the quotient topology. The mapping π_α is continuous. If α is a congruence on $(G,\cdot)$ (or on $(G,+)$), then the mapping π_α is open.

An equivalence relation α on G is called compact if the sets $\alpha(x)$ are compact.

Theorem 6. *Let $(G,+)$ be a commutative topological group, 0 be a zero of $(G,+)$, $c \in G$, φ and ψ be two automorphisms of the topological group $(G,+)$ and $(G,\cdot)=g(G,+,\varphi,\psi,0,c)$. If the space G contains a non-empty compact subset F of countable character, then for every open subset U of G containing 0 there exists a compact equivalence relation α_U on G such that:*

1. *$\alpha_U(0) \subseteq U$.*
2. *α_U is a congruence on $(G,\cdot)$.*
3. *α_U is a congruence on $(G,+)$.*
4. *The natural projection $\pi_U=\pi_{\alpha_U} : G \to G/\alpha_U$ is an open perfect mapping.*
5. *The space G/α_U is metrizable.*

Proof. We consider that $0 \in F \subseteq U$. Fix a sequence $\{U_n : n \in N\}$ of open subsets of G such that for every open set V containing F there exists $n \in N$ such that $F \subseteq U_n \subseteq V$. Suppose that $F \subseteq U_n$ and $U_{n+1} \subseteq U_n$ for every $n \in N$.

Then there exists a sequence $\{V_n : n \in N\}$ of open sets of G such that for every $n \in N$ we have:

- $V_{n+1} + V_{n+1} \subseteq V_n \subseteq U_n, \quad cl_G V_{n+1} \subseteq V_n$ and $V_n = -V_n$,
- $\varphi(V_{n+1}) \cup \psi(V_{n+1}) \subseteq V_n$.

We put $H = \cap\{V_n : n \in N\}$. By construction, H is a compact subgroup and the natural projection $\pi : G \to G/H$ is open and perfect. Let $\alpha(x) = x + H$ for every $x \in G$. Then α is a congruence on $(G, +)$. Suppose that $x\alpha z$ and $y\alpha v$. Then

$$\begin{aligned} x \cdot y &= \varphi(x) + \psi(y) + c, \\ z \cdot v &= \varphi(z) + \psi(v) + c, \\ \varphi(x) - \varphi(z) &\in H, \ \psi(y) - \psi(v) \in H. \end{aligned}$$

Thus

$$\begin{gathered} (x \cdot y) - (z \cdot v) = \\ = (\varphi(x) + \psi(y)) - (\varphi(z) + \psi(v)) = \\ = (\varphi(x) - \varphi(z)) + (\psi(y) - \psi(v)) \in H. \end{gathered}$$

Therefore α is a congruence on $(G, \cdot)$ too.

It is clear that the space G/H is metrizable. The proof is complete. $\square$

Corollary 2. *A first countable topological medial quasigroup is metrizable.*

A space X is called a *paracompact p-space* if there exists a perfect mapping $g : X \to Y$ onto some metrizable space Y (see [1]).

Corollary 3. *If a topological medial quasigroup contains a non-empty compact subset of countable character then it is a paracompact space p-space and admits an open perfect homomorphism onto a medial metrizable quasigroup.*

Corollary 4. *A Čech complete topological medial quasigroup is paracompact and admits an open perfect homomorphism onto a complete metrizable medial quasigroup.*

Corollary 5. *A locally compact medial quasigroup is paracompact and admits an open perfect homomorphism onto a metrizable locally compact medial quasigroup.*

6. On Haar measures on medial quasigroups

By $B(X)$ denote the family of all Borel subsets of the space X.

A non-negative real-valued function μ defined on the family $B(X)$ of Borel subsets of a space X is said to be a *Radon measure* on X if it has the following properties:

– $\mu(H) = \sup\{\mu(F) : F \subseteq H, F$ is a compact subset of $H\}$ for every $H \in B(X)$;

– for every point $x \in X$ there exists an open subset V_x containing x such that $\mu(V_x) < \infty$.

Definition 5. Let $(A, \cdot)$ be a topological quasigroup with the divisions $\{r, l\}$. A Radon measure μ on A is called:

– a *left invariant Haar measure*, if $\mu(U) > 0$ and $\mu(xH) = \mu(H)$ for every non-empty open set $U \subseteq A$, a point $x \in A$ and a Borel set $H \in B(A)$;

– a *right invariant Haar measure*, if $\mu(U) > 0$ and $\mu(Hx) = \mu(H)$ for every non-empty open set $U \subseteq A$, a point $x \in A$ and Borel set $H \in B(A)$;

– an *invariant Haar measure* if $\mu(U) > 0$ and $\mu(xH) = \mu(Hx) = \mu(l(x, H)) = \mu(r(H, x)) = \mu(H)$ for every non-empty open set $U \subseteq A$, a point $x \in A$ and a Borel set $H \in B(A)$;

Definition 6. We say that on a topological quasigroup $(A, \cdot)$ *there exists a unique left (right) invariant Haar measure*, if for every two left (right) invarinat Haar measures μ_1, μ_2 on A there exists a constant $c > 0$ such that $\mu_2(H) = c \cdot \mu_1(H)$ for every Borel set $H \in B(A)$.

If $(G, +)$ is a locally compact commutative group, then on G there exists a unique invariant Haar measure μ_G (see [6]).

Theorem 7. *Let $(G, \cdot)$ be a locally compact medial quasigroup, $(G, +)$ be a commutative topological group, $\varphi, \psi : G \to G$ be automorphisms of $(G, +), b \in G$ and $(G, \cdot) = g(G, +, \varphi, \psi, 0, b)$. On the group $(G, +)$ consider the invariant Haar measure μ_G. Then :*

1. *On $(G, \cdot)$ the right (left) invariant Haar measure is unique.*
2. *If μ is a left (right) invariant Haar measure on $(G, \cdot)$, then μ is a left (right) invariant Haar measure on $(G, +)$ too.*
3. *On $(G, \cdot)$ there exists some right invariant Haar measure if and only*

if $\mu_G(\varphi(H)) = \mu_G(H)$ *for every* $H \in B(A)$.

4. *If* $n < +\infty$, *and on* G *there exists some* $(n, +\infty)-$*identity, then on* $(G, \cdot)$ *the measure* μ_G *is a unique right invariant Haar measure.*
5. *If* $m < +\infty$, *and on* G *there exists some* $(+\infty, m)-$*identity, then on* $(G, \cdot)$ *the measure* μ_G *is a unique left invariant Haar measure.*
6. *If* $n, m < +\infty$, *and on* G *there exists some* $(n, m)-$*identity, then on* $(G, \cdot)$ *the measure* μ_G *is a unique invariant Haar measure.*

Proof. Let μ be a right invariant Haar measure on $(G, \cdot)$. Since $x \cdot y = \varphi(x) + \psi(y) + b$ for all $x, y \in G$, then $Hx = \varphi(H) + \psi(H) + b$. Thus μ is an invariant Haar measure on $(G, +)$ and there exists a constant $c > 0$ such that $\mu(H) = c \cdot \mu_G(H)$. Thus μ_G is a right invariant Haar measure on $(G, \cdot)$. The assertions 1,2 and 3 are proved.

Consider some topological automorphism h of $(G, +)$. Then $\mu_h(H) = \mu_G(h(H))$ is an invariant Haar measure on $(G, +)$. There exists a constant $c_h > 0$ such that $\mu_h(H) = \mu_G(h(H)) = c_h \cdot \mu_G(H)$ for every Borel subset $H \in B(G)$. In particular, $\mu_G\left(h^k(H)\right) = c_h^k \mu_G(H)$ for every $k \in N$. If $n < +\infty$ and 0 is an $(n, +\infty)-$identity, then $\varphi^n(x) = x$ for every $x \in G$ and $c_\varphi^n = 1$. Thus $c_\varphi = 1$, $\mu_G(H) = \mu_G(h(H))$ and μ_G is a right invariant Haar measure on $(G, \cdot)$. The assertions 4, 5 and 6 are proved. The proof is complete. □

In this way we can prove the following results.

Theorem 8. *Let* $(G, +)$ *be a topological quasigroup and* $(G, \cdot)$ *be an* (n, m)-*homogeneous isotope of* $(G, +)$. *Then:*

1. *On* $(G, +)$ *there exists a left (right) invariant Haar measure if and only if on* $(G, \cdot)$ *there exists a left (right) invariant Haar measure.*
2. *If on* $(G, +)$ *the a left (right) invariant Haar measure is unique, then on* $(G, \cdot)$ *the a left (right) invariant Haar measure is unique too.*

Theorem 9. *On a compact medial quasigroup* G *there exists a unique Haar measure* μ *for which* $\mu(G) = 1$.

Theorem 10. *Let* $(G, +)$ *be a locally compact group,* μ_G *be the left invariant Haar measure on* $(G, +)$ *and* $\varphi, \psi : G \to G$ *be the topological automorphism of* $(G, +)$. *Fix* $c \in G$ *and consider the binary operation* $x \cdot y = \varphi(x) + \psi(y) + c$. *Then:*

1. $(G, \cdot)$ *is a topological quasigroup.*
2. *If* $\mu_G(\psi(H)) = \mu_G(H)$ *for every Borel subset* $H \in B(G)$, *then* μ_G

is a left invariant Haare measure on $(G, \cdot)$.

3. *If* $m \in N$ *and* $\psi^m(x) = x$ *for every* $x \in G$, *then* μ_G *is a left invariant Haar measure on* $(G, \cdot)$.
4. *If* $(G, +)$ *is a compact group, then* μ_G *is an invariant Haar measure on* $(G, \cdot)$.

7. Examples

Example 2. Let $(R, +)$ be a topological commutative group of real numbers, $a > 0$, $b > 0$, $\varphi(x) = ax$, $\psi(y) = bx$ and $x \cdot y = \varphi(x) + \psi(y)$. Then $(R, \cdot)$ is a commutative locally compact medial quasigroup. If $H = [c, d]$, then $0 \cdot H = [ac, ad]$ and $H \cdot 0 = [bc, bd]$. Thus:

- on $(G, \cdot)$ there exists some right invariant Haar measure if and only if $a = 1$;
- on $(G, \cdot)$ there exists some left invariant Haar measure if and only if $b = 1$;
- if $a \neq 1$ and $b \neq 1$, then on $(G, \cdot)$ does not exist any left or right invariant Haar measure.

Example 3. Denote by $Z_p = Z/pZ = \{0, 1, ..., p-1\}$ the cyclic Abelian group of order n. Consider the Abelian group $(G, +) = (Z_5, +)$ and $\varphi(x) = 2x$, $\psi(x) = 4x$. Then $(G, \cdot) = g(G, +; \varphi, \psi)$ is a medial quasigroup and each element from $(G, \cdot)$ is $(2, 4)$-identity in G.

Example 4. Consider the Abelian group $(G, +) = (Z_5, +)$ and $\varphi(x) = \psi(x) = 3x$. Then $(G, \cdot) = g(G, +; \varphi, \psi)$ is medial quasigroup and all elements from $(G, \cdot)$ are the $(4, 4)$-identities in G.

Example 5. Consider the commutative group $(G, +) = (Z_5, +)$, $\varphi(x) = 2x, \psi(x) = 2x + 1$ and $x \cdot y = 2x + 2y + 1$. Then $(G, \cdot) = g(G, +; \varphi, \psi, 0, 1)$ is a commutative medial quasigroup and $(G, \cdot)$ does not contain (n, m)-identities.

Example 6. Consider the commutative group $(G, +) = (Z, +)$, $\varphi(x) = x, \psi(x) = x + 1$ and $x \cdot y = x + y + 1$. Then $(G, \cdot) = g(G, +; \varphi, \psi)$ is a medial quasigroup and $(G, \cdot)$ does not contain (n, m)-identities. On $(G, \cdot)$ there exists an invariant Haar measure.

Example 7. Let $(G,+)$ be an Abelian group and $x+x\neq 0$ for each $x\in G$. For example $(G,+)\in\{(Z_p,+) : p\in N, p\geqslant 2\}$. Denote $\varphi(x)=x$ and $\psi(x)=-x$ for each $x\in G$. Then $(G,\cdot)=g(G,+;\varphi,\psi)$ is a medial quasigroup and $(G,\cdot)$ contains the unique $(1,2)$-identity, which coincide with the zero element in $(G,+)$.

Example 8. Let $(G,+)=(Z_7,+)$, and $\varphi(x)=3x$ and $\psi(x)=5x$. Then $(G,\cdot)=g(G,+;\varphi,\psi)$ is a medial quasigroup. In this case 0 and 3 are $(12,6)$-identities.

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