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Product of the symmetric group with the alternating group on seven letters

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Abstract

We will find the structure of groups $G = AB$ where A and B are subgroups of G with A isomorphic to the alternating group on 7 letters and B isomorphic to the symmetric group on $n \geqslant 5$ letters.

1. Introduction

If A and B are subgroups of the group G and $G = AB$, then G is called a *factorizable group* and A and B are called *factors of the factorization*. We also say that G is the *product of its subgroups* A and B. If any of A or B is a non-proper subgroup of G, then $G = AB$ is called a *trivial factorization* of G, and by a *non-trivial* or a *proper factorization* we mean $G = AB$ with both A and B are proper subgroups of G. It is an interesting problem to know the groups with proper factorization. Of course not every group has a proper factorization, for example an infinite group with all proper subgroups finite has no proper factorization, also the Janko simple group J_1 of order 175 560 has no proper factorization. In what follows G is assumed to be a finite group. Now we recall some research papers towards the problem of factorization of groups under additional conditions on A and B. In [8] factorization of the simple group $L_2(q)$ are obtained and in [1] simple groups G with proper factorizations $G = AB$ such that $(|A|, |B|) = 1$ are given. Factorizations $G = AB$ with $A \cap B = 1$ are called *exact* and in [18] such factorizations for the alternating and symmetric groups are investigated. If A and B are maximal subgroups of G and $G = AB$, then this is called a *maximal factorization* of G. In [11] all

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maximal factorizations of the simple groups and their automorphism groups are obtained. Factorizations of sporadic simple groups and simple groups of Lie type with rank 1 or 2 as the product of two simple subgroups are obtained in [6] and [7] respectively. In [12] all groups with factorization $G = AB$, with A and B simple subgroups of G such that a Sylow 2-subgroup of A has rank 2 and a Sylow 2-subgroup of B is elementary abelian are completely classified.

Factorizations of groups involving alternating or symmetric groups have been investigated in some papers. In [10] groups G with factorization $G = AB$, where $A \cong B \cong \mathbf{A}_5$, are classified and in [13] groups $G = AB$ where A is a non-abelian simple group and B is isomorphic to to the alternating group on 5 letters are determined. In a series of papers G.L. Walls considered factorizations $G = AB$ of a group G with both factors simple [14], [15]. In [16] he began the study of factorizations when one factor is simple and the other is almost simple. To begin this study it is natural to start with the case where one factor is isomorphic to a simple alternating group and the other is isomorphic to a symmetric group. In [3] we classified all groups G with factorization $G = AB$ where $A \cong \mathbf{A}_6$ and B is isomorphic to a symmetric group on $n \geqslant 5$ letter, and in [4] we determined all groups G with factorization $G = AB$ where A is a simple group and $B \cong \mathbf{S}_6$. Motivated by the above results and to get a picture for the general case, in this paper we will study groups G with factorization $G = AB$, where $A \cong \mathbf{A}_7$ and B is isomorphic to a symmetric group on $n \geqslant 5$.

2. Preliminary results

In this section we obtain results which are needed in the proof of our main Theorem. Suppose Ω is a set of cardinality m and G is a k-homogeneous, $1 \leqslant k \leqslant m$, group on Ω. If H is a k-homogeneous subgroup of G, then it is easy to see that $G = G_{(\Delta)}H$ where Δ is a subset of cardinality k in Ω. We can give some factorization of groups using the previous observation. It is easy to verify that the order of a subgroup of $\mathbf{A}_7$ is one of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 18, 20, 21, 24, 36, 60, 72, 120, 168, 360 or 2520 and therefore the index of a proper subgroup of $\mathbf{A}_7$ is one of the numbers 7,15, 21, 35, 42, 70, 105, 120, 126, 140, 210, 252, 280, 315, 360, 420, 504, 630, 840, 1260 or 2520. Therefore $\mathbf{A}_7$ has transitive action on sets of cardinality equal to any of the latter numbers. Therefore we always have the factorization $\mathbf{S}_{n+1} = \mathbf{S}_n\mathbf{A}_7$ where $n =$ 6, 14, 20, 34, 41, 69, 104, 119, 125, 139, 209, 251, 279, 314, 359, 419, 503, 629, 839, 1259 or 2519. It is well-know that $\mathbf{A}_7$ has a 2-transitive action on 15 points and hence we have the factorization $\mathbf{S}_{15} = \mathbf{S}_{13}\mathbf{A}_7$. If we consider the 7-transitive action of $\mathbf{A}_7$ on 7 points we also have the factorizations $\mathbf{S}_7 = \mathbf{S}_n\mathbf{A}_7$, $2 \leqslant n \leqslant 7$. Therefore we have the following Lemma.

Lemma 1.

(*a*) $\mathbf{S}_{n+1} = \mathbf{A}_7\mathbf{S}_n$ *and* $\mathbf{A}_{n+1} = \mathbf{A}_7\mathbf{A}_n$ *for* $n = 6$, 14, 20, 34, 41, 69, 104, 119, 125, 139, 209, 251, 279, 314, 359, 419, 503, 629, 839, 1259 *or* 2519.

(*b*) $\mathbf{S}_{15} = \mathbf{A}_7\mathbf{S}_{13}$ *and* $\mathbf{A}_{15} = \mathbf{A}_7\mathbf{S}_{13}$.

(*c*) $\mathbf{S}_7 = \mathbf{A}_7\mathbf{S}_n$ *for* $2 \leqslant n \leqslant 7$.

In the following Lemmas we will find a special kind of factorizations for the alternating and symmetric groups.

Lemma 2. *Let $m, r, n \geqslant 5$ be natural numbers. If $\mathbf{A}_m = \mathbf{A}_r\mathbf{A}_n$ or $\mathbf{A}_m = \mathbf{A}_r\mathbf{S}_n$ are proper factorizations, then either $r = m-1$ and $\mathbf{A}_m$ has a transitive subgroup isomorphic to $\mathbf{A}_n$ or $\mathbf{S}_n$ which gives the factorizations $\mathbf{A}_m = \mathbf{A}_{m-1}\mathbf{A}_n$ and $\mathbf{A}_m = \mathbf{A}_{m-1}\mathbf{S}_n$, or $(m, r, n) = (10, 6, 8)$, $(15, 7, 13)$, $(15, 8, 13)$, $(10, 8, 6)$ giving the factorizations $\mathbf{A}_{10} = \mathbf{A}_6\mathbf{A}_8 = \mathbf{A}_6\mathbf{S}_8$, $\mathbf{A}_{15} = \mathbf{A}_7\mathbf{A}_{13} = \mathbf{A}_7\mathbf{S}_{13}$, $\mathbf{A}_{15} = \mathbf{A}_8\mathbf{A}_{13} = \mathbf{A}_8\mathbf{S}_{13}$ and $\mathbf{A}_{10} = \mathbf{A}_8\mathbf{S}_6$. Moreover all the above factorizations occurs.*

Proof. We use Theorem D of [11], but note that the case (ii) of this Theorem can not happen for these special factorizations of $\mathbf{A}_m$ stated in our Theorem. First we assume $\mathbf{A}_m = \mathbf{A}_r\mathbf{A}_n$. In this case without loss of generality we may assume $\mathbf{A}_{m-k} \trianglelefteq \mathbf{A}_r \leqslant \mathbf{S}_{m-k} \times \mathbf{S}_k$ for some $k, 1 \leqslant k \leqslant 5$, and $\mathbf{A}_n$ is k-homogeneous on m letter. Since the factorization is proper hence $m > r$ and $m > n$. If $m - k = 1$, then from $1 \leqslant k \leqslant 5$ we get $m = 6$ and we have the factorization $\mathbf{A}_6 = \mathbf{A}_5\mathbf{A}_5$. Hence $m - k = r$. If $k = 1$, then $\mathbf{A}_m = \mathbf{A}_{m-1}\mathbf{A}_n$ and $\mathbf{A}_n$ has a transitive action on m letters. If $k \geqslant 2$, then since $m > n$, from [9] and [5] we get $k = 2$ and the 2-transitive actions of $\mathbf{A}_r$ occurs if and only if $(r, m) = (5, 6), (6, 10), (7, 15), (8, 15)$. since we have assumed $m, r, n \geqslant 5$, so we obtain the triples listed in the Lemma.

Next we assume $\mathbf{A}_m = \mathbf{A}_r\mathbf{S}_n$. Again we use Theorem D in [11], but we must consider two cases

Case (*i*). $\mathbf{A}_{m-k} \trianglelefteq \mathbf{A}_r \leqslant \mathbf{S}_{m-k} \times \mathbf{S}_k$ for some $k, 1 \leqslant k \leqslant 5$, and $\mathbf{S}_n$ is k-homogeneous on m letters. Reasoning as above we must have $m - k = r$. If $k = 1$, then $\mathbf{A}_m = \mathbf{A}_{m-1}\mathbf{S}_n$ and $\mathbf{S}_n$ must have a transitive permutation representation on m points. Otherwise since $\mathbf{S}_n$ has no $k > 2$ transitive permutation representations except the trivial ones we don't get a possibility. However $\mathbf{S}_6$ has a 2-transitive permutation representations on 10 points giving the factorization $\mathbf{A}_{10} = \mathbf{A}_8\mathbf{S}_6$.

CASE (ii). $\mathbf{A}_{m-k} \trianglelefteq \mathbf{S}_n \leqslant \mathbf{S}_{m-k} \times \mathbf{S}_k$ for some $k, 1 \leqslant k \leqslant 5$, and $\mathbf{A}_r$ is k-homogeneous on m points. In this case $m - k = 1$ is not possible, hence $m - k = n$. Since we must have $m \geqslant n - 2$, so $k = 1$ is not possible. Natural k-homogeneous permutation representation of $\mathbf{A}_r$ don't give proper factorizations, therefore we must have $\mathbf{A}_r = \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7$ or $\mathbf{A}_8$ acting 2-transitively on sets of cardinality 6, 10, 15 and 15 respectively. In this case we obtain $(m, r, n) = (6, 5, 4), (10, 6, 8), (15, 7, 13), (15, 8, 13), (10, 8, 6)$ and the admissible triples are the ones listed in the Lemma. □

Lemma 3. *Let $m, r, n \geqslant 5$ be integers and $\mathbf{S}_m = \mathbf{A}_r\mathbf{S}_n$ be a non-trivial factorization of $\mathbf{S}_m$. Then we have one of the following possibilities:*

(a) $n = m - 1$ *and $\mathbf{A}_r$ has a transitive action on m points and the factorization $\mathbf{S}_m = \mathbf{A}_r\mathbf{S}_{m-1}$ occurs.*

(b) $r = m - 1$ *and $\mathbf{S}_n$ has a transitive action on r points and moreover* $2m \mid n!$.

(c) $(m, r, n) = (10, 6, 8), (15, 7, 13), (15, 8, 13)$ *and the factorizations* $\mathbf{S}_{10} = \mathbf{A}_6\mathbf{S}_8$*,* $\mathbf{S}_{15} = \mathbf{A}_7\mathbf{S}_{13}$*,* $\mathbf{S}_{15} = \mathbf{A}_8\mathbf{S}_{13}$ *all occurs.*

(d) $(m, n, r) = (10, 8, 6)$ *and* $\mathbf{S}_{10} = \mathbf{A}_8\mathbf{S}_6$.

Proof. Again we use Theorem D of [11], knowing that case (ii) of the Theorem does not hold in this special case. We consider two cases. Note that $m > r$ and $m > n$.

CASE (i). $\mathbf{A}_{m-k} \trianglelefteq \mathbf{S}_n \leqslant \mathbf{S}_{m-k} \times \mathbf{S}_k$ and $\mathbf{A}_r$ has a k-homogeneous action on m letters. If $k = 1$, then $n = m - 1$ and we have the factorization $\mathbf{S}_m = \mathbf{S}_{m-1}\mathbf{A}_r$ where $\mathbf{A}_r$ acts transitively on m letters. If $k \geqslant 2$, then by [9] and [5] the only non-trivial k-homogeneous representation of $\mathbf{A}_r$ on m letters occurs if and only if $k = 2$ and $(m, r) = (6, 5), (10, 6), (15, 7), (15, 8)$ and for these pairs we have $n =$ 4,8,13,13 respectively. Therefore cases (a) and (c) are proved and it is clear that the appropriate factorizations exists.

CASE (ii). $\mathbf{A}_{m-k} \trianglelefteq \mathbf{A}_r \leqslant \mathbf{S}_{m-k} \times \mathbf{S}_k$ and $\mathbf{S}_n$ has a k-homogeneous action on m letters. In this case $\mathbf{S}_n$ does not have a k-homogeneous action on m letters except the trivial ones if $k > 2$. In the case of $k = 2$, $\mathbf{S}_6$ has a non-trivial 2-transitive action on 10 letters. Therefore $k = 1$ which forces $r = m - 1$ and if we have the factorization $\mathbf{S}_m = \mathbf{A}_{m-1}\mathbf{S}_n$, then $\mathbf{S}_n$ must act on m letters transitively and order consideration yields $2m \mid n!$. In this way we obtain cases (b) and (d) and the Lemma is proved. □

3. Factorizations involving $\mathbf{A}_7$

To obtain our main result concerning groups with factorizations $G = \mathbf{A}_7\mathbf{S}_n$, $n \geqslant 5$ we need to know about simple primitive groups of certain degrees, and these degrees are indices of subgroups of $\mathbf{A}_7$ which are greater than 1. In section 2 we listed the 21 possible numbers, and we see that except 1260 and 2520 the rest of them are less than 1000. Simple primitive groups of degree up to 1000 are listed in [5] and we can obtain the simple primitive groups with the degree we want. These are listed in Table I. But we don't know about the simple primitive groups of degree 1260 and 2520 in the existing literature. The following Lemma deals with these cases.

Lemma 4. *Suppose G is not an alternating simple group but G is a simple permutation group of degree* 1260 *or* 2520*. Then it is not possible to decompose G as $G = \mathbf{A}_7\mathbf{A}_n$, for any n.*

Proof. According to the classification of finite simple groups any finite nonabelian simple group is isomorphic to either an alternating group, a sporadic group or a simple group of Lie type. Since G is written as the product of two simple groups results of [6] show that G can not be a sporadic simple group. If G is a simple group of Lie type, then by [7] the only possibility is $L_2(9) = \mathbf{A}_5\mathbf{A}_5$ which is not possible because $L_2(9)$ is not a permutation group of degree 1260 or 2520. Therefore we assume that G is a simple group of Lie type with Lie rank at least 3. Here we use results about the minimum index of a subgroup of a group of Lie type and consider the following cases.

(*i*). For $L_n(q), n \geqslant 4$, the proper subgroups have index at least $\frac{q^n-1}{q-1}$ and form $\frac{q^n-1}{q-1} \leqslant 2520$ we get the following possibilities: $L_4(2)$, $L_4(3)$, $L_4(4)$, $L_4(5)$, $L_4(7)$, $L_4(8)$, $L_4(9)$, $L_4(11)$, $L_4(13)$, $L_5(2)$, $L_5(3)$, $L_5(4)$, $L_5(5)$, $L_5(7)$, $L_6(2)$, $L_6(3)$, $L_6(4)$, $L_7(2)$, $L_8(2)$, $L_9(2)$, $L_{10}(2)$, $L_{11}(2)$. $L_4(2) \cong \mathbf{A}_8$ is not the case. If $L_4(3) = \mathbf{A}_7\mathbf{A}_n$, then by [2] we see that$L_4(3)$ does not contain a subgroup isomorphic to $\mathbf{A}_7$. If $L_4(4) = \mathbf{A}_7\mathbf{A}_n$, then since $17||L_4(4)|$ we must have $n \geqslant 17$, but in this case we must have $13||L_4(4)|$ which is not the case. Now using similar argument as above we rule out all the above possibilities.

(*ii*). For $U_n(q), n \geqslant 6$, the proper subgroups have index at least $(q^n - (-1)^n)(q^{n-1} - (-1)^{n-1})/(q^2 - 1)$ and for this number to be at least 2520 we get only $U_6(2)$. If $U_6(2) = \mathbf{A}_7\mathbf{A}_n$, then since $11||U_6(2)|$ we must have $n \geqslant 11$, but then from [2] we see that $U_6(2)$ does not have a subgroup isomorphic to A_{11}.

(*iii*). For $S_{2m}(q), m \geqslant 3$, the proper subgroup have index at least $\frac{q^{2m}-1}{q-1}$ when $q > 2$ and at least $2^m(2^m-1)$ when $q = 2$ and $m > 2$. In this case the following symplectic groups are the possibilities $S_6(2)$, $S_6(3)$, $S_6(4)$, $S_8(2)$, $S_{10}(2)$, $S_{12}(2)$, and again using [2] and order consideration we rule out the possibility $G = \mathbf{A}_7\mathbf{A}_n$.

(*iv*). For $O^{\epsilon}_{2m}(q), m \geqslant 4, \epsilon \equiv \pm$, the proper subgroups have index at least $(q^m-1)(q^{m-1}+1)/(q-1)$ when $\epsilon \equiv +$ and at least $(q^m+1)(q^{m-1}-1)/(q-1)$ when $\epsilon \equiv -$ except for the case $(q,\epsilon) = (2,+)$ when a proper subgroup has index at least $2^{m-1}(2^m-1)$. For $O_{2m+1}(q), m \geqslant 3, q$ odd, the proper subgroups have index at least $(q^{2m}-1)/(q-1)$ except when $q = 3$ and in this latter case the minimum index is $(q^{2m}-q^m)/2$. Now for this index to be at most 2520 we obtain the following orthogonal groups $O_7(3)$, $O^{\pm}_8(2)$, $O^{\pm}_8(3)$, $O^{\pm}_{10}(2)$, $O^{\pm}_{12}(2)$. Again order consideration rules out the possibility $G = \mathbf{A}_7\mathbf{A}_n$.

(*v*). For G to be an exceptional simple group of Lie type we use the argument in the proof of Theorem 9 in [5] from which only the possibilities $E_6(q)$ or $F_4(q)$ can arise and both of them are ruled out by order consideration. The Lemma is now proved. □

Theorem 5. *If $M = \mathbf{A}_7\mathbf{A}_n$ is a simple group, then*

(*a*) $M = \mathbf{A}_n$ *for* $n \geqslant 7$,

(*b*) $M = \mathbf{A}_{15} = \mathbf{A}_7\mathbf{A}_{13}$,

(*c*) $M = \mathbf{A}_{n+1} = \mathbf{A}_7\mathbf{A}_n$ *for* $n = 14, 20, 34, 41, 69, 104, 119, 125, 139,$ 219, 251, 279, 314, 359, 419, 503, 629, 839, 1259 *or* 2519.

Proof. Case (a) corresponds to the trivial factorization of M. Now suppose $M = \mathbf{A}_7\mathbf{A}_n$ is a non-trivial factorization of a simple group M. If C is a maximal subgroup of M containing $\mathbf{A}_n$, then we have $[M : C] | [\mathbf{A}_7 : \mathbf{A}_7 \cap C]$. Therefore M is a simple primitive group of degree equal to the index of a proper subgroup of $\mathbf{A}_7$. If M is an alternating group, then by Lemma 2 we get cases (b) and (c). Therefore we assume M is not an altrnating group. By Lemma 4 M can not be a primitive group of degree 1260 or 2520. Therefore we may assume M is a simple primitive group of degree less than 1000. Simple primitive groups of these special degrees are listed in Table I. Now using [6] and [7] the only cases that need to be considered are $S_6(2), S_8(2), O^+_8(2)$ or J_2. If $S_6(2) = \mathbf{A}_7\mathbf{A}_n$, then since $2^9 || S_6(2)|$ we must have $n \geqslant 8$ and by [2] we get $n = 8$. But in this case if $S_6(2) = \mathbf{A}_7\mathbf{A}_8$, then $|\mathbf{A}_7 \cap \mathbf{A}_8| = 35$ which is a contradiction because $\mathbf{A}_7$ does not contain a subgroup of order 35. Order consideration rules out the possibilities $S_8(2)$

or $O_8^+(2)$ to be factorized as $\mathbf{A}_7\mathbf{A}_n$, for any n. By [2] the group J_2 does not contain a subgroup isomorphic to $\mathbf{A}_7$, and the Lemma is proved now. □

Table I. Simple primitive groups of certain degrees

degree	Groups
7	$\mathbf{A}_7, L_2(7)$
15	$\mathbf{A}_{15}, \mathbf{A}_6, \mathbf{A}_7, \mathbf{A}_8$
21	$\mathbf{A}_{21}, \mathbf{A}_7, L_2(7), L_3(4)$
35	$\mathbf{A}_{35}, \mathbf{A}_7, \mathbf{A}_8$
42	$\mathbf{A}_{42}$
70	$\mathbf{A}_{70}$
105	$\mathbf{A}_{105}, \mathbf{A}_{15}, L_3(4)$
120	$\mathbf{A}_{120}, \mathbf{A}_9, \mathbf{A}_{10}, L_2(16), L_3(4), S_4(4), S_6(2), S_8(2), O_8^+(2)$
126	$\mathbf{A}_{126}, \mathbf{A}_9, \mathbf{A}_{10}, L_2(125), U_3(5), U_4(3)$
140	$\mathbf{A}_{140}, L_2(139)$
210	$\mathbf{A}_{210}, \mathbf{A}_{10}, \mathbf{A}_{21}$
252	$\mathbf{A}_{252}, L_2(251)$
280	$\mathbf{A}_{280}, \mathbf{A}_9, L_3(4), U_4(3), J_2$
315	$\mathbf{A}_{315}, S_6(2), J_2$
360	$\mathbf{A}_{360}, L_2(359)$
420	$\mathbf{A}_{420}, L_2(419)$
504	$\mathbf{A}_{504}, L_2(503)$
630	$\mathbf{A}_{630}, \mathbf{A}_{36}$
840	$\mathbf{A}_{840}, \mathbf{A}_9, L_2(839), J_2$

Lemma 6. *There is no non-trivial factorization $G = \mathbf{A}_7\mathbf{S}_n$ with G simple except* $G = \mathbf{A}_{15} = \mathbf{A}_7\mathbf{S}_{13}$.

Proof. Let G be a simple group with a non-trivial factorization $G = \mathbf{A}_7\mathbf{S}_n$ for some natural number n. If G is isomorphic to an alternating group, then by Lemma 2 the only possibility is $G = \mathbf{A}_{15} = \mathbf{A}_7\mathbf{S}_{13}$. Hence we assume G is not an alternating group. As in the proof of Lemmas if C is a maximal subgroup of G containing $\mathbf{S}_n$, then $[M : C]|[\mathbf{A}_7 : \mathbf{A}_7 \cap C] = d$ and so G is represented as a simple primitive group of degree d where d is the index of a proper subgroup of $\mathbf{A}_7$. First we consider simple primitive groups of degree $d \leqslant 1000$ which are listed in Table I. We can exclude the linear groups $L_2(q)$ and the groups $L_3(4), S_4(4)$ and J_2 as by [2] they don't contain the alternating group of degree 7. Therefore we have to examine the groups $S_6(2), S_8(2), O_8^+(2), U_3(5)$ or $U_4(3)$ for appropriate decomposition. If $S_8(2) = \mathbf{A}_7\mathbf{S}_n$, then since $17||S_8(2)|$ we must have $n \geqslant 17$, but in this case we must have $13||S_n|$ which is a contradiction. If $O_8^+(2) = \mathbf{A}_7\mathbf{S}_n$. then order consideration implies $n \geqslant 12$ which is a contradiction because by [2] $O_8^+(2)$ does not contain a subgroup isomorphic to $\mathbf{S}_{12}$. For $S_6(2) = \mathbf{A}_7\mathbf{S}_n$ order consideration yields $n = 8$, but then $|\mathbf{A}_7 \cap \mathbf{S}_8| = 70$ contradicting the fact that $\mathbf{A}_7$ does not have a subgroup of order 70. If $U_3(5) = \mathbf{A}_7\mathbf{S}_n$, then since $5^3||U_5(3)|$ we must have $n \geqslant 10$ which is not possible because, by [2], the group $U_3(5)$ does not contain a subgroup isomorphic to $\mathbf{S}_{10}$. If $U_4(3) = \mathbf{A}_7\mathbf{S}_n$, then order consideration will imply $n \geqslant 9$, which is not possible because, by [2], $U_4(3)$ does not contain a subgroup is isomorphic to $\mathbf{S}_9$.

Secondly we should consider simple primitive groups G of degree $d > 1000$ which can be written as $G = \mathbf{A}_7\mathbf{S}_n$, and these degrees are 1260 and 2520. But by Lemma 4 we know a list of simple groups which possibly have this property. Now case by case examination of these groups, with the same method as used in the proof of Lemma 4, will end to a contradiction. The Lemma is proved now. □

Lemma 7. *Let $H = AB, A \cong \mathbf{A}_7, B \cong \mathbf{A}_n$, be a proper factorization of a group H and $H \ncong A \times B$. Then H is isomorphic to an alternating group $\mathbf{A}_m$ for possible m.*

Proof. Let N be a normal subgroup of H. Since A is a simple group therefore $N \cap A = A$ or 1. If $N \cap A = A$, then $A \subseteq N$ and we will have $H = AB = NB$ and by [15] we must have $H = B$, $H \cong A \times B$ or $H =$ Hol $(\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2)$, and non of them is the case. Therefore $N \cap A = 1$ and similarly $N \cap B = 1$.

Now assume N is a maximal normal subgroup of G. We have $\frac{H}{N} =$

$(\frac{AN}{N})(\frac{BN}{N})$ and $\frac{AN}{N} \cong A \cong \mathbf{A}_7$ and $\frac{BN}{N} \cong B \cong \mathbf{A}_n$, and hence the simple group $\frac{H}{N}$ is the product of $\mathbf{A}_7$ and $\mathbf{A}_n$ and by Theorem 5 we must have $\frac{H}{N} \cong \mathbf{A}_m$ for suitable m. Now by [17] we must have $N = 1$ and $H \cong \mathbf{A}_m$. This completes the proof. □

Now we state and prove our final result.

Theorem 8. *Let G be a group such that $G = AB$, $A \cong \mathbf{A}_7$ and $B \cong \mathbf{S}_n$, $n \geqslant 5$, then one of the following cases occurs:*

(*a*) $G \cong \mathbf{A}_7$,

(*b*) $G \cong \mathbf{S}_n, n \geqslant 7$,

(*c*) $G \cong \mathbf{A}_7 \times \mathbf{S}_n$,

(*d*) $G \cong \mathbf{A}_{15} = \mathbf{A}_7\mathbf{S}_n$,

(*e*) $G \cong \mathbf{S}_{n+1} = \mathbf{A}_7\mathbf{S}_n$, $n = 14, 20, 34, 41, 69, 104, 119, 125, 139, 219,$ 251, 279, 314, 359, 419, 503, 629, 839, 1259 *or* 2519,

(*f*) $G \cong \mathbf{S}_{15} = \mathbf{A}_7\mathbf{S}_{13}$ *or* $G \cong \mathbf{A}_{15} \times \mathbf{Z}_2 = \mathbf{A}_7\mathbf{S}_{13}$,

(*g*) $G \cong (\mathbf{A}_7 \times \mathbf{A}_7)\langle\tau\rangle$, τ *an automorphism of order* 2 *of* $\mathbf{A}_7$ *and* $\mathbf{A}_7 \times \mathbf{A}_7$ *is the minimal normal subgroup of* G,

(*h*) $G \cong (\mathbf{A}_7 \times \mathbf{A}_n)\langle\tau\rangle$, $n \neq 7$, *where* τ *acts as an automorphism of order* 2 *on both factors* (*in this case* $\mathbf{A}_7$ *or* $\mathbf{A}_n$ *is the minimal normal subgroup of* G).

Proof. We will use Lemma 4 of [3]. Let M be a minimal normal subgroup of $G = AB$, $G \ncong A \times B$, where $A \cong \mathbf{A}_7$ and $B \cong \mathbf{S}_n$, $n \geqslant 5$. Then we have the following possibilities.

(*i*). $M = G = AB$ is simple . In this case by Lemma 6 only $G \cong \mathbf{A}_{15} = \mathbf{A}_7\mathbf{S}_{13}$ is possible which is case (d) of our Theorem.

(*ii*). $G = MB, M = AB'$ is simple, where $B' \cong \mathbf{A}_n$ denotes the commutator subgroup of B. If $M = A$ or B', then we get trivial factorizations which are case (a) and case (b) of our Theorem. Therefore we assume $M = AB', A \cong \mathbf{A}_7, B' \cong \mathbf{A}_n$, is a simple group with non-trivial factorization. By Lemma 5 we must have either $M \cong \mathbf{A}_{15} = \mathbf{A}_7\mathbf{A}_{13}$ or $M \cong \mathbf{A}_{n+1} = \mathbf{A}_7\mathbf{A}_n$ for the n's specified in the Theorem. In the latter case $[G : M] = 2$, hence $G = M\langle\tau\rangle$ where τ is an element of order 2 in $\mathbf{S}_n \backslash \mathbf{A}_n$. Now in the latter case the same reasoning as used in the proof of Theorem 4 in [16] yields $G = \mathbf{S}_{n+1}$, which is the case (e) of the Theorem. In the case of $M \cong \mathbf{A}_{15} = \mathbf{A}_7\mathbf{A}_{13}$ if τ acts as an inner automorphism on M we obtain $\mathbf{A}_{15} \times \mathbf{Z}_2 \cong \mathbf{A}_7\mathbf{S}_{13}$ and if τ acts as an outer automorphism on M we obtain $\mathbf{S}_{15} \cong \mathbf{A}_7\mathbf{S}_{13}$ which are included in case (f) of the Theorem.

(*iii*). $G = MB, B \cong \mathbf{S}_n, M \cong \mathbf{A}_7 \times \mathbf{A}_7$. In this case $n = 7$ and therefore $G \cong (\mathbf{A}_7 \times \mathbf{A}_7)\langle\tau\rangle$, with τ an automorphism of order 2 and $\mathbf{A}_7 \times \mathbf{A}_7$ a minimal normal subgroup of G, which is the case (g) of the Theorem.

(*iv*). $M = A$ or B', $AB' \cong A \times B' \cong \mathbf{A}_7 \times \mathbf{A}_n$, $[G : AB'] = 2$. In this case $G = (\mathbf{A}_7 \times \mathbf{A}_n)\langle\tau\rangle$ where τ acts as an automorphism of order 2 on both factors with $\mathbf{A}_7$ or $\mathbf{A}_n$ as the minimal normal subgroup of G. This is the case (h) in our Theorem.

(*v*). Finally we must have $M \cap A = M \cap B = 1$, $|M|\,|[A : A \cap B]$ and $|M|\,|[B : A \cap B]$, furthermore $|M|.|A \cap B| = |\frac{AM}{M} \cap \frac{BM}{M}|$. We will show that no new possibilities arise in this case and the proof of our Theorem will be completed. M is isomorphic to the direct product of isomorphic simple groups. From $|M|\,|[A : A \cap B]$ it follows that if M is abelian, then $|M| = 2$, 3, 4, 5, 7, 8, 9 and if M is non-abelian, then $M \cong \mathbf{A}_5, \mathbf{A}_6, L_2(7)$ or $\mathbf{A}_7$.

Now the groups A, B and G act on M by conjugation with the kernels $C_A(M) \trianglelefteq A, C_B(M) \trianglelefteq B$ and $C = C_G(M)$ respectively. If $C_A(M) = 1$, then A would be isomorphic to a subgroup of $\mathrm{Aut}(M)$ and by the structure of M the only possibility is $M = A$ which has been considered above. Therefore $C_A(M) = A$ which implies $A \leqslant C_G(M) = C$. Now $C_B(M) = 1, B'$ or B because $B \cong \mathbf{S}_n, n \geqslant 5$. Since $A \leqslant C$ we must have $G = AB = CB$ and hence $|A||C \cap B| = |C||A \cap B|$. We have $C \cap B = C_B(M)$, and if $C_B(M) = 1$, then $A = C$ and $A \cap B = 1$. Since $C \trianglelefteq G$ we consider the group $H = AB' = CB'$ where A and B' are simple alternating groups and $[G : H] = 2$. Now by Lemma 6 either this factorization is not proper or $H \cong A \times B'$ or H isomorphic to an alternating group. If the factorization is not proper, then either $A \subseteq B'$ or $B' \subseteq A$ contradicting $A \cap B = 1$. The other cases force G to be a symmetric group which is considered above. If $C_B(M) = B$, then $B \leqslant C_G(M)$ and we will get $M \leqslant Z(G)$.

Finally we will assume $C_B(M) = B \cap C = B'$. Now from $G = AB = CB$ we get $[G : C] = 2$. We know that $[A \cap B : A \cap B'] = 1$ or 2. If $[A \cap B : A \cap B'] = 2$, then $|AB'| = |AB| = |G|$ implying $G = AB' \subseteq C$ or $G = C$ which is a contradiction. Therefore $A \cap B = A \cap B'$ from which we obtain $|AB'| = \frac{1}{2}|G| = |C|$, hence $C = AB'$.

Our arguments so far show that either $M \leqslant Z(G)$ or $C = AB'$ where $A \cong \mathbf{A}_7$ and $B' \cong \mathbf{A}_n$. If $C = AB'$ then the factorization must be proper because $M \trianglelefteq C$ and A and B' are simple groups, therefore by Lemma 7 either $C \cong A \times B'$ or $C \cong \mathbf{A}_m$ for suitable m. If $C \cong A \times B'$, then as $[G : C] = 2$ we will obtain case (f) again. The case $C \cong \mathbf{A}_m$ can not happen because $M \trianglelefteq C$. Now we will deal with the case $M \leqslant Z(G)$. We have $\frac{G}{M} \cong (\frac{AM}{M})(\frac{BM}{M})$ with $\frac{AM}{M} \cong A \cong \mathbf{A}_7$ and $\frac{BM}{M} \cong B \cong \mathbf{S}_n$ and by induction either $G = MB \cong M \times B$ or $\frac{G}{M} = \mathbf{S}_{n+1}$ for n's as in case (e) of the Theorem. Now the same reasoning as used in the proof of Theorem 4 in [16] leads to a contradiction. The Theorem is proved now. □

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