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A note on the Akivis algebra of a smooth hyporeductive loop

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Abstract

Using the fundamental tensors of a smooth loop and the differential geometric characterization of smooth hyporeductive loops, the Akivis operations of a local smooth hyporeductive loop are expressed through the two binary and the one ternary operations of the hyporeductive triple algebra (h.t.a.) associated with the given hyporeductive loop. Those Akivis operations are also given in terms of Lie brackets of a Lie algebra of vector fields with the hyporeductive decomposition which generalizes the reductive decomposition of Lie algebras. A nontrivial real two-dimensional h.t.a. is presented.

1. Introduction

A *quasigroup* is a set Q with a binary operation of multiplication denoted by $\circ$ or juxtaposition such that the knowledge of any two of x, y, z in the equation $x \circ y = z$ uniquely specifies the third. A *loop* is a quasigroup $(Q, \circ)$ with a two-sided identity e. In the case when Q is a neighborhood of the fixed point e in a smooth (real finite-dimensional) manifold and the operation $\circ$ is a smooth function $Q \times Q \to Q$, then $(Q, \circ)$ is called a *local smooth* loop.

As for Lie groups, an infinitesimal theory for smooth quasigroups is considered by M. A. Akivis (see [1], [2], [3]). If $(Q, \circ)$ is a smooth loop then in a sufficiently small neighborhood of e, the binary operation $\circ$ has the following Taylor expansion [1]:

$$(x \circ y)^i = x^i + y^i + \tau^i_{jk} x^j y^k + \mu^i_{jkl} x^j x^k y^l + \nu^i_{jkl} x^j y^k y^l + ...$$

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where the quantities μ^i_{jkl} and ν^i_{jkl} have the properties $\mu^i_{jkl} = \mu^i_{kjl}$ and $\nu^i_{jkl} = \nu^i_{jlk}$. The so-called *fundamental tensors* α^i_{jk}, β^i_{ljk} of the given smooth loop $(Q, \circ, e)$ are defined as follows:

$$\alpha^i_{jk} = \frac{1}{2}\left(\tau^i_{jk} - \tau^i_{kj}\right), \qquad \beta^i_{ljk} = 2\mu^i_{jkl} - 2\nu^i_{jkl} + \alpha^s_{jk}\alpha^i_{sl} - \alpha^i_{js}\alpha^s_{kl}\ .$$

The *commutator* and the *associator* at the identity e of $(Q, \circ, e)$ are expressed in terms of the fundamental tensors α^i_{jk} and β^i_{ljk} as follows:

$$(x \circ y)^i - (y \circ x)^i = 2\alpha^i_{jk} x^i y^k + o(\rho^2),$$

$$[(x \circ y) \circ z]^i - [x \circ (y \circ z)]^i = \beta^i_{ljk} x^l y^j z^k + o(\rho^3),$$

where $\rho = max(|x^i|, |y^i|)$.

Therefore the tensor α^i_{jk} (respectively β^i_{ljk}) characterizes the principal part of the deviation degree from commutativity (respectively associativity) of the loop $(Q, \circ, e)$. It should be noted that these expressions of the commutator and the associator hold in any smooth loop (more precisely, in a sufficiently small neighborhood of any element of that loop) and the tensors α^i_{jk} and β^i_{ljk} are defined at any point of the manifold Q (cf. [1]). For $\alpha^i_{jk} = 0$ and $\beta^i_{ljk} = 0$, the loop $(Q, \circ, e)$ becomes locally an abelian group and for $\beta^i_{ljk} = 0$ it is a local Lie group.

Using the fundamental tensors, the tangent space T_eQ may be provided with a structure of a binary-ternary algebra (the *tangent algebra of the smooth loop*) if define

$$(X \diamond Y)^i = 2\alpha^i_{jk} X^j Y^k\,, \qquad [X, Y, Z]^i = \beta^i_{ljk} X^l Y^j Z^k\,, \tag{1}$$

for all $X, Y, Z \in T_eQ$. It is shown [2] that $\diamond$ and $[-,-,-]$ satisfy the following identities

$$X \diamond X = 0, \tag{2}$$

$$[X, X, X] = 0, \tag{3}$$

$$\sigma\{XY \diamond Z\} = \sigma\{[X, Y, Z]\} - \sigma\{[Y, X, Z]\}, \tag{4}$$

where σ denotes the cyclic sum with respect to X, Y, Z and juxtaposition is used to reduce the number of brackets, that is $XY \diamond Z$ means $(X \diamond Y) \diamond Z$. Following [4], a (real finite-dimensional) vector space is called an *Akivis algebra* if it carries a bilinear operation $\diamond$ and a trilinear operation $[-,-,-]$ satisfying the identities $(2) - (4)$. The identity (4) is known as the *Akivis* *identity.* Hereafter we shall refer to the operations $\diamond$ and $[-,-,-]$ as defined in (1) as to the Akivis operations.

We will be interested in the situation when a smooth loop $(Q,\circ,e)$ is related to an affine connection space (Q,∇). In [8], [11] a construction of a loop centered at a fixed point e of (Q,∇) is given. Such a loop is called the *geodesic loop* of (Q,∇) at the point e (it turns out that e is the two-sided identity element of that loop). Moreover the geodesic loop operation $\circ$ is supplemented by an unary multiplication $(t,x)\mapsto tx$ of any element $x \in (Q,\circ,e)$ by a real scalar t, giving rise to the concept of a *geodesic odule* (see [11]). The identity

$$((t+u)x)\circ y = tx\circ(ux\circ y) \tag{5}$$

is called the *left monoalternative property*, where t and u are real numbers; likewise is defined the *right* monoalternative property. The right monoalternative property plays a key role in the differential geometric theory of some classes of loops. It turns out that (see [3]) for a geodesic loop $(Q,\circ,e)$ of an affine connection space (Q,∇), its fundamental tensors are expressed in terms of the torsion and curvature of the space (Q,∇) as follows:

$$\alpha^i_{jk} = -\frac{1}{2}\,T^i_{jk}(e), \qquad \beta^i_{ljk} = \frac{1}{2}\left(R^i_{l,jk} - \nabla_k T^i_{lj}\right)(e)\,. \tag{6}$$

Accordingly the Akivis operations of $(Q,\circ,e)$ are also expressed in terms of the torsion and curvature of (Q,∇).

For the general theory of specific classes of smooth loops it is sometimes convenient to give the explicit form of their Akivis operations. This is easy, according to (6), whenever a suitable differential geometric theory is built for a given class of smooth loops. The tangent algebra to a smooth *Bol loop* is called a *Bol algebra* (see [10], [15]) while the tangent algebra to a smooth *homogeneous loop* is called a *Lie triple algebra* (see [9], [12]). One observes that a Bol algebra (resp. a Lie triple algebra) is an Akivis algebra of a smooth Bol loop (resp. a smooth homogeneous loop) with additional conditions.

In [5] the Lie triple algebra of a smooth homogeneous loop was related to its Akivis algebra. It is our purpose in this note to do the same for hyporeductive loops since they are a generalization both of Bol loops and homogeneous loops ([13], [14]). Here the approach is geometric in the sense of (6) (see Section 2) and algebraic meaning that the Akivis operations of a smooth hyporeductive loop are expressed in terms of the Lie brackets of a Lie algebra satisfying some specific conditions (see Section 3). We wonder whether the method of the algebraic calculus of formal power series, developed in [5] for the case of smooth homogeneous loops, could be applied to smooth hyporeductive loops.

2. Tangent algebras to smooth hyporeductive loops: hyporeductive triple algebras (h.t.a.)

A loop $(Q, \circ, e)$ is said *left hypospecial* if there exists $b(x, y) \in Q$ with $x, y \in Q$ such that $b(x, e) = e = b(e, x)$ and the mapping $\phi(x, y) = L_{b(x,y)} l_{x,y}$ has the property

$$\phi L_z \phi^{-1} = L_{(\phi z)/b(x,y)}$$

where $L_u v = u \circ v$, $l_{u,v} = L^{-1}_{u \circ v} L_u L_v$ and / denotes the right division in $(Q, \circ, e)$. A *left hyporeductive loop* is a left hypospecial loop with the left monoalternative property (5). Similarly is defined a *right* hyporeductive loop. An infinitesimal theory for smooth hyporeductive loops is initiated by L.V. Sabinin in [13], [14], where he constructed a tangent algebra for such loops that is called a *hyporeductive algebra*. It should be noted that there is a one-to-one correspondance between hyporeductive algebras and smooth hyporeductive loops. In [6] (see also [7]) a differential geometric study for smooth hyporeductive loops is suggested. In particular it is shown that a smooth hyporeductive loop $(Q, \circ, e)$ can locally be seen as an affine connection space (Q, ∇) with zero curvature satisfying the following structure equations

$$d\omega^i = \frac{1}{2}\ T^i_{jk}\ \omega^j\ \wedge\ \omega^k, \tag{7}$$

$$dT^i_{jk} = \left(T^i_{ls}(T^s_{jk} + a^s_{jk}) - r^i_{l,jk}\right)\omega^l, \tag{8}$$

where a^s_{jk} and $r^i_{l,jk}$ are constants and $a^i_{jk} = -a^i_{kj}$, $r^i_{l,jk} = -r^i_{l,kj}$. Moreover, the geodesic loop at a fixed point of an affine connection space with structure equations (7), (8) is a (right) hyporeductive loop. Using the known differential geometric techniques we obtained [6] that the integrability criteria of (7), (8) constitute the determining identities of a hyporeductive algebra if we set

$$(X.Y)^i = a^i_{jk} X^j Y^k\,, \quad (X * Y)^i = (-T^i_{jk}(e) - a^i_{jk}) X^j Y^k\,,$$
$$< Z; X, Y >^i = -r^i_{l,jk} X^j Y^k Z^k\,, \tag{9}$$

for $X, Y, Z \in T_e Q$. The operations $*$, . and $< -; -, - >$ are linked by a certain set of identities ([6], [7], [14]). They are as follows:

$$\sigma\{\ \xi\,.\,(\eta\,.\,\zeta) - <\xi;\eta,\zeta>\} = 0\,,$$

$$\sigma\{\ \zeta * (\xi\,.\,\eta)\,\} = 0\,,$$

$$\sigma\{<\theta;\zeta,\ \xi\,.\,\eta>\} = 0\,,$$

$$\begin{aligned}&\kappa. <\zeta;\xi,\eta> -\zeta. <\kappa;\xi,\eta> + <\zeta.\kappa;\xi,\eta> =\\ &\quad = <\xi*\eta;\zeta,\kappa> - <\zeta*\kappa;\xi,\eta> +\zeta* <\kappa;\xi,\eta> -\kappa* <\zeta;\xi,\eta> +\\ &\quad + (\xi*\eta)*(\zeta*\kappa) + (\xi*\eta).(\zeta*\kappa)\,,\end{aligned}$$

$$\begin{aligned}&\chi.(\kappa. <\zeta;\xi,\eta> -\zeta. <\kappa;\xi,\eta> + <\zeta.\kappa;\xi,\eta>) +\\ &\quad + << \chi;\xi,\eta>;\zeta,\kappa> - << \chi;\zeta,\kappa>;\xi,\eta> +\\ &\quad + <\chi;\zeta,<\kappa;\xi,\eta>> - <\chi;\kappa,<\zeta;\xi,\eta>> = 0\,,\end{aligned}$$

$$\chi * (\kappa\,. <\zeta;\xi,\eta> -\zeta\,. <\kappa;\xi,\eta> + <\zeta\,.\,\kappa;\xi,\eta>)\ = 0\,,$$

$$<\theta;\chi,\ \kappa\,. <\zeta;\xi,\eta> -\zeta\,. <\kappa;\xi,\eta> + <\zeta\,.\,\kappa;\xi,\eta>> = 0\,,$$

$$\begin{aligned}&\kappa\,. <\zeta;\xi,\eta> -\zeta\,. <\kappa;\xi,\eta> + <\zeta\,.\,\kappa;\xi,\eta> +\\ &\quad + \eta\,. <\xi;\zeta,\kappa> -\xi\,. <\eta;\zeta,\kappa> + <\xi\,.\,\eta;\zeta,\kappa> = 0\,,\end{aligned}$$

$$\zeta* <\kappa;\xi,\eta> -\kappa* <\zeta;\xi,\eta> +\xi* <\eta;\zeta,\kappa> +\eta* <\xi;\zeta,\kappa> = 0\,,$$

$$\begin{aligned}&\Sigma\{< (<\xi\,.\,\eta;\zeta,\kappa> +\eta\,. <\xi;\zeta,\kappa> -\xi\,. <\eta;\zeta,\kappa>);\lambda,\mu> +\\ &\quad + <\lambda.\mu;<\eta;\zeta,\kappa>,\xi> +\mu\,. <\lambda;<\eta;\zeta,\kappa>,\xi> -\\ &\quad -\lambda\,. <\mu;<\eta;\zeta,\kappa>,\xi> -(<\lambda.\mu;<\xi;\zeta,\kappa>,\eta> +\\ &\quad +\mu\,. <\lambda;<\xi;\zeta,\kappa>,\eta> -\lambda <\mu;<\xi;\zeta,\kappa>,\eta>)\,\} = 0\,,\end{aligned}$$

$$\begin{aligned}&\Sigma\{\ (<\mu;<\eta;\zeta,\kappa>,\xi> - <\mu;<\xi;\zeta,\kappa>,\eta>) * \lambda +\\ &\quad +(<\lambda;<\xi;\zeta,\kappa>,\eta> - <\lambda;<\eta;\zeta,\kappa>,\xi>) * \mu\} = 0\,,\end{aligned}$$

$$\begin{aligned}&\Sigma\{<\theta;(<\mu;<\eta;\zeta,\kappa>,\xi> - <\mu;<\xi;\zeta,\kappa>,\eta>),\lambda> +\\ &\quad + <\theta;(<\lambda;<\xi;\zeta,\kappa>,\eta> - <\lambda;<\eta;\zeta,\kappa>,\xi>),\mu>\,\} = 0\,,\end{aligned}$$

where σ denotes the cyclic sum with respect to ξ,η,ζ and Σ the one with respect to pairs (ξ,η), (ζ,κ), (λ,μ). Any (real finite-dimensional) vector space with two anticommutative bilinear operations and one trilinear, skew-symmetric with respect to the two last variables, operation satisfying those identities is called an *abstract hyporeductive triple algebra* (h.t.a. for short).

It is worthy of note that such identities are obtained [14] if work out the Jacobi identities of the Lie algebra of vector fields enveloping the given hyporeductive algebra and satisfying some specific conditions.

We give an example of a nontrivial real 2-dimensional h.t.a.

Example. Let m be a 2-dimensional algebra over the field of real numbers with basis $\{u, v\}$. Define on m the following operations:

$$u * v = u, \quad u.v = v, \quad < u; u, v >= v, \quad < v; u, v >= 0$$

with the symmetries $u * u = 0 = u.u$, $< t; u, u >= 0$, where $t = u$ or v. Then it could be checked that m is a nontrivial h.t.a. that is not a Bol algebra nor a Lie triple algebra. □

We have the following theorem whose proof is somewhat elementary in view of structure equations (7), (8) above.

Theorem 1. *Let* $(Q, \circ, e)$ *be a given smooth local hyporeductive loop and* $(T_eQ, ., *, < -; -, - >)$ *be the corresponding (up to an isomorphism) h.t.a. Then the Akivis operations* $\diamond$ *and* $[-, -, -]$ *of* $(Q, \circ, e)$ *are linked with* $.$, $*$, $< -; -, - >$ *as follows:*

(i) $X \diamond Y = X.Y + X * Y$,

(ii) $[X, Y, Z] = -\frac{1}{2}\big(< Z; X, Y > + Z \diamond (X * Y)\big)$

for all $X, Y, Z \in T_eQ$.

Proof. Let $(X * Y)^i = b^i_{jk} X^j Y^k$, that is $b^i_{jk} = -T^i_{jk}(e) - a^i_{jk}$. Then from (1), (6) and (9) we get (i).

Next, from (8) we know that $-r^i_{l,jk} = (\nabla_l T^i_{jk} + T^i_{ls} b^s_{jk})(e)$. Therefore, since $< Z; X, Y >^i = -r^i_{l,jk} X^j Y^k Z^l = ((\nabla_l T^i_{jk} + T^i_{ls} b^s_{jk})(e)) X^j Y^k Z^l$, from (1), (6) we get (ii) (recall that $R^i_{l,jk} = 0$). □

Remark 1. (a) Using (i) the Akivis operation $[X, Y, Z]$ in (ii) can also be expressed by $\diamond$ and $.$ as follows:

(iii) $[X, Y, Z] = -\frac{1}{2}\big(< Z; X, Y > + Z \diamond (X \diamond Y) - Z \diamond (X.Y)\big)$.

(b) From (i) and (ii) we see that if $X.Y = 0$ for all $X, Y \in T_eQ$, then $X \diamond Y = X * Y$ and $[X, Y, Z] = (-1/2)(< Z; X, Y > + Z \diamond (X \diamond Y))$ and we are in the situation of Bol algebras (see [10], [15]). Likewise for $X * Y = 0$ for all $X, Y \in T_eQ$ we get $X \diamond Y = X.Y$ and $[X, Y, Z] = (-1/2) < Z; X, Y >$

and we have the case of Lie triple algebras [5].

With the remarks above one could think of the operation . (resp. $*$) as of a deviation degree of a h.t.a. from a Bol algebra (resp. a Lie triple algebra). Although the transformations are somewhat tedious and lengthy, one could write down the determining identities of a h.t.a. in terms of the Akivis operations $\diamond$, $[-,-,-]$ and the operation . (or $*$).

3. An alternative approach

Let m be a (real finite-dimensional) vector space of covariantly constant vector fields of an affine connection space with zero curvature (Q,∇) and $e \in Q$ a fixed point. Let g be the Lie algebra of vector fields generated by m and such that $g = m + [m,m]$ (here $[m,m]$ denotes the subset of g generated by all $[X,Y]$ with $X,Y \in m$) and let h be the subalgebra of g defined by $h = \{X \in g : X(e) = o\}$. Then

$$g = m \dot{+} h \tag{10}$$

(direct sum of vector spaces; see [16]). Additionally let assume that there exists in g a subspace n such that

$$g = m \dot{+} n \;\; \text{(direct sum of subspaces)}, \tag{11}$$

$$[n,m] \subset m. \tag{12}$$

A pair (g,h) with the decomposition (10) such that (11), (12) hold is said *hyporeductive* ([13], [14]).

Proposition 2. *The hyporeductive pair (g,h) with conditions* (10) – (12) *induces on m a structure of a h.t.a.*

Proof. If $X,Y \in m$ then $[X,Y] \in g$ and the decomposition (11) induces a binary operation, say ., on m

$$X_i.X_j = [X_i,X_j]^n_m \tag{13}$$

(here and in the sequel $[X,Y]^w_v$ denotes the projection on v parallely w), where X_s $(s = 1,...,l,\ l = dim\ m)$ constitute a basis of m. We denote by $D(X_i,X_j) = [X_i,X_j] - X_i.X_j$ $(i \neq j)$ the basis elements of n. Further, using (10) and (12), we define on m a binary operation

$$X_i * X_j = [X_i,X_j]^h_m - X_i.X_j \tag{14}$$

and a ternary operation

$$< X_k; X_i, X_j >= -[X_k, D(X_i, X_j)]. \tag{15}$$

Now using the procedure described in [13], [14] one could write down the Jacobi identities in g with respect to the set $\{X_\alpha,\ D(X_\beta, X_\gamma)\}$ of basis elements. This in turn leads to the set of determining identities of a h.t.a. so that $(m, ., *, < -; -, - >)$ becomes a h.t.a. of vector fields. □

Above we considered m as the linear space of covariantly constant vector fields on an affine connection manifold (Q, ∇) with zero curvature; this is intended for a relation with local smooth loops with the right monoalternative property and, further, with local smooth hyporeductive loops. Specifically we mean the following. If e is a fixed point on (Q, ∇), then m may be identified with the tangent space T_eQ and therefore, in the case when m is a h.t.a., T_eQ is a h.t.a. Moreover, since (Q, ∇) has zero curvature, the geodesic loop $(Q, ., e)$ of (Q, ∇) centered at the point e has the right monoalternative property [15] and, if T_eQ is a h.t.a., $(Q, ., e)$ has the (right) hypospecial property ([6], [7]). Thus we get a (right) hyporeductive geodesic loop $(Q, ., e)$ with T_eQ as its tangent algebra. But then from (6), (8), (9), (13), (14) and (15) we see that its Akivis operations have the following expressions through the Lie brackets of g:

$$X \diamond Y = [X, Y]^h_m\,, \tag{16}$$

$$[X, Y, Z] = \frac{1}{2}\big([Z, [X, Y]^m_n\,] - [Z, [X, Y]^h_m\,]^h_m + [Z, [X, Y]^n_m\,]^h_m \big). \tag{17}$$

Thus we have the following

Theorem 3. *Let g be a real finite-dimensional Lie algebra generated by a subspace of vector fields and let (g, h) be the hyporeductive pair with the hyporeductive decomposition* (10) − (12)*. Then the Akivis operations of the local smooth hyporeductive loop corresponding (up to an isomorphism) to the h.t.a. in g are expressed as in* (16)*,* (17)*.* □

One observes that we have worked with an h.t.a. of covariantly constant vector fields in a smooth affine connection space with zero curvature. But one can also start from a structure of abstract h.t.a. given on the tangent space W to a fixed point e of that connection space and then extend this structure to the one of a h.t.a. of covariantly constant vector fields V through the identification of W with $V = \{X_\xi : X_\xi(e) = \xi \in W\}$.

We conclude with the following remarks in full analogy with the ones we done in Section 2.

Remark 2. (a) We get the Bol theory ([10], [15]) if $n = [m,m]$, i.e. $[X,Y]^n_m = 0$ in which case we have $g = m \dot{+} h$, and $[[m,m],m] \subset m$ so that (17) reads

$$[X,Y,Z] = \frac{1}{2}\big(\,[Z,[X,Y]] - [Z,[X,Y]^h_m\,]^h_m\,\big)$$

((16) remains the same).

(b) The hyporeductive pair (g,h) (see (10) – (12)) becomes reductive when n coincides with h, i.e. $g = m \dot{+} h$ and $[h,m] \subset m$. Therefore the Akivis operation (17) reduces to the following

$$[X,Y,Z] = \frac{1}{2}\,[Z,[X,Y]^m_h\,]$$

(again (16) remains the same) and one observes that we get precisely the Akivis operations of the local smooth loop associated with the corresponding reductive decomposition ([5]).

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