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A note on trimedial quasigroups

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Abstract

The purpose of this brief note is to sharpen a result of Kepka [2] [3] about the axiomatization of the variety of trimedial quasigroups.

A groupoid is *medial* if it satisfies the identity $wx \cdot yz = wy \cdot xz$. A groupoid is *trimedial* if every subgroupoid generated by 3 elements is medial. Medial groupoids and quasigroups have also been called abelian, entropic, and other names, while trimedial quasigroups have also been called triabelian, terentropic, etc. (See [1], especially p. 120, for further background.)

In [2] [3], Kepka showed that a quasigroup satisfying the following three identities must be trimedial.

$$xx \cdot yz = xy \cdot xz \tag{1}$$

$$yz \cdot xx = yx \cdot zx \tag{2}$$

$$(x \cdot xx) \cdot uv = xu \cdot (xx \cdot v) \tag{3}$$

The converse is trivial, and so these three identities characterize trimedial quasigroups. Here, we show that, in fact, (2) and (3) are sufficient to characterize this variety (as a subvariety of the variety of quasigroups). Note that in the theorem we only assume left cancellation, not the full strength of the quasigroup axioms.

Theorem. *A groupoid with left cancellation which satisfies* (2) *and* (3) *must also satisfy* (1).

Proof. $(x{\cdot}xz)(xx{\cdot}yz) = (x{\cdot}xx)(xz{\cdot}yz) = (x{\cdot}xx)(xy{\cdot}zz) = (x{\cdot}xy)(xx{\cdot}zz) = (x \cdot xy)(xz \cdot xz) = (x \cdot xz)(xy \cdot xz)$. Now cancel. □

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In [2] [3], Kepka showed that the following single identity characterizes trimedial quasigroups:

$$[(xx \cdot yz)]\{[xy \cdot uu][(w \cdot ww) \cdot zv]\} = [(xy \cdot xz)]\{[xu \cdot yu][wz \cdot (ww \cdot v)]\}.$$

Using the theorem we can sharpen this.

Corollary. *The following identity characterizes trimedial quasigroups*:

$$[(xy \cdot uu)][(w \cdot ww) \cdot zv] = [(xu \cdot yu)][wz \cdot (ww \cdot v)].$$

Proof. To obtain (2) set $z = ww$ and use right cancellation. To obtain (3) set $y = u$ and use left cancellation. □

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