Abel-Grassmann’s bands

Petar V. Protić and Nebojša Stevanović

Abstract

Abel-Grassmann’s groupoids or shortly AG-groupoids have been considered in a number of papers, although under the different names. In some papers they are named LA-semigroups [3] in others left invertive groupoids [2]. In this paper we deal with AG-bands, i.e., AG-groupoids whose all elements are idempotents. We introduce a few congruence relations on AG-band and consider decompositions of Abel-Grassmann’s bands induced by these congruences. We also give the natural partial order on Abel-Grassmann’s band.

1. Introduction

A groupoid $S$ in which the following

$$ (\forall a, b, c \in S) \quad ab \cdot c = cb \cdot a, \quad (1) $$

is true is called an Abel-Grassmann’s groupoid, [5]. It is easy to verify that in every AG-groupoid the medial law $ab \cdot cd = ac \cdot bd$ holds.

Abell-Grassmann’s groupoids are not associative in general, however they have a close relation with semigroups and with commutative structures. Introducing a new operation on AG-groupoid makes it a commutative semigroup. On the other hand introducing a new operation on a commutative inverse semigroup turns it into an AG-groupoid.

Abel-Grassmann’s groupoid satisfying $(\forall a, b, c \in S) \ ab \cdot c = b \cdot ca$ (called weak associative law in [4]) is an AG*-groupoid. It is easy to prove that any AG*-groupoid satisfies the permutation identity of a next type

$$ a_1a_2 \cdot a_3a_4 = a_{\pi(1)}a_{\pi(2)} \cdot a_{\pi(3)}a_{\pi(4)}, $$

2000 Mathematics Subject Classification: 20N02
Keywords: AG-groupoid, antirectangular AG-band, AG-band decompositions
Supported by Grant 1379 of Ministry of Science through Math. Inst.SANU
where \( \pi \) is any permutation on a set \( \{1, 2, 3, 4\} \), \([5]\).

Let \((S, \cdot)\) be \(AG\)-groupoid, \(a \in S\) be a fixed element. We can define the "sandwich" operation on \(S\) as follows:

\[
x \circ y = xa \cdot y, \quad x, y \in S.
\]

It is easy to verify that \(x \circ y = y \circ x\) for any \(x, y \in S\), in other words \((S, \circ)\) is a commutative groupoid. If \(S\) is \(AG^*\)-groupoid and \(x, y, z \in S\) are arbitrary elements, then

\[
(x \circ y) \circ z = ((xa \cdot y)a)z = za \cdot (xa \cdot y)
\]

and

\[
x \circ (y \circ z) = xa \cdot (y \circ z) = xa \cdot (ya \cdot z) = za \cdot (ya \cdot x) = za \cdot (xa \cdot y),
\]

whence \((x \circ y) \circ z = x \circ (y \circ z)\). Consequently \((S, \circ)\) is a commutative semigroup.

Let \(S\) be the commutative inverse semigroup. We define a new operation on \(S\) as follows:

\[
a \bullet b = ba^{-1}, \quad a, b \in S.
\]

It has been shown in \([3]\) that \((S, \bullet)\) is Abel-Grassmann’s groupoid. Connections mentioned above makes \(AG\)-groupoid to be among the most interesting nonassociative structures. Same as in Semigroup Theory bands and band decompositions appears as the most fruitful methods for research on \(AG\)-groupoids.

If in \(AG\)-groupoid \(S\) every element is an idempotent, then \(S\) is an \(AG\)-band.

An \(AG\)-groupoid \(S\) is an \(AG\)-band \(Y\) of \(AG\)-groupoids \(S_\alpha\) if

\[
S = \bigcup_{\alpha \in Y} S_\alpha,
\]

\(Y\) is an \(AG\)-band, \(S_\alpha \cap S_\beta = \emptyset\) for \(\alpha, \beta \in Y, \alpha \neq \beta\) and \(S_\alpha S_\beta \subseteq S_{\alpha \beta}\).

A congruence \(\rho\) on \(S\) is called band congruence if \(S/\rho\) is a band.

2. Some decompositions of \(AG\)-bands

Let \(S\) be a semigroup and for each \(a \in S\), \(a^2 = a\). That is, let \(S\) be an associative band. If for all \(a, b \in S\), \(ab = ba\), then \(S\) is a semilattice. If for all \(a, b \in S\), \(a = aba\), then \(S\) is the rectangular band. It is a well known
result in Semigroup Theory that the associative band \( S \) is a semilattice of rectangular bands. It is not hard to prove that a commutative \( AG \)-band is a semilattice.

Let us now introduce the following notion.

**Definition 2.1.** Let \( S \) be an \( AG \)-band, we say that \( S \) is an **antirectangular \( AG \)-band** if for every \( a, b \in S \), \( a = ba \cdot b \).

Let us remark that in that case it holds

\[
a = ba \cdot b = ba \cdot bb = bb \cdot ab = b \cdot ab.
\]

(2)

From above it follows that each antirectangular \( AG \)-band is a quasigroup.

**Example 2.1.** Let a groupoid \( S \) be a given by the following table.

<table>
<thead>
<tr>
<th>( \cdot )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

Then \( S \) is an antirectangular \( AG \)-band and a quasigroup. Let us remark that \( S \) is the unique \( AG \)-band of order 4 and we shall see below that it appears frequently in band decompositions both as an \( AG \)-band into which other bands can be decomposed and like a component. For this reasons from now on we shall call this band \( Traka \) 4 or simply \( T4 \). We also remark that nonassociative \( AG \)-bands of order \( \leq 3 \) do not exist.

An \( AG \)-band is **anticommutative** if for all \( a, b \in S \), \( ab = ba \) impies that \( a = b \).

**Lemma 2.1.** Every antirectangular \( AG \)-band is anticommutative.

*Proof.* Let \( S \) be an antirectangular band, \( a, b \in S \) and \( ab = ba \). Then

\[
a = ba \cdot b = ab \cdot b = bb \cdot a = ba = ab = aa \cdot b = ba \cdot a = ab \cdot a = b.
\]

\( \square \)

**Theorem 2.1.** If \( S \) is an \( AG \)-band, then \( S \) is an \( AG \)-band \( Y \) of, in general case nontrivial, antirectangular \( AG \)-bands \( S_{\alpha} \), \( \alpha \in Y \).

*Proof.* Let \( S \) be an \( AG \)-band. Then we define the relation \( \rho \) on \( S \) as

\[
apb \iff a = ba \cdot b, \ b = ab \cdot a.
\]

(3)
Clearly, the relation \(\rho\) is reflexive and symmetric. If \(a \rho b\), \(b \rho c\), then by (2) and (3) we have

\[
ac \cdot a = ac \cdot (ba \cdot b) = ((ba \cdot b)c)a = (cb \cdot ba)a
= (a \cdot ba) \cdot cb = b \cdot cb = c.
\]

Similarly, \(a = ca \cdot c\) thus the relation \(\rho\) is transitive. Hence, \(\rho\) is an equivalence relation.

Let \(a \rho b\) and \(c \in S\). Then by (1) and the medial law we have

\[
ac = (ba \cdot b)c = cb \cdot ba = (cc \cdot b) \cdot ba = (bc \cdot c) \cdot ba
= (ba \cdot c) \cdot bc = (ba \cdot cc) \cdot ba = (bc \cdot ac) \cdot bc.
\]

Dually, \(bc = (ac \cdot bc) \cdot ac\) and so \(ac \rho bc\). Also,

\[
ca = cc \cdot a = ac \cdot c = ((ba \cdot b)c)c = (cb \cdot ba)c = (c \cdot ba) \cdot cb
= (cc \cdot ba) \cdot cb = (cb \cdot ca) \cdot cb.
\]

Dually, \(cb = (ca \cdot cb) \cdot ca\) and so \(ca \rho cb\). Hence, \(\rho\) is a congruence on \(S\).

Since \(S\) is a band we have that \(\rho\) is a band congruence on \(S\). From \(apb\) we have \(a = a^2 \rho ab\), whence it follows that \(\rho\)-classes are closed under the operation. By the definition of \(\rho\) it follows that \(\rho\)-classes are antirectangular AG-bands. By Lemma 2.1, \(\rho\)-classes are anticommutative AG-bands.

In Example 2.1. we have \(S = S_\alpha \cup S_\beta \cup S_\gamma\) where \(S_\alpha = \{1\}\), \(S_\beta = \{3\}\), \(S_\gamma = \{2, 4, 5, 6\}\) are equivalence classes \(mod \rho\) and \(Y = \{\alpha, \beta, \gamma\}\) is a semilattice. Obviously, \(S_\alpha, S_\beta\) are trivial AG-bands and \(S_\gamma\) is anti-isomorphic with AG-band \(T_4\) (as is Example 2.1.).

**Example 2.2.** Let AG-band \(S\) be given by the following table.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>5</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>5</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

Now, \(S = S_\alpha \cup S_\beta \cup S_\gamma\) where \(S_\alpha = \{1\}\), \(S_\beta = \{3\}\), \(S_\gamma = \{2, 4, 5, 6\}\) are equivalence classes \(mod \rho\) and \(Y = \{\alpha, \beta, \gamma\}\) is a semilattice. Obviously, \(S_\alpha, S_\beta\) are trivial AG-bands and \(S_\gamma\) is anti-isomorphic with AG-band \(T_4\) (as is Example 2.1.).

**Lemma 2.2.** Let \(S\) be an AG-band and \(e, a, b \in S\). Then \(ea = eb\) implies that \(ae = be\) and conversely.
Proof. Suppose that \( ea = eb \), then
\[
\begin{align*}
ea = ee \cdot a &= ee \cdot ba = ea \cdot ba = eb \cdot ba \\
= (ee \cdot b) \cdot ba &= (be \cdot e) \cdot ba = (ba \cdot e) \cdot be = (ea \cdot b) \cdot be \\
= (eb \cdot b) \cdot be &= (bb \cdot e) \cdot be = be \cdot be = be.
\end{align*}
\]
Conversely, suppose that \( ae = be \), then
\[
\begin{align*}
ea &= ee \cdot a = ee \cdot e = ee \cdot b = eb.
\end{align*}
\]

Remark 2.1. As a consequence of Lemma 2.2, \( e = ef \) and so \( e = fe \) and conversely.

Theorem 2.2. Let \( S \) be an AG-band. Then the relation \( \nu \) defined on \( S \) by
\[
\forall a, b \in S \quad a \nu b \iff (\exists e \in S) \quad ea = eb
\]
is a band congruence relation on \( S \).

Proof. Reflexivity and symmetry is obvious. Suppose that \( a \nu b \) and \( b \nu c \) for some \( a, b, c \in S \). Then there exist elements \( e, f \in S \) such that \( ea = eb \) and \( fb = fc \). According to the Lemma 2.2 we also have \( ae = be \), \( bf = cf \). Now
\[
\begin{align*}
fe \cdot a &= ae \cdot f = be \cdot f = be \cdot ff = be \cdot ef = ce \cdot ef \\
= ce \cdot f f &= ce \cdot f = fe \cdot c
\end{align*}
\]
implies that \( \nu \) is transitive.

It remains to prove compatibility. Suppose \( a \nu b \) and let \( c \in S \) be an arbitrary element. Then there exists \( e \in S \) such that \( ea = eb \). We have, now
\[
\begin{align*}
c \cdot ea &= c \cdot eb \implies cc \cdot ea = cc \cdot eb \implies ce \cdot ca = ce \cdot eb,
\end{align*}
\]
so \( \forall c \in S \). Similarly
\[
\begin{align*}
ea \cdot c &= eb \cdot c \implies ea \cdot cc = eb \cdot cc \implies ec \cdot ac = ec \cdot be,
\end{align*}
\]
so \( \forall c \in S \). □

In Example 2.1 we have \( \nu \equiv \Delta \), since \( S \) is a quasigroup. In Example 2.2, \( S = S_\alpha \cup S_\beta \cup S_\gamma \cup S_\delta \), where \( S_\alpha = \{1, 2, 3\} \), \( S_\beta = \{4\} \), \( S_\gamma = \{5\} \), \( S_\delta = \{6\} \) are the equivalence classes \( mod \nu \). Let us remark that AG-band \( Y = \{\alpha, \beta, \gamma, \delta\} \) is anti-isomorphic with \( T4 \).

Lemma 2.3. Let \( S \) be an AG-groupoid. Then the relation \( \sigma \) on \( S \) defined by the formula
\[
\forall a, b \in S \quad a \sigma b \iff ab = ba
\]
is reflexive, symmetric and compatible.
Proof. Clearly $\sigma$ is reflexive and symmetric. If $a\sigma b$ and $c \in S$, then by medial law we have

\begin{align*}
ac \cdot bc &= ab \cdot cc = ba \cdot cc = bc \cdot ac, \\
ca \cdot cb &= cc \cdot ab = cc \cdot ba = cb \cdot ca.
\end{align*}

Hence $ac\sigma bc$, $ca\sigma cb$, and so $\sigma$ is left and right compatible. This means that $\sigma$ is compatible.

Definition 2.2. Let $S$ be an AG-band. Then $S$ is transitivity commutative if for every $a, b, c \in S$ from $ab = ba$ and $bc = cb$ it follows that $ac = ca$.

It is easy to verify that AG-bands in examples 2.1 and 2.2 are transitively commutative.

Theorem 2.3. Let $S$ be a transitively commutative AG-band. Then $S$ is an AG-band $Y$ of, in general case nontrivial, semilattices $S_\alpha$, $\alpha \in Y$.

Proof. In this way the relation $\sigma$ defined by (3) is transitive. Now, by Lemma 2.3 we have that relation $\sigma$ is a band congruence on $S$. Clearly, $\sigma$-classes are commutative AG-bands, i.e., semilattices.

In Example 2.2 we have that $S = S_\alpha \cup S_\beta \cup S_\gamma \cup S_\delta$, AG-band $Y = \{\alpha, \beta, \gamma, \delta\}$ is anti-isomorphic with AG-band $T4$, $S_\alpha = \{1, 2, 3\}$ is nontrivial semilattice and $S_\beta = \{4\}$, $S_\gamma = \{5\}$, $S_\delta = \{6\}$ are trivial semilattices.

Now, let $S$ be a transitively commutative AG-band, and let $a\sigma b \iff ab = ba$. Then from

\begin{align*}
ab \cdot a &= ba \cdot a = aa \cdot b = aa \cdot bb = ab \cdot ab, \\
ab \cdot b &= bb \cdot a = bb \cdot aa = ba \cdot ba = ab \cdot ab
\end{align*}

it follows that $ab \cdot a = ab \cdot b$, and so $a\sigma b$. Hence, if $S$ is an transitively commutative AG-band, then $\sigma \subseteq \nu$.

3. The natural partial order of AG-band

Theorem 3.1. If $S$ is AG-band, then the relation $\leq$ defined on $E(S)$

\[ e \leq f \iff e = ef \]

is a (natural) partial order relation and $\leq$ is compatible with the right and with the left with multiplication.
Proof. Clearly, \( e \leq e \) and relation \( \leq \) is reflexive. Let \( e \leq f, f \leq e \), then \( e = ef, f = fe \) and by the Remark 2.1 we have \( e = f \) so relation \( \leq \) is antisymmetric. If \( e \leq f, f \leq g \) then \( e = ef, f = fg \) also by the Remark 2.1 it holds that \( f = gf \). Now by (1) it follows that

\[
eg e = ef \cdot g = gf \cdot e = fe = e.
\]

Hence, \( e \leq g \) and relation \( \leq \) is transitive thus \( \leq \) is a partial order relation. Now, \( e \leq f \iff e = ef \) and \( g \in S \) yields

\[
eg e = ef \cdot g = ef \cdot gg = eg \cdot fg,
\]

\[
eg g = g \cdot ef = gg \cdot ef = ge \cdot gf
\]

so \( eg \leq fg, \ ge \leq gf \). Hence, the relation \( \leq \) is left and right compatible with multiplication. \( \square \)

In Example 2.1, \( \leq \equiv \Delta \). In Example 2.2 we have \( 2 < 1, \ 2 < 3 \) while other elements are incomparable.

References


