Quasigroups and Related Systems **11** (2004), 115 – 125

Extensions of Latin subsquares and local embeddability of groups and group algebras

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Abstract

We will show that any "self-adjoint" Latin subsquare with constant diagonal can be extended to a Latin square with the same property. As a consequence, every loop with inverses satisfying the identity $(xy)^{-1} = y^{-1}x^{-1}$ (an IAA loop for short) is locally embeddable into finite IAA loops, and its loop algebra is locally embeddable into loop algebras of finite IAA loops. The IAA property enables to extend this result to loop algebras with the natural involution arising from the inverse map on the loop. In particular, this is true for groups and their group algebras.

1. Introduction

This paper arises from the study of groups locally embeddable into finite groups (LEF groups) and algebras locally embeddable into finite dimensional algebras (LEF algebras). Both notions were introduced and investigated by Gordon and Vershik in [8]. A relation between local embeddability of a group and its group algebra was established by the present author in [12], solving a problem formulated in [8].

A more general notion of approximability of topological groups by finite ones was introduced by E. Gordon in connection with his study of approximation of operators in spaces of functions on topological groups (cf. [6, 7]). However, not all topological groups are approximable by finite ones, in particular, by far not all (discrete) groups are LEF. This raises the issue of approximation of groups by some finite grupoids, retaining as much of the group structure as possible. L. Glebsky and E. Gordon, in [5], proved that the approximability of locally compact groups by finite semigroups is equivalent to their approximability by finite groups. This indicates that in order

2000 Mathematics Subject Classification: 20N05, 20E25, 16S34, 05B15
Keywords: group, loop, Latin square, group algebra, loop algebra, local embeddability
The paper was supported by VEGA - the grant agency of Slovak Republic

to extend the class of LEF groups one has to sacrifice the associativity of the binary operation. In the mentioned paper the study of approximability of groups by finite quasigroups was commenced.

We will show that every group is even locally embeddable into finite loops with inverses satisfying the identity $(xy)^{-1} = y^{-1}x^{-1}$, which we call IAA loops for short. The last property enables to extend the above mentioned result from [12] to group algebras with involution. In fact, we will be working within a slightly more general scope. Given an IAA loop L and a field K with an involutive automorphism, we will prove that L is locally embeddable into finite IAA loops, and its loop algebra KL, with the natural involution arising from the inverse map on L, is locally embeddable into loop algebras of finite loops with natural involution.

The proof utilizes the well known relation between quasigroups and Latin squares. Its key ingredient is a kind of embedding theorem for Latin subsquares (Theorem 2.4). It gives some sufficient conditions guaranteeing the extendability of a Latin subsquare, symmetric with respect to some involutive permutation of the set of its elements and with constant diagonal, to a Latin square with the same property.

2. α-symmetric Latin squares

A $p \times q$ matrix $R = (r_{ij})$ with elements from a set A is called a *Latin rectangle of size* $p \times q$ *over* A if every element of A occurs at most once in each row as well as in each column. If $p = q$ then the Latin rectangle is called a *Latin subsquare of order* p. If $p = q$ equals the number n of elements of the finite set A then the Latin rectangle is called a *Latin square of order* n *over* A.

Definition 2.1. Let $\alpha : A \to A$ be an involutive permutation of the set A, i.e., $\alpha^2 = id$. A Latin (sub)square $R = (r_{ij})$ over A is called *α-symmetric* if $\alpha(r_{ij}) = r_{ji}$ for all i, j.

Obviously, if (r_{ij}) is an α-symmetric Latin (sub)square then $\alpha(r_{ii}) = r_{ii}$, in other words, all the diagonal elements are fixed by α.

We will make use of the following results. The number of occurrences of an element $a \in A$ in a Latin rectangle R will be denoted by $N_R(a)$.

Lemma 2.2. [11, Ch. 6, Theorem 2.2] *A Latin rectangle R of size $p \times q$ over an n element set A can be extended to a Latin square of over A if and only if $N_R(a) \geq p + q - n$ for all $a \in A$.*

Lemma 2.3. [4, Corollary II.10.9] *Let $m \le n$, $\mathcal{U} = \{U_1, U_2, \dots, U_n\}$ and $\mathcal{V} = \{V_1, V_2, \dots, V_m\}$ be two collections having systems of distinct representatives (SDR). Then some SDRs $\hat{U}$ of $\mathcal{U}$ and $\hat{V}$ of $\mathcal{V}$, satisfying $\hat{V} \subseteq \hat{U}$, exist if and only if*

$$|\mathcal{U}'| + |\mathcal{V}'| \le m + \left|\bigcup \mathcal{U}' \cap \bigcup \mathcal{V}'\right|$$

for all $\mathcal{U}' \subseteq \mathcal{U}$ and $\mathcal{V}' \subseteq \mathcal{V}$.

The next theorem is a partial case, for $\alpha = id$, of Cruse theorem on extensions of commutative Latin squares — cf. [3, Theorem 1] or [9, Theorem 4.1]. The other way round, it can be regarded as a generalization of the special case ($r_{ii} = 1$) of the quoted result from commutative Latin squares to the α-symmetric ones.

Theorem 2.4. *Let n be even, α be an involutive permutation of the set $A = \{1, \dots, n\}$ with $\alpha(1) = 1$, and $R = (r_{ij})$ be an α-symmetric Latin subsquare over A of order $m < n$ such that $r_{ii} = 1$ for all $i \le m$. Then R can be extended to an α-symmetric Latin square $S = (s_{ij})$ over A satisfying $s_{ii} = 1$ for all $i \le n$ if and only if $N_R(k) \ge 2m - n$ for all $k \in A$.*

Proof. Obviously, the inequality is necessary by Lemma 2.2. In the reversed direction we will proceed by induction, showing that the Latin subsquare R satisfying the assumptions can be extended to an α-symmetric Latin subsquare $\tilde{R} = (r_{ij})$ of order $m+1$ over A such that $N_{\tilde{R}}(k) \ge 2(m+1) - n$ for all $k \in A$ and $r_{ii} = 1$ for all $i \le m+1$ (the elements of the extension $\tilde{R}$ will be still denoted by r_{ij}). This way R can be extended to an α-symmetric Latin square S of order n with the desired property, in $n - m$ steps.

The case $m = n - 1$ is trivial. So we can assume $m < n - 1$.

Let U_i ($i = 1, 2, \dots m$) be the set of elements of A, not occuring in the ith row of R, and $\mathcal{U} = \{U_1, U_2, \dots, U_m\}$. Set

$$\begin{aligned}
\mathcal{V}_0 &= \{\{k\};\, N_R(k) = 2m - n\},\\
\mathcal{V}_1 &= \{\{k, \alpha(k)\};\, N_R(k) = 2m - n + 1\},\\
\mathcal{V} &= \mathcal{V}_0 \cup \mathcal{V}_1.
\end{aligned}$$

Now it suffices to show that there exist SDRs $\hat{U} = \{u_1, u_2, \dots, u_n\}$ of $\mathcal{U}$ and $\hat{V}$ of $\mathcal{V}$ such that $\hat{V} \subseteq \hat{U}$. Indeed, adding $(u_1, u_2, \dots, u_m)^T$ to R as a new last column and $(\alpha(u_1), \alpha(u_2), \dots, \alpha(u_m), 1)$ as a new last row, we get the following matrix of order of order $m+1$:

$$\tilde{R} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1m} & u_1 \\ r_{21} & r_{22} & \cdots & r_{2m} & u_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mm} & u_m \\ \alpha(u_1) & \alpha(u_2) & \cdots & \alpha(u_m) & 1 \end{pmatrix}.$$

As u_i is a representative of the set U_i, it does not occur in the ith row of R, hence by the α-symmetry $\alpha(u_i)$ does not occur in its ith column. Thus $\tilde{R}$ is a Latin subsquare over A. The α-symmetry and $r_{ii} = 1$ for all $i \leq m+1$ are clear from the construction.

The inequality $N_{\tilde{R}}(k) \geq 2(m+1) - n$ is automatically satisfied for the elements of $A \setminus \bigcup \mathcal{V}$. The same will be verified for the elements of $\bigcup \mathcal{V}_0$ and $\bigcup \mathcal{V}_1$ separately.

The α-symmetry of R implies $N_R(k) = N_R(\alpha(k))$ for all $k \in A$. Then

$$\alpha(k) \in \mathcal{V}_0 \Leftrightarrow k \in \mathcal{V}_0.$$

As $\bigcup \mathcal{V}_0 \subseteq \hat{V} \subseteq \hat{U}$, we have $\{k, \alpha(k)\} \subseteq \hat{U} \cap \alpha(\hat{U})$, consequently $N_{\tilde{R}}(k) = N_R(k) + 2 = 2m - n + 2$ for all $k \in \bigcup \mathcal{V}_0$.

If $\{k, \alpha(k)\} \in \mathcal{V}_1$ then eighter $k \in \hat{V}$ or $\alpha(k) \in \hat{V}$. In any case $\{k, \alpha(k)\} \subseteq \hat{U} \cup \alpha(\hat{U})$, hence $N_{\tilde{R}}(k) \geq N_R(k) + 1 \geq 2m - n + 2$ for all $k \in \bigcup \mathcal{V}_1$.

Finally, it remains to prove the existence of suitable SDRs $\hat{U}$ and $\hat{V}$. To this end we use Lemma 2.3, thus we have to verify its assumptions.

We show $|\mathcal{V}| \leq |\mathcal{U}| = m$ first. Assume that $|\mathcal{V}| = m + x$, where $x \geq 1$ is an integer.

If $\alpha(k) = k$ and $k \neq 1$ then, by α-symmetry, $N_R(k)$ is even. As n is even, too, $N(k) \neq 2m - n + 1$. The last inequality is true for $k = 1$, as well, because $N_R(1) = m \neq 2m - n + 1$. (Recall that $m < n - 1$.) Hence $|V| = 2$ for all $V \in \mathcal{V}_1$. Then the number of fields of the Latin subsquare R which can be filled by elements of A is at most

$$M = |\mathcal{V}_0|(2m - n) + 2|\mathcal{V}_1|(2m - n + 1) + (n - 2|\mathcal{V}_1| - |\mathcal{V}_0|)m.$$

Then $|\mathcal{V}_1| = m + x - y$, where $y = |\mathcal{V}_0|$. Thus

$$\begin{aligned} M &= M(x, y) \\ &= y(2m - n) + 2(m + x - y)(2m - n + 1) + (n - 2(m + x - y) - y)m \\ &= (n - m - 2)y - 2(n - m - 1)x + m(2m - n + 2) \end{aligned}$$

can be regarded as a function of the arguments x and y, decreasing in x and nondecreasing in y. As y takes the values from the set $\{0, 1, \dots, m + x\}$, only,

$$M(x, m + x) = (m - n)x + m^2$$

is the maximal value of $M(x, y)$ for a fixed x. This is still a decreasing function of x, hence its maximum is

$$M(1, m + 1) = m - n + m^2 < m^2.$$

In other words, not all fields of R can be filled by elements of A. Thus the assumption $|\mathcal{V}| > |\mathcal{U}|$ leads to a contradiction, and we have $|\mathcal{V}| \le |\mathcal{U}|$.

As $V_1 \cap V_2 = \emptyset$ for any distinct $V_1, V_2 \in \mathcal{V}$, the collection $\mathcal{V}$ has some SDR. The existence of an SDR for $\mathcal{U}$ follows from Lemma 2.2.

It remains to show the inequality

$$|\mathcal{U}'| + |\mathcal{V}'| \le m + \left|\bigcup \mathcal{U}' \cap \bigcup \mathcal{V}'\right| \tag{1}$$

for all $\mathcal{U}' \subseteq \mathcal{U}$ and $\mathcal{V}' \subseteq \mathcal{V}$. Take some fixed $\mathcal{U}'$, $\mathcal{V}'$, and consider the bipartite graph $\Gamma = \left(\bigcup \mathcal{V}', \mathcal{U}', E\right)$ with the edge set $E = \left\{(v, U) \in \bigcup \mathcal{V}' \times \mathcal{U};\ v \in U\right\}$. One can readily see that the degrees of its vertices satisfy the following conditions:

$$\begin{aligned} \deg(v) &= \quad n - m, && v \in \textstyle\bigcup \mathcal{V}_0'; \\ \deg(v) &= \quad n - m - 1, && v \in \textstyle\bigcup \mathcal{V}_1'; \\ \deg(U) &\le \quad n - m, && U \in \mathcal{U}, \end{aligned}$$

where $\mathcal{V}_i' = \mathcal{V} \cap \mathcal{V}_i$ for $i = 0, 1$. Denoting

$$p = \left|\bigcup \mathcal{V}_0' \cap \bigcup \mathcal{U}'\right|, \qquad q = \left|\bigcup \mathcal{V}_1' \cap \bigcup \mathcal{U}'\right|,$$

we have $|\bigcup \mathcal{U}' \cap \bigcup \mathcal{V}'| = p + q$. Now, one can give an upper bound for the number of edges ending in $\mathcal{U} \setminus \mathcal{U}'$:

$$\left(\left|\bigcup \mathcal{V}_0'\right| - p\right)(n - m) + \left(\left|\bigcup \mathcal{V}_1'\right| - q\right)(n - m - 1) \le \left(m - |\mathcal{U}'|\right)(n - m).$$

Realizing $|\bigcup \mathcal{V}_0'| = |\mathcal{V}_0'|$ and $|\bigcup \mathcal{V}_1'| = 2|\mathcal{V}_1'|$, the last inequality can be written in the following form

$$\left(|\mathcal{V}_0'| - p\right)(n - m) + \left(2\,|\mathcal{V}_1'| - q\right)(n - m - 1) \le \left(m - |\mathcal{U}'|\right)(n - m).$$

An elementary computation shows that this one is equivalent to

$$|\mathcal{V}_0'| + |\mathcal{V}_1'| + |\mathcal{U}| \le m + p + \frac{n - m - 1}{n - m} q - \frac{n - m - 2}{n - m}\,|\mathcal{V}_1'|\,.$$

As $|\mathcal{V}_0'| + |\mathcal{V}_1'| = |\mathcal{V}'|$, $\frac{n-m-1}{n-m} < 1$ and $\frac{n-m-2}{n-m}\,|\mathcal{V}_1'| \geq 0$, the last inequality implies (1).

Hence, by Lemma 2.3, there exist SDRs $\hat{U}$ of $\mathcal{U}$ and $\hat{V}$ of $\mathcal{V}$ such that $\hat{V} \subseteq \hat{U}$. □

The idea of the presented proof of Theorem 2.4, based on Lemma 2.3 and the proof of the above mentioned Cruse Theorem [3], was suggested by the referee. The core of author's original, and considerably longer, proof consisted of an algorithm written in a computer-like language. Its entry was an arbitrary extension of the original Latin subsquare R to a Latin square R' over A, existing by the virtue of Lemma 2.2. The algorithm transformed the $(m+1) \times (m+1)$ upper left corner of R' into a Latin subsquare $\tilde{R}$ extending R, still satisfying the assumptions of the theorem. Having checked the extendability of $\tilde{R}$, the desired Latin square S could have been obtained by repeating the algorithm $n-m$ times, again.

3. IAA loops and groups

A *quasigroup* is a grupoid Q satisfying both the left and the right cancellation law, i.e.,

$$(xy_1 = xy_2 \ \vee \ y_1x = y_2x) \Rightarrow y_1 = y_2$$

for all $x, y_1, y_2 \in Q$. A quasigroup with a unit 1 (which is necessarily unique) is called a *loop*. If a loop L possess two-sided inverses then, due to the cancellation, they are uniquely determined, so that the notation x^{-1} is unambiguous.

Definition 3.1. A loop L with (two-sided) inverses has the *inverse antiautomorphism property* if the mapping $x \mapsto x^{-1}$ is an antiautomorphism of $(L, \cdot)$, i.e.,

$$(xy)^{-1} = y^{-1}x^{-1}, \tag{2}$$

for every $x, y \in L$.

A loop with the inverse antiautomorphism property is briefly called an *IAA loop*. Obviously, every group is an IAA loop. On the other hand, an IAA loop does not necessarily satisfy the conditions $x^{-1}(xy) = y$ and $(xy)y^{-1} = x$.

The following definition goes back to Mal'tsev [10], where it can be found in a more general universal-algebraic setting.

Definition 3.2. Let Q be a grupoid and $\boldsymbol{F}$ be some class of grupoids. Then Q is said to be *locally embeddable into the class* $\boldsymbol{F}$ if for any finite set $M \subseteq Q$ there is an $F \in \boldsymbol{F}$ and an injective map $\varphi : (M \cup M^2) \to F$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for every $x, y \in M$.

In this section we will prove that every IAA loop, in particular every group, is locally embeddable into the class of finite IAA loops. To this end we will exploit the representation of quasigroups by Latin squares: Enumerating the elements of a finite quasigroup Q, its multiplication table can readily be turned into a Latin square over Q. Fixing an element 1 of a quasigroup Q and changing the order of some rows and columns, if necessary, we can transform its Latin square into the multiplication table of some loop with the unit 1. Expressed in the quasigroups terminology: *Every quasigroup is isotopic to a loop* (cf. [1]).

For technical convenience we will formulate the results on embeddability of IAA loops, announced in the introduction within a more general framework of "partial IAA loops with a root".

Definition 3.3. A structure $\left(L, \sqrt{L}, \cdot\right)$, where $\cdot$ is a partial binary operation on L and $\sqrt{L} \subseteq L$, is said to be a *partial IAA loop with the root* $\sqrt{L}$ if

(a) The operation $\cdot$ satisfies the cancellation law, whenever defined.

(b) There exists an element $1 \in \sqrt{L}$ such that $x \cdot 1 = 1 \cdot x = x$ for all $x \in L$.

(c) The product xy is defined for all $x, y \in \sqrt{L}$ and $L = \left(\sqrt{L}\right)^2$, i.e., each $z \in L$ has the form $z = xy$ for some $x, y \in \sqrt{L}$.

(d) For every $x \in L$ there exists an $x^{-1} \in L$ such that $xx^{-1} = x^{-1}x = 1$.

(e) If xy is defined then so is $y^{-1}x^{-1}$ and $(xy)^{-1} = y^{-1}x^{-1}$.

Theorem 3.4. *Let $\left(L, \sqrt{L}, \cdot\right)$ be a partial IAA loop with a finite root $\sqrt{L}$. Then there exists a finite IAA loop F and an injective map $\varphi : L \to F$ such that $\varphi(1) = 1$ and $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \sqrt{L}$.*

In other words, every finite root of a partial IAA loop can be extended to a finite IAA loop. It can be easily seen that such a partial embedding φ satisfies the condition $\varphi(x^{-1}) = \varphi(x)^{-1}$, as well.

Proof. Denote the elements of L by $1, 2, \dots, m'$ (with 1 denoting the unit). We can assume $\sqrt{L} = \{1, 2, \dots m\}$ for some $m \leq m'$. For $i, j = 1, 2, \dots, m$

we put $r_{ij} = k$ if and only if $i \cdot j = k$. Choose an even $n \geq \max(2m, m')$ and define a permutation α of $A = \{1, 2, \ldots, n\}$ as follows:

$$\alpha(k) = \begin{cases} k^{-1}, & \text{if } k \leq m'; \\ k, & \text{if } k > m'. \end{cases}$$

Without loss of generality we can assume that $R = (r_{ij})$ is an α-symmetric Latin subsquare over A of order m satisfying the assumptions of Theorem 2.4 (if not, we can always achieve this by changing the order of some rows in R).

Hence there is an α-symmetric Latin square $S = (s_{ij})$ over A of order n, extending R such that $s_{ii} = 1$ for all $i \leq n$.

Define a binary operation $\cdot$ on the set $F = A$ by putting $p \cdot q = k$ if and only if $p = s_{i1},\ q = s_{1j},\ k = s_{ij}$ for some (uniquely determined) $i, j \leq n$. This definition is independent of the order of rows and columns in S. The fact that $(F, \cdot)$ is a loop with unit 1 could be visualized by interchanging the order of some rows an columns in S yielding the multiplication table of F. Moreover we have $k^{-1} = \alpha(k)$ for each $k \in F$. So it suffices to verify the IAA property, i.e.,

$$\alpha(pq) = \alpha(q)\alpha(p)$$

for all $p, q \in F$.

Let $p = s_{i1}$ and $q = s_{1j}$. Then $pq = s_{ij}$. By the α-symmetry of S we have $\alpha(pq) = s_{ji}$, $\alpha(q) = s_{1i}$ and $\alpha(p) = s_{j1}$. Hence $\alpha(q)\alpha(p) = s_{ji} = \alpha(pq)$.

Now it is enough to take for $\varphi : L \to F$ the identity map. □

Corollary 3.5. *Every IAA loop, in particular, every group, is locally embeddable into the class of finite IAA loops.*

Proof. Given an IAA loop L and a finite $M \subseteq L$, put $\bar{M} = M \cup M^{-1} \cup \{1\}$. Then $(\bar{M} \cup \bar{M}^2, \bar{M}, \cdot)$ is the partial IAA loop with the root $\bar{M}$. □

Applying a standard model-theoretic compactness argument to the last corollary we get (see, e.g., [2])

Corollary 3.6. *Every IAA loop, in particular, every group, can be embedded into an ultraproduct of a system of finite IAA loops.*

4. Quasialgebras and loop algebras

A linear space A over a field K with a bilinear (not necessarily associative) binary operation $\cdot$ will be called a *quasialgebra over* K. We avoid the wide spread term *non-associative algebra*, as the operation $\cdot$ may (but need not) be associative. A quasialgebra with a unit element 1 is called *unitary*.

The definition of a *quasigroup algebra* KQ *of a quasigroup* Q *over* K is analogous to that of a group algebra: It is the linear space over K formed by formal linear combinations $\sum_{x\in Q} a_x x$ of elements of Q with just finitely many nonzero coefficients $a_x \in K$. Their product is defined by the usual convolution formula

$$\Big(\sum_{x\in Q} a_x x\Big)\cdot\Big(\sum_{y\in Q} b_y y\Big) = \sum_{x,y\in Q}(a_x b_y)xy = \sum_{z\in Q}\sum_{xy=z} a_x b_y z.$$

A quasigroup algebra of a loop L will be called a *loop algebra*; it is obviously unitary, with the unit $1 \in L$.

Given an involutive automorphism $a \mapsto \bar{a}$ of the field K, a unary operation * on a quasialgebra A is called an *involution* if for all $u, v \in A$ and $a, b \in K$ we have

(a) $(au+bv)^* = \bar{a}u^* + \bar{b}v^*$;

(b) $(u^*)^* = u$;

(c) $(uv)^* = v^*u^*$.

In what follows K will be some field with an involutive automorphism $a \mapsto \bar{a}$, and we will be dealing just with quasialgebras over K.

The following observation can be verified by some straightforward computations.

Proposition 4.1. *Let L be an IAA loop. Then*

(i) the operation $\Big(\sum_{x\in L} a_x x\Big)^* = \sum_{x\in L} \bar{a}_x x^{-1}$ *is an involution on* KL;

(ii) $xx^* = x^*x = 1 \quad$ *for all* $x \in L$.

The above defined operation $u \mapsto u^*$ will be referred to as the *natural involution* of the loop algebra KL.

The notion of local embeddability from Definition 3.2 can be modified to quasialgebras (with involution) as follows:

Definition 4.2. Let A be a quasialgebra with involution and $\boldsymbol{H}$ be some class of quasialgebras with involution. Then A is said to be *locally embeddable into the class* $\boldsymbol{H}$ if for any finite set $M \subseteq A$ there is an $H \in \boldsymbol{H}$ and an injective linear map $\psi : \operatorname{span}(M \cup M^* \cup M^2) \to H$ such that for every $u, v \in M$ we have

(a) $\psi(uv) = \psi(u)\psi(v)$;

(b) $\psi(u^*) = \psi(u)^*$.

Theorem 4.3. *Let $A = KL$ be the loop algebra of an IAA loop L, endowed with the natural involution. Then KL is locally embeddable into the class of loop algebras of finite IAA loops, with natural involution.*

Proof. Let $M \subseteq A$ be finite. Then there is a finite set $M_0 \subseteq L$ such that $M \subseteq \operatorname{span}(M_0)$ and $M_0 = {M_0}^{-1}$. By Corollary 3.5, there is an injective map $\varphi : M_0 \cup M_0^2 \to F$ into some finite IAA loop F.

Let $H = KF$ be the loop algebra of F. As $M_0 \cup M_0^2 \subseteq L$, it is linearly independent in H. Hence the map φ can be extended to an injective linear map $\lambda : \operatorname{span}(M_0 \cup M_0^2) \to H$. Now it suffices to take the restriction ψ of λ to $\operatorname{span}(M \cup M^* \cup M^2) \subseteq \operatorname{span}(M_0 \cup M_0^2)$. Then one can readily see that $\psi : \operatorname{span}(M \cup M^* \cup M^2) \to H$ is an injective linear map, satisfying the conditions (a) and (b) of Definition 4.2. □

Corollary 4.4. *Let $A = KG$ be the group algebra of a group G, endowed with the natural involution. Then KG is locally embeddable into the class of loop algebras with natural involution of finite IAA loops.*

Similarly as in Corollary 3.6 one can obtain from Theorem 4.3

Corollary 4.5. *Every loop algebra of an IAA loop, in particular, every group algebra, can be embedded into an ultraproduct of a system of loop algebras of finite IAA loops with natural involution.*

The question whether Theorem 4.3 can be extended beyond the class of loop algebras of IAA loops remains open. Let us close with the following conjecture.

Conjecture. *Let A be a unitary quasialgebra with involution which is spanned by a set $U(A) = \{u \in A;\ u^*u = uu^* = 1\}$. Then A is locally embeddable into the class of finite dimensional quasialgebras with involution.*

Obviously, if the set $U(A) \subseteq A$ is closed under multiplication then it forms an IAA loop with the inverse $x^{-1} = x^*$. It is not clear if the above conjecture is true under this additional assumption. If A is an algebra (i.e., it is associative) then $U(A)$ is a group. Even this special case of our conjecture remains open.

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Received April 24, 2003