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Quasi p-ideals of quasi BCI-algebras

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Abstract

As a continuation of our previous study of fuzzy subquasigroups and fuzzy ideals of BCI-algebras, the notion of a quasi p-ideal is introduced. Characterizations of quasi p-ideals of the set of all fuzzy points in BCI-algebras are obtained. Next, using special chains of reals we determine the number of non-equivalent fuzzy p-ideals of some types of BCI-algebras (especially BCI-algebras which are quasigroups) and give the method of computation of fuzzy p-ideals.

1. Introduction

The fundamental concept of a fuzzy set, introduced by Zadeh [10] in 1965, provides a natural generalization for treating mathematically the fuzzy phenomena which exist pervasively in our real world and for building new branches of fuzzy mathematics. In the area of fuzzy BCK/BCI-algebra, several researches have been carried out since 1991. The connection between some BCI-algebras, quasigroups and commutative groups motivated us to study connections between fuzzy ideals of BCI-algebras and fuzzy subgroups of the corresponding groups (see for example [2] and [3]).

On the other hand, in [7], Lele et al. used the notion of fuzzy point to study some properties of BCK-algebras. Jun and Lele [5] used the notion of fuzzy points for establishing quasi ideal. As a continuation of [5] and our previous study, in this paper, we introduce the notion of quasi p-ideal in the set of all fuzzy points of a fixed BCI-algebra, and give some characterizations of this ideal.

Next, using special sequences of real numbers, we determine the number of non-equivalent fuzzy p-ideals of some types of BCI-algebras (especially

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these BCI-algebras which are quasigroups) and give the method of computation of such fuzzy p-ideals.

2. Preliminaries

An algebra $(X, *, 0)$ of type $(2, 0)$ is said to be a *BCI-algebra* if for all $x, y, z \in X$ it satisfies:

(1) $((x * y) * (x * z)) * (z * y) = 0$,
(2) $(x * (x * y)) * y = 0$,
(3) $x * x = 0$,
(4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

A non-empty subset A of a BCI-algebra X is called an *ideal* of X if

- $0 \in A$,
- $x * y \in A$ and $y \in A$ imply $x \in A$.

A non-empty subset A of a BCI-algebra X is called a *p-ideal* of X if

- $0 \in A$,
- $(x * z) * (y * z) \in A$ and $y \in A$ imply $x \in A$.

A p-ideal is an ideal. The converse is not true [6], but every ideal is a subset of some p-ideal (see [11]). In BCI-algebras which are quasigroups, i.e. in BCI-algebras isotopic to commutative groups (see [1]), these ideals coincide. Such quasigroups are medial and a finite subset of such BCI-algebra is an ideal if and only if it is a subgroup of the corresponding group. For infinite ideals it is not true.

A fuzzy set μ in a BCI-algebra X is called a *fuzzy ideal* of X if for all $x, y \in X$ we have

- $\mu(0) \geqslant \mu(x)$,
- $\mu(x) \geqslant \min\{\mu(x * y), \mu(y)\}$.

A fuzzy set μ in a BCI-algebra X is called a *fuzzy p-ideal* of X if for all $x, y, z \in X$ we have

- $\mu(0) \geqslant \mu(x)$,
- $\mu(x) \geqslant \min\{\mu((x * z) * (y * z)), \mu(y)\}$.

Any fuzzy p-ideal is a fuzzy ideal. The converse does not hold in general [6]. But basing on the results obtained in [1] it is not difficult to see that in a BCI-quasigroup a fuzzy set μ is a fuzzy ideal if and only if it is a fuzzy p-ideal.

A fuzzy set μ in a set X is called a *fuzzy point* if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is $\alpha \in (0, 1]$ we denote this fuzzy point by x_α, where the point x is called its *support*.

Let $FP(X)$ denote the set of all fuzzy points in X and define a binary operation $\odot$ on $FP(X)$ by

$$x_\alpha \odot y_\beta = (x * y)_{\min\{\alpha,\beta\}}\,,$$

where $*$ is a binary operation on X. If $(X,*)$ is a quasigroup, then $(FP(X),\odot)$ is not a quasigroup in general.

If $(X,*,0)$ is a BCI-algebra, then

(p_1) $((x_\alpha \odot y_\beta) \odot (x_\alpha \odot z_\gamma)) \odot (z_\gamma \odot y_\beta) = 0_{\min\{\alpha,\beta,\gamma\}}$,
(p_2) $(x_\alpha \odot (x_\alpha \odot y_\beta)) \odot y_\beta = 0_{\min\{\alpha,\beta\}}$,
(p_3) $x_\alpha \odot x_\alpha = 0_\alpha$,

for all $x_\alpha, y_\beta, z_\gamma \in FP(X)$. But the following does not hold:

(p_4) $x_\alpha \odot y_\beta = y_\beta \odot x_\alpha = 0_{\min\{\alpha,\beta\}}$ imply $x_\alpha = y_\beta$.

Hence we know (see [5]) that $FP(X)$ may not be a BCI-algebra, and so we call $FP(X)$ the *quasi BCI-algebra*.

3. Quasi p-ideals

For a fuzzy set μ in a BCI-algebra X we define the set $FP(\mu)$ of all fuzzy points in X *covered* by μ to be the set

$$FP(\mu) = \{x_q \in FP(X) \mid q \leqslant \mu(x), \quad 0 < q \leqslant 1\}.$$

Example 3.1. Let $X = \{0,a,b,c,d\}$ be a BCI-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	d	c	b
a	a	0	d	c	b
b	b	b	0	d	c
c	c	c	b	0	d
d	d	d	c	b	0

For a fuzzy set μ in X defined by $\mu(0) = 1$, $\mu(a) = 0.6$ and $\mu(b) = \mu(c) = \mu(d) = 0.3$, we have

$$FP(\mu) = \{0_\alpha, a_\beta, b_\gamma, c_\delta, d_\sigma \,|\, \alpha \in (0,1],\ \beta \in (0,\ 0.6],\ \gamma,\delta,\sigma \in (0,\ 0.3]\}.$$

Definition 3.2. For a fuzzy set μ in a BCI-algebra X, the set $FP(\mu)$ of all fuzzy points in X covered by μ is called a *quasi p-ideal* of $FP(X)$ if for all $\delta \in Im(\mu)$ and $x_\alpha, y_\beta, z_\gamma \in FP(X)$:

(i) $0_\delta \in FP(\mu)$

(ii) $(x_\alpha \odot z_\gamma) \odot (y_\beta \odot z_\gamma),\ y_\beta \in FP(\mu) \Longrightarrow x_{\min\{\alpha,\beta,\gamma\}} \in FP(\mu)$.

It is not difficult to see that in the above example $FP(\mu)$ is a quasi p-ideal of $FP(X)$.

Note that in [5] and [7] Jun and Lele et al. described ideals of $FP(X)$ of the second type which are called *quasi ideals.*

Definition 3.3. A subset $FP(\mu)$ of $FP(X)$ is called a *quasi ideal* of $FP(X)$ if $0_\alpha \in FP(\mu)$ for all $\alpha \in Im(\mu)$ and

(iii) $x_\alpha \odot y_\beta,\ y_\beta \in FP(\mu) \Longrightarrow x_{\min\{\alpha,\beta\}} \in FP(\mu)$

for all $x_\alpha, y_\beta \in FP(X)$.

Proposition 3.4. *Every quasi p-ideal of $FP(X)$ is also a quasi ideal.*

Proof. Let $x_\alpha, y_\beta \in FP(X)$ be such that $x_\alpha \odot y_\beta \in FP(\mu)$ and $y_\beta \in FP(\mu)$. Then $(x_\alpha \odot y_\beta) \odot (y_\beta \odot y_\beta) = x_\alpha \odot y_\beta \in FP(\mu)$ and $y_\beta \in FP(\mu)$. Since $FP(\mu)$ is a quasi p-ideal of $FP(X)$, it follows that $x_{\min\{\alpha,\beta\}} \in FP(\mu)$. Hence $FP(\mu)$ is a quasi ideal of $FP(X)$. □

The converse of Proposition 3.4 may not be true as seen in the following example.

Example 3.5. Let $X = \{0, a, b, c, d\}$ be a set with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	c	d	a	0

Then $(X, *, 0)$ is a BCK-algebra and hence a BCI-algebra. Let μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} 0.9 & \text{if} \quad x \in \{0,\, b\}, \\ 0.3 & \text{if} \quad x \in \{a,\, c,\, d\}. \end{cases}$$

Consider the set

$$FP(\mu) = \{0_\alpha, a_\beta, b_\gamma, c_\delta, d_\sigma \mid \alpha, \gamma \in (0,\, 0.9],\, \beta, \delta, \sigma \in (0,\, 0.3]\,\}.$$

Then $FP(\mu)$ is a quasi ideal of $FP(X)$. Note that

$$(a_{0.4} \odot c_{0.5}) \odot (b_{0.7} \odot c_{0.5}) = (a * c)_{0.4} \odot (b * c)_{0.5} = 0_{0.4} \odot 0_{0.5} = 0_{0.4} \in FP(\mu)$$

and $b_{0.7} \in FP(\mu)$. But $a_{\min\{0.4,\, 0.5,\, 0.7\}} = a_{0.4} \notin FP(\mu)$. This shows that $FP(\mu)$ is not a quasi p-ideal of $FP(X)$.

The converse of Proposition 3.4 is true only in some very limited cases. One of such cases is given in following theorem.

Theorem 3.6. *Let μ be a fuzzy set in a BCI-algebra X. If $FP(\mu)$ is a quasi ideal of $FP(X)$ such that for all $x_\alpha, y_\beta, z_\gamma \in FP(X)$*

$$(x_\alpha \odot z_\gamma) \odot (y_\beta \odot z_\gamma) \in FP(\mu) \Longrightarrow x_\alpha \odot y_\beta \in FP(\mu),$$

then $FP(\mu)$ is a quasi p-ideal of $FP(X)$.

Proof. Let $x_\alpha, y_\beta, z_\gamma \in FP(X)$ be such that $(x_\alpha \odot z_\gamma) \odot (y_\beta \odot z_\gamma) \in FP(\mu)$ and $y_\beta \in FP(\mu)$. Then by hypothesis, we have $x_\alpha \odot y_\beta \in FP(\mu)$ and $y_\beta \in FP(\mu)$, and so $x_{\min\{\alpha,\beta\}} \in FP(\mu)$ since $FP(\mu)$ is a quasi ideal of $FP(X)$. But $\min\{\alpha, \beta, \gamma\} \leqslant \min\{\alpha,\, \beta\}$ and $x_{\min\{\alpha,\beta\}} \in FP(\mu)$ imply (according to the definition of $FP(\mu)$) that $x_{\min\{\alpha,\beta,\gamma\}} \in FP(\mu)$. Hence $FP(\mu)$ is a quasi p-ideal of $FP(X)$. □

Now we describe the connection between fuzzy p-ideals of a BCI-algebra X and quasi p-ideals of $FP(X)$.

Theorem 3.7. *If μ is a fuzzy p-ideal of a BCI-algebra X, then $FP(\mu)$ is a quasi p-ideal of $FP(X)$.*

Proof. Since $\mu(0) \geqslant \mu(x)$ for all $x \in X$, we have $\mu(0) \geqslant \alpha$ for all $\alpha \in Im(\mu)$. Hence $0_\alpha \in FP(\mu)$.

Let $x_\alpha, y_\beta, z_\gamma \in FP(X)$ be such that $(x_\alpha \odot z_\gamma) \odot (y_\beta \odot z_\gamma) \in FP(\mu)$ and $y_\beta \in FP(\mu)$. Then $\mu((x * z) * (y * z)) \geqslant \min\{\alpha, \beta, \gamma\}$ and $\mu(y) \geqslant \beta$. Since μ is a fuzzy p-ideal of X, it follows that

$$\begin{aligned} \mu(x) &\geqslant \min\{\mu((x * z) * (y * z)),\, \mu(y)\} \\ &\geqslant \min\{\min\{\alpha, \beta, \gamma\},\, \beta\} = \min\{\alpha, \beta, \gamma\} \end{aligned}$$

so that $x_{\min\{\alpha,\beta,\gamma\}} \in FP(\mu)$. This completes the proof. □

We now consider the converse of Theorem 3.7.

Theorem 3.8. *Let μ be a fuzzy set in a BCI-algebra X such that $FP(\mu)$ is a quasi p-ideal of $FP(X)$. Then μ is a fuzzy p-ideal of X.*

Proof. Obviously $\mu(0) \geqslant \mu(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $\mu((x * z) * (y * z)) = \alpha$ and $\mu(y) = \beta$. Then $y_\beta \in FP(\mu)$ and

$$(x_\alpha \odot z_\alpha) \odot (y_\beta \odot z_\alpha) = ((x * z) * (y * z))_{\min\{\alpha,\beta\}} \in FP(\mu).$$

Since $FP(\mu)$ is a quasi p-ideal, it follows that $x_{\min\{\alpha,\beta\}} \in FP(\mu)$ so that

$$\mu(x) \geqslant \min\{\alpha, \beta\} = \min\{\mu((x * z) * (y * z)),\ \mu(y)\}.$$

Therefore μ is a fuzzy p-ideal of X. □

Lemma 3.9. [6] *A fuzzy set μ in a BCI-algebra X is a fuzzy p-ideal of X if and only if the level set $L(\mu;\alpha) = \{x \in X \mid \mu(x) \geqslant \alpha\}$ is a p-ideal of X when it is non-empty.*

Combining Lemma 3.9 and Theorems 3.7 and 3.8, we have

Theorem 3.10. *Let μ be a fuzzy set in a BCI-algebra X. Then the following statements are equivalent.*

(*i*) *μ is a fuzzy p-ideal of X,*
(*ii*) *$FP(\mu)$ is a quasi p-ideal of $FP(X)$,*
(*iii*) *$L(\mu;\alpha)$ is a p-ideal of X for every $\alpha \in Im(\mu)$.* □

4. Fuzzy p-ideals with a finite set of values

Results of this section are motivated by the corresponding results obtained for fuzzy subgroups and different types of fuzzy ideals of algebras connected with logic (cf. for example [2], [4] and [6]).

In the sequel we will consider only fuzzy sets with a finite image, i.e. fuzzy sets for which $2 \leqslant |Im(\mu)| < \infty$. Similarly as in the group theory, we assume that the empty set $\emptyset$ is a subalgebra (a subgroup, respectively). Moreover, we assume also that every fuzzy set takes value 1 on the empty set. Thus a fuzzy point x_α can be defined as a fuzzy set x_α on X such that

$$x_\alpha(z) = \begin{cases} 1 & \text{for} \quad z \in \emptyset \\ \alpha & \text{for} \quad z = x \\ 0 & \text{for} \quad z \neq x \end{cases}$$

We start with the following.

Proposition 4.1. *Let $\{X_\omega : \omega \in \Omega\}$, where $\emptyset \neq \Omega \subseteq [0,1]$, be a collection of p-ideals of a BCI-algebra X such that*

$$(i) \quad X = \bigcup_{\omega\in\Omega} X_\omega,$$

$$(ii) \quad \alpha > \beta \iff X_\alpha \subset X_\beta \quad \forall\, \alpha, \beta \in \Omega.$$

Then a fuzzy set μ in X defined by

$$\mu(x) = \sup\{\omega \in \Omega : x \in X_\omega\}$$

is a fuzzy p-ideal of X.

Proof. In view of Lemma 3.9, it is sufficient to show that every nonempty level set $L(\mu;\alpha)$ is a p-ideal of X. Assume $L(\mu;\alpha) \neq \emptyset$ for some $\alpha \in [0,1]$. Then

$$\alpha = \sup\{\beta \in \Omega : \beta < \alpha\} = \sup\{\beta \in \Omega : X_\alpha \subset X_\beta\}$$

or

$$\alpha \neq \sup\{\beta \in \Omega : \beta < \alpha\} = \sup\{\beta \in \Omega : X_\alpha \subset X_\beta\}.$$

In the first case we have $L(\mu;\alpha) = \bigcap_{\beta<\alpha} X_\beta$, because

$$x \in L(\mu;\alpha) \iff x \in X_\beta \text{ for all } \beta < \alpha \iff x \in \bigcap_{\beta<\alpha} X_\beta\,.$$

In the second case, there exists $\varepsilon > 0$ such that $(\alpha-\varepsilon, \alpha) \cap \Omega = \emptyset$. We prove that in this case $L(\mu;\alpha) = \bigcup\limits_{\beta\geqslant\alpha} X_\beta$. Indeed, if $x \in \bigcup\limits_{\beta\geqslant\alpha} X_\beta$, then $x \in X_\beta$ for some $\beta \geqslant \alpha$, which gives $\mu(x) \geqslant \beta \geqslant \alpha$. Thus $x \in L(\mu;\alpha)$, i.e. $\bigcup\limits_{\beta\geqslant\alpha} X_\beta \subseteq L(\mu;\alpha)$. Conversely, if $x \notin \bigcup\limits_{\beta\geqslant\alpha} X_\beta$, then $x \notin X_\beta$ for all $\beta \geqslant \alpha$, which implies that $x \notin X_\beta$ for all $\beta > \alpha - \varepsilon$, i.e. if $x \in X_\beta$ then $\beta \leqslant \alpha - \varepsilon$. Thus $\mu(x) \leqslant \alpha - \varepsilon$. Therefore $x \notin L(\mu;\alpha)$. Hence $L(\mu;\alpha) \subseteq \bigcup\limits_{\beta\geqslant\alpha} X_\beta$, and in the consequence $L(\mu;\alpha) = \bigcup\limits_{\beta\geqslant\alpha} X_\beta$. This completes our proof because $\bigcup\limits_{\beta\geqslant\alpha} X_\beta$ and $\bigcap\limits_{\beta<\alpha} X_\beta$ are p-ideals. □

Proposition 4.2. *Let μ be a fuzzy set in X and let $Im(\mu) = \{\lambda_0, \lambda_1, ..., \lambda_n\}$, where $\lambda_0 > \lambda_1 > ... > \lambda_n$. If $X_0 \subset X_1 \subset ... \subset X_n = X$ are p-ideals of X such that $\mu(X_k \setminus X_{k-1}) = \lambda_k$ for $k = 0, 1, ..., n$, where $X_{-1} = \emptyset$, then μ is a fuzzy p-ideal in X.*

Proof. Since X_0 is a p-ideal, then $0 \in X_0$ and $\mu(0) = \mu(X_0 \setminus X_{-1}) = \lambda_0$, which gives $\mu(0) \geqslant \mu(x)$ for all $x \in X$.

To prove that μ satisfies the second condition of the definition of fuzzy p-ideals we consider the following four cases:

1^o $(x*z)*(y*z)\in X_k\setminus X_{k-1},\;\; y\in X_k\setminus X_{k-1}\,,$

2^o $(x*z)*(y*z)\in X_k\setminus X_{k-1},\;\; y\notin X_k\setminus X_{k-1}\,,$

3^o $(x*z)*(y*z)\notin X_k\setminus X_{k-1},\;\; y\in X_k\setminus X_{k-1}\,,$

4^o $(x*z)*(y*z)\notin X_k\setminus X_{k-1},\;\; y\notin X_k\setminus X_{k-1}\,.$

In the first case $x\in X_k$, because X_k is a p-ideal. Thus

$$\mu(x)\geqslant\lambda_k=\mu((x*z)*(y*z))=\mu(y)=\min\{\mu((x*z)*(y*z)),\,\mu(y)\}\,.$$

In the second case $y\in X_{k-1}\subset X_k$ or $y\in X_m\setminus X_{m-1}\subset X_m\setminus X_k$ for some $m>k$. This together with $(x*z)*(y*z)\in X_k\setminus X_{k-1}$ implies $x\in X_k$ or $x\in X_m\setminus X_k$. Thus

$$\mu(x)\geqslant\lambda_k=\mu((x*z)*(y*z))=\min\{\mu((x*z)*(y*z)),\,\mu(y)\}$$

for $x\in X_k,\; y\in X_{k-1}$. Similarly

$$\mu(x)\geqslant\lambda_m=\mu(y)=\min\{\mu((x*z)*(y*z)),\,\mu(y)\}$$

for $y\in X_m\setminus X_{m-1},\; x\in X_m\setminus X_k$.

In the last two cases the process of verification is analogous. □

Corollary 4.3. *Let μ be a fuzzy set in X and let $Im(\mu)=\{\lambda_0,\lambda_1,...,\lambda_n\}$, where $\lambda_0>\lambda_1>...>\lambda_n$. If $X_0\subset X_1\subset...\subset X_n=X$ are p-ideals of X such that $\mu(X_k)\geqslant\lambda_k$ for $k=0,1,...,n$, then μ is a fuzzy p-ideal in X.*

Corollary 4.4. *If $Im(\mu)=\{\lambda_0,\lambda_1,...,\lambda_n\}$, where $\lambda_0>\lambda_1>...>\lambda_n$, is the the set of values of a fuzzy p-ideal μ in X, then all $L(\mu;\lambda_k)$ are p-ideals of X such that $\mu(L(\mu;\lambda_0))=\lambda_0$ and $\mu(L(\mu;\lambda_k)\setminus L(\mu;\lambda_{k-1}))=\lambda_k$ for $k=1,2,...,n$.*

Proposition 4.5. *If a fuzzy p-ideal μ in a BCI-algebra X has the finite set of values, then every descending chain of p-ideals of X terminates at finite step.*

Proof. Suppose there exists a strictly descending chain $X_1\supset X_2\supset X_3\supset...$ of p-ideals of a BCI-algebra X which does not terminate at finite step. We prove that μ defined by

$$\mu(x)=\begin{cases}\dfrac{n}{n+1} & \text{for}\quad x\in X_n\setminus X_{n+1},\; n=1,2,\dots\\ 1 & \text{for}\quad x\in\bigcap X_n,\; n=1,2,\dots\end{cases}$$

where $X_1 = X$, is a fuzzy p-ideal with an infinite number of values.

Clearly $\mu(0) \geqslant \mu(x)$ for all $x \in X$. Let $x, y, z \in X$. Assume that $(x * z) * (y * z) \in X_n \setminus X_{n+1}$ and $y \in X_k \setminus X_{k+1}$ for some k and some n. (Without loss of generality, we can assume $n \leqslant k$.) Then $y \in X_n$, and in the consequence, $x \in X_n$ because X_n is a p-ideal. Hence

$$\mu(x) \geqslant \frac{n}{n+1} = \min\{\mu((x * z) * (y * z)),\, \mu(y)\}\,.$$

If $(x * z) * (y * z)$ and y are in $\bigcap X_n$, then $x \in \bigcap X_n$. Thus

$$\mu(x) = 1 = \min\{\mu((x * z) * (y * z)),\, \mu(y)\}\,.$$

If $(x * z) * (y * z) \notin \bigcap X_n$ and $y \in \bigcap X_n$, then $(x * z) * (y * z) \in X_k \setminus X_{k+1}$ for some k. Hence $x \in X_k$ and

$$\mu(x) \geqslant \frac{k}{k+1} = \min\{\mu((x * z) * (y * z)),\, \mu(y)\}\,.$$

If $(x * z) * (y * z) \in \bigcap X_n$ and $y \notin \bigcap X_n$, then $y \in Y_t \setminus X_{t+1}$ for some t, which implies $x \in X_t$ and

$$\mu(x) \geqslant \frac{t}{t+1} = \min\{\mu((x * z) * (y * z)),\, \mu(y)\}\,.$$

This proves that μ is a fuzzy p-ideal. Obviously μ has an infinite number of different values. Obtained contradiction completes our proof. $\square$

For finite BCI-algebras the following proposition is true (cf. [6]).

Proposition 4.6. *Let μ and ν be a fuzzy p-ideals of a finite BCI-algebra X such that the families of level p-ideals of μ and ν are identical. Then $\mu = \nu$ if and only if $Im(\mu) = Im(\nu)$.*

5. Equivalences of fuzzy p-ideals

Results of this section are motivated by the corresponding results obtained for fuzzy subgroups [9] and by the connection of some BCI-algebras [1] with groups.

In the set $F(X)$ of all fuzzy sets on X we can introduce (sf. [9]) the equivalence relation based on the heuristic principle that the distinction or similarity of fuzzy sets is really based on the relative membership degrees of elements with respect to each other rather than the absolute membership degree of each element to the fuzzy set under consideration. Thus two fuzzy sets are similar if they maintain the same relative degrees of membership with respect to two elements. This gives the motivation to the following relation [9]:

$$\mu \sim \nu \Longleftrightarrow \begin{cases} \mu(x) > \mu(y) & \Leftrightarrow \quad \nu(x) > \nu(y) \\ \mu(x) = 1 & \Leftrightarrow \quad \nu(x) = 1 \\ \mu(x) = 0 & \Leftrightarrow \quad \nu(x) = 0 \end{cases}$$

for all $x, y \in X$.

It is not difficult to see that this relation is an equivalence relation on $F(X)$ and coincides with the equality of subsets in 2^X.

The condition $\mu(x) = 0$ if and only if $\nu(x) = 0$ says that the supports of μ and ν are equal. This condition cannot be redundant since it is an essential part of the equivalence relation as seen in the example below.

If in the above definition we replace the strict inequality by $\geqslant$ we obtain the new equivalence relation which has the same equivalence classes as the above equivalence.

Example 5.1. Let $K = \{1, -1, i, -i\}$ be a group. Then $(K, \cdot, 1)$ is a BCI-algebra (a BCI-quasigroup in fact) with $0 = 1$. Define two fuzzy sets μ and ν putting

$$\mu(x) = \begin{cases} 1 & \text{for} \quad x = 1 \\ 0.5 & \text{for} \quad x = -1 \\ 0.3 & \text{for} \quad x \in \{i, -i\} \end{cases} \qquad \text{and} \qquad \nu(x) = \begin{cases} 1 & \text{for} \quad x = 1 \\ 0.5 & \text{for} \quad x = -1 \\ 0 & \text{for} \quad x \in \{i, -i\} \end{cases}$$

Then these fuzzy sets are fuzzy p-ideals satisfying only two first condition of the above definition. Hence μ and ν are not equivalent.

Proposition 5.2. *If μ and ν are equivalent fuzzy p-ideals (fuzzy ideals), then* $|Im(\mu)| = |Im(\nu)|$.

Proof. The proof is analogous to the proof of Proposition 2.2 in [9]. □

Note that the converse of Proposition 5.2 is not true.

Example 5.3. Let $X = \{0, a, b, c\}$ be a BCI-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	b	b
a	a	0	c	b
b	b	b	0	0
c	c	b	a	0

Define fuzzy sets μ and ν in X as follows:

$$\mu(x) = \begin{cases} 1 & \text{for} \quad x = 0 \\ 0.5 & \text{for} \quad x = a \\ 0.3 & \text{for} \quad x \in \{b, c\} \end{cases} \qquad \nu(x) = \begin{cases} 1 & \text{for} \quad x = 0 \\ 0.5 & \text{for} \quad x = b \\ 0.3 & \text{for} \quad x \in \{a, c\} \end{cases}$$

Then these two fuzzy sets are fuzzy ideals with the same supports and the same images. But μ and ν are not equivalent because $\mu(a) > \mu(b)$, but $\nu(a) \not> \nu(b)$.

Between level p-ideals of equivalent fuzzy p-ideals there is a one-to-one correspondence. Namely, the following theorem is valid.

Theorem 5.4. *Two fuzzy p-ideals μ and ν of a BCI-algebra X are equivalent if and only if for each $\alpha > 0$ there exists $\beta > 0$ such that $L(\mu;\alpha) = L(\nu;\beta)$.*

Proof. Let μ and ν be equivalent. If $\mu(x) = 1$ for all x, then also $\nu(x) = 1$ for all x. In this case we put $\beta = \alpha$. Analogously when $\mu(x) = 0$ for all $x \in X$. Now, if $|Im(\mu)| \geqslant 2$, then, according to the Proposition 4.2, fuzzy p-ideals μ and ν have the same number of values. Thus $Im(\mu) = \{\alpha_1, \ldots, \alpha_n\}$ and $Im(\nu) = \{\beta_1, \ldots, \beta_n\}$ for some $\alpha_i < \alpha_{i+1}$ and $\beta_i < \beta_{i+1}$. Hence $L(\mu;\alpha_i) \supsetneq L(\mu;\alpha_{i+1})$ and $L(\nu;\beta_i) \supsetneq L(\nu;\beta_{i+1})$. This together with the condition $\mu(x) > \mu(y) \Leftrightarrow \nu(x) > \nu(y)$ gives $L(\mu;\alpha_i) = L(\nu;\beta_i)$.

Conversely, since by the assumption $|Im(\mu)| \geqslant 2$, there exists $x \in X$ such that $\mu(x) > 0$. Thus $x \in L(\mu;\alpha)$ for some $\alpha > 0$. But by hypothesis there is $\beta > 0$ such that $L(\mu;\alpha) = L(\nu;\beta)$. Hence $\nu(x) \geqslant \beta > 0$. Similarly we can show that $\nu(x) > 0$ implies $\mu(x) > 0$. Therefore $\mu(x) = 0$ if and only if $\nu(x) = 0$.

Now let $\alpha = \mu(x) > \mu(y)$ for some $x, y \in X$. In this case, by hypothesis $x \in L(\mu;\alpha) = L(\nu;\beta)$. If $\nu(x) \leqslant \nu(y)$, then obviously $\nu(y) \geqslant \beta$ and $y \in L(\nu;\beta) = L(\mu;\alpha)$, which is impossible. Thus $\nu(x) > \nu(y)$. Similarly $\nu(x) > \nu(y)$ implies $\mu(x) > \mu(y)$.

If $\mu(x) = 1$, then also $\mu(0) = 1$, by the definition of fuzzy p-ideals, and, in the consequence $0, x \in L(\mu;1) = L(\nu;\beta)$ for some $\beta > 0$. Hence $\nu(0) = \nu(x)$ for all $x \in L(\nu;\beta) = L(\mu;\alpha)$ because $\nu(0) > \nu(x)$ implies $1 = \mu(0) > \mu(x)$. But for $\nu(0) < 1 = \nu(\emptyset)$ we have also $\mu(0) < \mu(\emptyset) = 1$, which is a contradiction. Therefore $\beta = 1$. Hence $\mu(x) = 1$ if and only if $\nu(x) = 1$. This completes the proof. □

Now let

$$\emptyset \subset X_1 \subset X_2 \subset \ldots \subset X_n = X$$

be a maximal chain of p-ideals of a BCI-algebra X. Putting $\mu(\emptyset)=1$ and $\mu(X_k\setminus X_{k-1})=\lambda_k$ for all $k=1,\ldots,n$, where

$$1\geqslant\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_n\geqslant 0$$

we can obtain a fuzzy p-ideal μ on X. Such fuzzy p-ideal can be identified with the sequence

$$\lambda_1\,\lambda_2\,\ldots\,\lambda_n.$$

It is clear that non-equivalent fuzzy p-ideals have distinct sequences.

Example 5.5. Let $(X,*,0)$ be a BCI-algebra induced by $\mathbb{Z}_5$, i.e. let $X=\mathbb{Z}_5$ and $x*y=(x+4y)(mod\,5)$. Then $(X,*,0)$ is a group-like BCI-algebra (BCI-quasigroup) in which all p-ideals are subgroups of $\mathbb{Z}_5$ (cf. [1]). Thus a maximal chain of p-ideals of X has the form $\emptyset\subset X_1\subset X_2$, where $X_1=\{0\}$ and $X_2=\mathbb{Z}_5$ and corresponds to the sequence $\lambda_1\,\lambda_2$.

Using Theorem 5.4 it is not difficult to see that any fuzzy p-ideal of X corresponds to a fuzzy p-ideal determined by one of the following three sequences: $1\,1$, $1\,\lambda$, $1\,0$, where $1>\lambda>0$. The first sequence determines a fuzzy p-ideal μ_1 such that $\mu_1(x)=1$ for all $x\in X$. The second corresponds to μ_2 such that $\mu_2(0)=1$ and $\mu_2(x)=\lambda$ for all $x\neq 0$. The sequence $1\,0$ represents μ_3 such that $\mu_3(0)=1$ and $\mu_3(x)=0$ for all $x\neq 0$. (Fuzzy p-ideals μ_2 and μ_3 are non-equivalent because they have different supports.)

Note that the number of fuzzy p-ideals of this BCI-algebra is $1+2=2^2-1$, i.e., one fuzzy p-ideal whose support is X_1 and two whose support is X_2.

Example 5.6. Now let $(X,*,0)$ be a BCI-algebra induced by $\mathbb{Z}_4$. Then $x*y=(x+3y)(mod\,4)$ and $\emptyset\subset X_1\subset X_2\subset X_3$, where $X_1=\{0\}$, $X_2=\{0,2\}\simeq\mathbb{Z}_2$, $X_3=\mathbb{Z}_4$, is a maximal chain of p-ideals of X. This chain corresponds to the sequence $\lambda_1\,\lambda_2\,\lambda_3$.

Similarly as in the previous case, it is not difficult to see that all non-equivalent fuzzy p-ideals of X correspond to one of the following sequences: $1\,1\,1$, $1\,1\,\lambda_1$, $1\,1\,0$, $1\,\lambda_1\,\lambda_1$, $1\,\lambda_1\,\lambda_2$, $1\,\lambda_1\,0$, $1\,0\,0$, where $1>\lambda_1>\lambda_2>0$.

The sequence $1\,\alpha\,\beta$ represents a fuzzy p-ideal

$$\mu(x)=\begin{cases}1 & \text{for} \quad x\in X_1\\ \alpha & \text{for} \quad x\in X_2\setminus X_1\\ \beta & \text{for} \quad x\in X_3\setminus X_2\end{cases}$$

In this case the number of fuzzy p-ideals is $1+2+2^2=2^3-1$, i.e. one fuzzy p-ideal whose support is X_1, 2 whose support is X_2, and 2^2 whose support is X_3.

Basing on the above two examples we can formulate the following theorem, which can be proved by induction.

Theorem 5.7. *A chain* $X_1 \subset X_2 \subset \ldots \subset X_n = X$ *of p-ideals of a* BCI*-algebra* X *induces* $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ *non-equivalent fuzzy p-ideals of* X*.*

Corollary 5.8. *A* BCI*-algebra* X *in which all its p-ideals can be ordered in the chain* $X_1 \subset X_2 \subset \ldots \subset X_n = X$ *has exactly* $2^n - 1$ *non-equivalent fuzzy p-ideals.*

6. Fuzzy p-ideals of group-like BCI-algebras

Group-like BCI-algebras are described in [1]. Such BCI-algebras are quasigroups induced by commutative groups, i.e. for every group-like BCI-algebra $(X, *, 0)$ there exists a commutative group $(X, +, 0)$ such that $x * y = x - y$ holds for all $x, y \in X$. The maximal chain of p-ideals of BCI-algebra X induced by a cyclic p-group $\mathbb{Z}_{p^n}$ coincides with the maximal chain of subgroups of $\mathbb{Z}_{p^n}$ and has the form $\{0\} \subset X_1 \subset \ldots \subset X_n$, where $X_k = \mathbb{Z}_{p^k}$. Thus, as a consequence of our Theorem 5.7 or Proposition 3.3 from [9], we obtain

Corollary 6.1. *A* BCI*-algebra induced by a cyclic p-group* $\mathbb{Z}_{p^n}$ *has exactly* $2^{n+1} - 1$ *non-equivalent fuzzy p-ideals.*

Similarly, as a consequence of our Theorem 5.7 and results obtained in [9] (Theorem 3.4 and Proposition 3.6), we obtain

Corollary 6.2. *A* BCI*-algebra induced by the group* $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$*, where* $p \neq q$ *are primes, has exactly* $2^{n+1}(n+2) - 1$ *non-equivalent fuzzy p-ideals.*

Corollary 6.3. *A* BCI*-algebra induced by the group* $\mathbb{Z}_q \times \mathbb{Z}_q$*, where* q *is a prime, has exactly* $4q + 7$ *non-equivalent fuzzy p-ideals.*

Thus, for example, BCI-algebras induced by $\mathbb{Z}_p$, where p is a prime, have only 3 non-equivalent fuzzy p-ideals. All these fuzzy p-ideals are described in Example 5.5. BCI-algebras induced by $\mathbb{Z}_{p^2}$ have 7 non-equivalent fuzzy p-ideals (see Example 5.6), but BCI-algebras induced by $\mathbb{Z}_{12}$, $\mathbb{Z}_{18}$ and $\mathbb{Z}_{20}$ have 31 such fuzzy p-ideals.

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