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Characterization of division μ-LA-semigroups

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Abstract

Let G be a left almost semigroup (LA-semigroup), also known as Abel-Grassman's groupoid and a left invertive groupoid. In this paper we have shown that G is a division μ-LA-semigroup if and only if it has a linear form. Characterization of division μ-LA-semigroups is also done by using permutations.

1. Introduction

A left almost semigroup [2], abbreviated as LA-semigroup, is an algebraic structure midway between a groupoid and a commutative semigroup. Although the structure is non-associative and non-commutative, nevertheless, it possesses many interesting properties which we usually find in associative and commutative algebraic structures.

Kazim and Naseerudin have introduced the concept of an LA-semigroup and have investigated some basic but important characteristics of this structure in [2]. They have generalized some useful results of semigroup theory. Relationships between LA-semigroups and quasigroups, semigroups, loops, monoids, and groups have been established.

Later, Mushtaq and others in [1], [5], [6], [7], [8], and [10] have studied the structure further and added many results to the theory of LA-semigroups. Holgate [1], has called the same structure as left invertive groupoid. It is also known as Abel-Grassman's groupoid or AG-groupoid. In this paper we shall call it LA-semigroup.

Kepka [4] has done extensive study of quasigroups satisfying some weak forms of the medial law. In this paper we have extended some of his results to LA-semigroups.

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A groupoid $\mathcal{G} = (G, \cdot)$ is called a *left almost semigroup*, abbreviated as *LA-semigroup*, if its elements satisfy the *left invertive law*: $(ab)c = (cb)a$. Examples of LA-semigroups can be found in [5] and [6].

An element $e \in G$ is called a *left identity* if $ea = a$ for all $a \in G$. An element $a' \in G$ is called a *left inverse* of a if $\mathcal{G}$ contains left identity e and $a'a = e$. As in the case of semigroups, both e and a' are unique [5]. In [5] it is proved also that if $\mathcal{G}$ contains a left identity then $ab = cd$ implies $ba = dc$ for all $a, b, c, d \in G$. As in the case of semigroups, an element a of an LA-semigroup $\mathcal{G}$ is called *left cancellative* if $ab = ac$ implies $b = c$. Similarly, it is *right cancellative* if $ba = ca$ implies $b = c$. If it is both left and right, it is called *cancellative*.

It is also known [2] that in an LA-semigroup $\mathcal{G}$, the *medial law*: $(ab)(cd) = (ac)(bd)$ holds for every $a, b, c, d \in G$. An LA-semigroup with a left identity is called an *LA-monoid*. In [9] an LA-monoid with a left inverse is called an *LA-group*. Because in an LA-group every left inverse is a right inverse, therefore, we can re-define an LA-group as follows: An LA-monoid G is called a left almost group, abbreviated as LA-group, if it contains inverses.

Suppose that $(G, \cdot)$ is a commutative group. Then it is easy to see that $(G, *)$, where $a * b = ba^{-1}$, is an example of an LA-group.

Let $\mathcal{G}$ be an LA-semigroup and $a \in G$. A mapping $L_a : G \to G$, defined by $L_a(x) = ax$, is called the *left translation by a*. Similarly a mapping $R_a : G \to G$, defined by $R_a(x) = xa$ is called the *right translation by a*. An LA-semigroup $\mathcal{G}$ is called a *division LA-semigroup* if the mappings L_a and R_a are onto for all $a \in G$.

An LA-semigroup $\mathcal{G}$ is called a *μ-LA-semigroup* if there are two mappings α, β of the set G onto G and an LA-monoid $(G, \circ)$ such that $ab = \alpha(a) \circ \beta(b)$ for all $a, b \in G$. Note that if we take α, β to be identity maps and $(G, \circ) = (G, \cdot)$, then an LA-monoid $(G, \cdot)$ is trivially a μ-LA-monoid.

Let $\mathcal{G}$ be a division μ-LA-semigroup. Then $((G, \circ), \alpha, \psi, g)$ is said to be a right linear form of $\mathcal{G}$ if $(G, \circ)$ is an LA-group, α a mapping of G onto G, ψ an endomorphism of $(G, \circ)$, $g \in G$ and $ab = \alpha(a) \circ (g \circ \psi(b))$ for all $a, b \in G$. Similarly $((G, \circ), \psi, \alpha, g)$ is said to be a *left linear form* of $\mathcal{G}$ if $ab = \psi(a) \circ (g \circ \alpha(b))$ for all $a, b \in G$. If $\varphi = \alpha$ is an endomorphism of $\mathcal{G}$, then $((G, \circ), \varphi, \psi, g)$ is a called a *linear form* of $\mathcal{G}$.

2. Division LA-semigroups

Having set the terminology and given the basic definitions we are now in a position to prove the following results.

Proposition 2.1. *Every LA-group is a division μ-LA-group.*

Proof. Let $\mathcal{G}$ be an LA-group and L_a its left translation. Then

$$ab = (ea)b = (ba)e$$

yields

$$\begin{aligned} L_a((xe)a^{-1}) &= a((xe)a^{-1}) = (((xe)a^{-1})a)e = ((aa^{-1})(xe))e \\ &= (e(xe))e = (xe)e = (ee)x = ex = x. \end{aligned}$$

Thus for every $x \in G$ there exists $(xe)a^{-1} \in G$ such that $L_x((xe)a^{-1}) = x$. Hence L_a is onto. Also R_a is onto because $R_a(xa^{-1}) = (xa^{-1})a = (aa^{-1})x = ex = x$ for every $x \in G$. Hence $\mathcal{G}$ is a division LA-group. Thus, the observation that every LA-monoid is trivially a μ-LA-monoid, and Theorem 9 in [3], imply that $\mathcal{G}$ is in fact a division μ-LA-group. □

Let $C(G, \circ)$ denote the centre of LA-semigroup $(G, \circ)$.

Theorem 2.2. *If $\mathcal{G}$ is an LA-semigroup, then the following statements are equivalent:*

(*i*) *$\mathcal{G}$ is a division μ-LA-semigroup,*
(*ii*) *$\mathcal{G}$ has a linear form $((G, \circ), \varphi, \psi, g)$ such that $\varphi\psi(a) \circ g = g \circ \psi\varphi(a)$ for every $a \in G$. In this case $C(G, \circ) = G$.*

Proof. $(i) \Rightarrow (ii)$. Since $\mathcal{G}$ is a division μ-LA-semigroup satisfying the medial law, by Theorem 15 in [3], $((G, \circ), \varphi, \psi, g)$ is the linear form of $\mathcal{G}$ such that $\varphi\psi(a) \circ h = h \circ \psi\varphi(a)$ for all $a \in G$, where $h = \psi\varphi(x) \circ g$ for some $x \in G$. But by Theorem 15 in [3], we can assume that x is the left identity of $(G, \circ)$. Thus $h = x \circ g = g$.

$(ii) \Rightarrow (i)$. Since $\mathcal{G}$ has a linear form $((G, \circ), \varphi, \psi, g)$, therefore by the definition, $\mathcal{G}$ is a division μ-LA-semigroup and so $ab = \varphi(a) \circ (g \circ \psi(b))$ for all $a, b \in G$, where $(G, \circ)$ is an LA-group. If e is the left identity in $(G, \circ)$, then this last equation can be written as $\varphi(a) \circ (e \circ \psi(b) = \varphi(a) \circ \psi(b)$, which implies that $\mathcal{G}$ is a division μ-LA-semigroup.

Let $x \in C(G, \circ)$. We wish to show that $x \in G$. Let $a, b, c \in G$, then

$$\begin{aligned} (ax)(bc) &= (\varphi(a) \circ (g \circ \psi(x))\, (\varphi(b) \circ (g \circ \psi(c)) \\ &= \varphi(\varphi(a) \circ (g \circ \psi(x)) \circ (g \circ \psi(\varphi(b) \circ (g \circ \psi(c)))) \\ &= \varphi^2(a) \circ (\varphi(g) \circ \varphi\psi(x)) \circ (g \circ (\psi\varphi(b) \circ (\psi(g) \circ \psi^2(c))). \end{aligned}$$

Since $(G, \circ)$ is an LA-group, we can apply the medial and the left invertive laws (which hold in $(G, \circ)$) to the above identity. Hence

$$(ax)(bc) = (\varphi^2(a) \circ g) \circ ((\psi\varphi(x) \circ (\psi(g) \circ \psi^2(c))) \circ (\varphi(g) \circ \varphi\psi(b))).$$

Since $(\varphi\psi(a) \circ g) \circ (g \circ \psi\varphi(b)) = (\psi\varphi(b) \circ g) \circ (g \circ \psi\varphi(a))$, therefore

$$\begin{aligned}(ax)(bc) &= (\varphi^2(a) \circ g) \circ ((\psi\varphi(b) \circ \psi(g)) \circ ((\psi(g) \circ \psi^2(c)) \circ \varphi\psi(x))) \\ &= (\varphi^2(a) \circ g) \circ ((\varphi(g) \circ \varphi\psi(b)) \circ ((\psi\varphi(x) \circ (\psi(g) \circ \psi^2(c))).\end{aligned}$$

Applying the medial law again, we get

$$\begin{aligned}(ax)(bc) &= (\varphi^2(a) \circ (\varphi(g) \circ \varphi\psi(b))) \circ (g \circ (\psi\varphi(x) \circ (\psi(g) \circ \psi^2(c)))) \\ &= \varphi(\varphi(a) \circ (g \circ \psi(b))) \circ (g \circ \psi(\varphi(x) \circ (g \circ \psi(c)))) \\ &= (\varphi(a) \circ (g \circ \psi(b)))\,(\varphi(x) \circ (g \circ \psi(c))) = (ab)(xc).\end{aligned}$$

Thus $x \in G$, and so $C(G, \circ) \subseteq G$.

Conversely, let $y \in G$. Then

$$(\varphi\psi(a) \circ g) \circ \psi\varphi(y) = (\psi\varphi(y) \circ g) \circ \varphi\psi(a).$$

Since $\varphi\psi(a) \circ g = g \circ \psi\varphi(a)$, therefore the above identity gives

$$(g \circ \psi\varphi(a)) \circ \psi\varphi(y) = (g \circ \varphi\psi(y)) \circ \varphi\psi(a),$$

i.e.

$$(\psi\varphi(y) \circ \psi\varphi(a)) \circ g = (\varphi\psi(a) \circ \varphi\psi(y)) \circ g.$$

Since $(G, \circ)$ is cancellative, $\psi\varphi(y) \circ \psi\varphi(a) = \varphi\psi(a) \circ \varphi\psi(y)$. But $\psi\varphi = \varphi\psi$, by Theorem 16 in [3]. So $\psi\varphi(y) \circ \psi\varphi(a) = \psi\varphi(a) \circ \psi\varphi(y)$. Thus $\psi\varphi(y) \in C(G, \circ)$. This together with the fact that $\psi\varphi : G \to G$ is a homomorphism, imply $y \in G$. Hence $G \subseteq C(G, \circ)$, and in consequence $G = C(G, \circ)$. □

Corollary 2.3. *A division μ-LA-semigroup $\mathcal{G}$ is commutative if it has a linear form $((G, +), \varphi, \psi, g)$ such that $(G, +)$ is a commutative group and $\psi\varphi = \varphi\psi$.*

Proof. If a division μ-LA-semigroup $\mathcal{G}$ has a linear form as above, then $\varphi\psi(a) + g = \psi\varphi(a) + g = g + \psi\varphi(a)$. Therefore $G = C(G, \circ)$ by Theorem 2.2. □

Theorem 2.4. *For any division μ-LA-semigroup $\mathcal{G}$ there are mappings α, β of G onto G such that $\alpha(a)\beta(b) = \alpha(b)\beta(a)$ for every $a, b \in G$.*

Proof. Since $\mathcal{G}$ is a division μ-LA-semigroup, therefore $\alpha = L_c$ and $\beta = R_c$ are onto mappings (for all $c \in G$), and $\alpha(a)\beta(b) = L_c(a)R_c(b) = (ca)(bc) = (bc)(ca) = (cb)(ac) = L_c(b)R_c(a) = \alpha(b)\beta(a)$. □

Theorem 2.5. *A division μ-LA-semigroup $\mathcal{G}$ is commutative if and only if the mapping $a \mapsto aa$ is an endomorphism of $\mathcal{G}$.*

Proof. If $a \mapsto aa$ is an endomorphism of $\mathcal{G}$. Then $(ab)(ab) = (aa)(bb)$ for every $a, b \in G$, because $\mathcal{G}$ is medial, and so $G = C(G, \circ)$ by Theorem 2.2.

Conversely, if $\mathcal{G}$ is commutative, then $(ab)(ab) = (aa)(bb)$ implies that the mapping $a \mapsto aa$ is an endomorphism of $\mathcal{G}$. □

Proposition 2.6. *The mapping $a \mapsto aa$ is an endomorphism of $\mathcal{G}$ if $\mathcal{G}$ is an LA-semigroup.*

Proof. The proof is a trivial consequence of the medial law. □

Note here that the converse is not true because there are medial groupoids, which are not LA-semigroups.

An LA-semigroup $\mathcal{G}$ is called *idempotent* if $aa = a$ for all $a \in G$. An LA-semigroup $\mathcal{G}$ in which $aa = bb$ for all $a, b \in G$ is called *unipotent*.

Proposition 2.7. *Let $\mathcal{G}$ be a left cancellative LA-semigroup. Then:*

(i) *α and ψ are permutations of G, if $((G, \circ), \alpha, \psi, g)$ is a right linear form of $\mathcal{G}$, ,*

(ii) *φ and β are permutations of G, if $((G, \circ), \varphi, \beta, g)$ is a left linear form of $\mathcal{G}$.*

Proof. (i) Since $((G, \circ), \alpha, \psi, g)$ is a right linear form of a left cancellative LA-semigroup $\mathcal{G}$, therefore α is a mapping from G onto G and ψ is an endomorphism of $\mathcal{G}$. We prove that α and ψ are one-to-one.

Let $\alpha(a) = (aj) \circ g^{-1} = R_j(a) \circ g^{-1}$. If $\alpha(a) = \alpha(b)$, then $R_j(a) \circ g^{-1} = R_j(b) \circ g^{-1}$. Since $(G, \circ)$ is cancellative, therefore $R_j(a) = R_j(b)$, which by Theorem 2.6 from [5], implies $a = b$. Hence α is one-to-one.

Let $\psi(a) = L_y(a)$, where $y = \alpha^{-1}(g^{-1})$. Since $\alpha(a) = R_j(a) \circ g^{-1}$, therefore $\alpha(y) = R_j(y) \circ g^{-1}$. But $\alpha(y) = g^{-1}$ implies $g^{-1} = R_j(y) \circ g^{-1}$, i.e. $y = \alpha^{-1}(R_j(y) \circ g^{-1}) = \alpha^{-1}(g^{-1})$. Now $\psi(a) = L_y(a) = \alpha^{-1}(R_j(y) \circ g^{-1})a$. If $\psi(a) = \psi(b)$, then $\alpha^{-1}(y\,j \circ g^{-1})a = \alpha^{-1}(y\,j \circ g^{-1})b$. Since α is one-to-one, therefore $(y\,j \circ g^{-1})a = (y\,j \circ g^{-1})b$, which by Theorem 2.6 from [5] implies $a = b$. Thus ψ is one-to-one.

(ii) Analogously as (i). □

Theorem 2.8. *Let $\mathcal{G}$ be an LA-semigroup. Then the following conditions are equivalent:*

(i) *$\mathcal{G}$ is a division μ-LA-semigroup,*

(ii) *$\mathcal{G}$ has a linear form $((G, +), \sigma, \psi, g)$ such that $(G, +)$ is a commutative group and $\sigma(\psi(a) + g) = \sigma(g) + \psi\sigma(a)$.*

Proof. Since a division LA-semigroup $\mathcal{G}$ is medial, by Theorem 16 in [3], $\mathcal{G}$ has a linear form $((G,+),\sigma,\psi,g)$ such that $(G,+)$ is a commutative group and $\sigma\psi=\psi\sigma$. Thus $\sigma(\psi(a)+g)=\sigma(g)+\sigma\psi(a)=\sigma(g)+\psi\sigma(a)$ because σ is an endomorphism.

Conversely, if an LA-semigroup $\mathcal{G}$ has a linear form as in (ii), then $ab=\sigma(a)+g+\psi(b)$, which for $g=0$ shows that $\mathcal{G}$ is a division μ-LA-semigroup. □

Theorem 2.9. *Let an LA-semigroup $\mathcal{G}$ has a linear form $((G,\circ),\varphi,\psi,g)$. Then $\mathcal{G}$ is a commutative group, if φ, ψ are central automorphism of $(G,\circ)$ and $\varphi\psi=\psi\varphi$.*

Proof. If φ,ψ are central automorphisms of $(G,\circ)$ such $\varphi\psi=\psi\varphi$, then $\varphi(a),\psi(a)\in C(G,\circ)$ for every $a\in G$. Thus $\varphi\psi(a)\in C(G,\circ)$ and $\varphi\psi(a)\circ g=g\circ\psi\varphi(a)$ for every $g\in G$. Theorem 2.2 completes the proof. □

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