Extending sloops of cardinality 16 to SQS-skeins with all possible congruence lattices

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Abstract

It is well known that each STS(15) with a sub-$STS(7)$ is derived [11]. In this article, we will improve this result by showing that each non-simple sloop L of cardinality 16 with any possible congruence lattice $C(L)$ can be extended to a non-simple SQS-skein S of cardinality 16 with all possible congruence lattices for $C(S)$. Accordingly, we may say that any triple system $STS(15)$ with m sub-$STS(7)$s is a derived triple system from an $SQS(16)$ having n sub-$SQS(8)$s for all possible non-zero numbers of m and n.

1. Introduction

A *Steiner quadruple* (*triple*) *system* is a pair $(L;B)$, where L is a finite set and B is a collection of 4-subsets (3-subsets) called blocks of L such that every 3-subset (2-subset) of L is contained in exactly one block of B [9], [10]. Let $SQS(m)$ denote a Steiner quadruple system (briefly: quadruple system) of cardinality m and $STS(n)$ denote Steiner triple system (briefly: triple system) of cardinality n.

It is well known that $SQS(m)$ exists iff $m \equiv 2$ or 4 (mod 6) and $STS(n)$ exists iff $n \equiv 1$ or 3 (mod 6) (cf. [9], [10]).

Let $\mathbf{L} = (L;B)$ be a quadruple system. If one considers $L_x = L - \{x\}$ for any point $x \in L$ and deletes that point from all blocks which contain it then the resulting system $(L_x;B(x))$ is a triple system, where $B(x) = \{b^{‘} = b - \{x\} : b \in B$ and $x \in b\}$. Now, $(L_x;B(x))$ is called a derived triple system (or briefly DTS) of $(L;B)$ (cf. [9], [10]).

There is one to one correspondence between STSs and sloops. A sloop $\mathbf{L} = (L;\cdot,1)$ is a groupoid with a neutral element 1 satisfying the identities:

$$x \cdot y = y \cdot x, \quad 1 \cdot x = x, \quad x \cdot (x \cdot y) = y.$$

2000 Mathematics Subject Classification: 05B30, 08A99, 05B07, 20N05
Keywords: Steiner triple system, derived sloop, Steiner quadruple system, SQS-skein.

Notice that for any a and $b \in L$ the equation $a \cdot x = b$ has the unique solution $x = a \cdot (a \cdot x) = a \cdot b$, i. e., $\mathbf{L}$ is a quasigroup.

A sloop $\mathbf{L}$ is called *Boolean* if it satisfies in addition the associative law.

Also, there is one to one correspondence between SQSs and SQS-skeins (cf. [9], [10]). An SQS-skein $(S; q)$ is an algebra with a unique ternary operation q satisfying:

$$\begin{aligned} q(x, y, z) &= q(x, z, y) = q(z, x, y), \\ q(x, x, y) &= y, \\ q(x, y, q(x, y, z)) &= z. \end{aligned}$$

Since the equation $q(a, b, x) = c$ has the unique solution $q(a, b, c) = x$ for $a, b, c \in S$, it follows that an SQS-skein $(S; q)$ is a ternary quasigroup (3-quasigroup).

An SQS-skein $(S; q)$ is called *Boolean* if it satisfies in addition the identity: $q(a, x, q(a, y, z)) = q(x, y, z)$.

The sloop associated with a derived triple system is also called derived.

A subsloop $\mathbf{N}$ of $\mathbf{L}$ (sub-SQS-skein of $\mathbf{S}$) is called *normal* if and only if $\mathbf{N} = [1]\theta$ ($\mathbf{N} = [x]\theta$) for a congruence θ on $\mathbf{L}(respectively, \mathbf{S})$ (cf. [1], [12]).

A subsloop $\mathbf{N}$ is called normal if and only if

$$x \cdot (y \cdot N) = (x \cdot y) \cdot N$$

for all $x, y \in L$ [12].

There is an isomorphism between the lattice of normal subsloops (sub-SQS-skeins containing a fixed element) and the congruence lattice of the sloop (SQS-skein) (cf. [1], [12]). Quackenbush in [12] and similarly the author in [1] have proven that the congruences of sloops (of SQS-skeins) are permutable, regular and uniform. Moreover, they proved the following property well known from groups.

Theorem 1. *Every subsloop (sub-SQS-skein) of a finite sloop $\mathbf{L} = (L; \cdot, 1)$ (SQS-skein $\mathbf{S} = (S; q)$) with cardinality $\frac{1}{2}|L|$ (respectively, $\frac{1}{2}|S|$) is normal.*

The variety of all sloops (SQS-skeins) is a Mal'cev variety. Any Boolean group is a sloop that is called a *Boolean sloop*. If $(G; +)$ is a Boolean group, then $(G; q(x, y, z) = x + y + z)$ is a Boolean SQS-skein [1]. The class of all Boolean sloops (Boolean SQS-skeins) is the smallest non-trivial subvariety of the variety of all sloops (SQS-skeins).

In section 2, we will do an algebraic classification of the class of all sloops of cardinality 16 according to the shape of its congruence lattice and the concepts of solvability and nilpotence. We will show that this classification coincides with the combinatorial classification based on the number of subsystems of cardinality 7 (cf. [5], [7]) and the classification of the class of all SQS-skeins of cardinality 16 (cf. [1]).

Let $\mathbf{L}$ be a derived sloop from an SQS-skein $\mathbf{S}$, then the congruence lattice $C(\mathbf{S})$ of $\mathbf{S}$ is a sublattice of the congruence lattice $C(\mathbf{L})$ of $\mathbf{L}$. We are faced with the question: is any sloop $\mathbf{L}$ of cardinality 16 derived from an SQS-skein $\mathbf{S}$ for all possible sublattice $C(\mathbf{S})$ of the lattice $C(\mathbf{L})$?

Among the $DTS(15)$s determined in [11], there are 23 systems having a subsystem of order 7. In this article, it will be shown that any $STS(15)$ with n sub-$STS(7)$s can be extended to an $SQS(16)$ with $2n$ sub-$SQS(8)$s in particular and to an $SQS(16)$ with all possible number of sub-$SQS(8)$s in general.

Clearly any Boolean sloop is derived from a Boolean SQS-skein and both have the same congruence lattice. In subsection 3.1, we will show that any non-simple sloop $\mathbf{L}$ of cardinality 16 can be derived from an SQS-skein $\mathbf{S}$ in which both $\mathbf{L}$ and $\mathbf{S}$ have the same congruence lattice.

In [8] Guelzow constructed a semi-Boolean SQS-skein of cardinality 16 all of whose derived sloops are Boolean. Then, we may say that if the congruence lattices of all derived sloops of an SQS-skein are isomorphic, it is not necessary that the congruence lattice of this SQS-skein is isomorphic to them.

Subsection 3.2 is devoted to the proof that any non-simple sloop $\mathbf{L}$ of cardinality 16 can be extended to an SQS-skein $\mathbf{S}$ with any proper sub-lattice $C(\mathbf{S})$ of the lattice $C(\mathbf{L})$.

2. Algebraic classification of sloops of cardinality 16

We define the solvability of sloops similarly as the definition of solvability of SQS-skeins given in [1]. A congruence θ of a sloop $\mathbf{L}$ (an SQS-skein $\mathbf{S}$) will be called *Boolean* if $\mathbf{L}/\theta$ ($\mathbf{S}/\theta$) is Boolean. Clearly, the largest congruence of any sloop (SQS-skein) is Boolean and the intersection of any two Boolean congruences is Boolean.

A *Boolean series* of congruences on a sloop $\mathbf{L}$ (an SQS-skein $\mathbf{S}$) is a series of congruences

$$1 := \theta_0 \supseteq \theta_1 \supseteq \theta_2 \supseteq ... \supseteq \theta_n := 0$$

such that the factor algebra $[1]\theta_i/\theta_{i+1}$ (respectively, $[x]\theta_i/\theta_{i+1}$) is a Boolean sloop (respectively, SQS-skein) for all $i = 0, 1, ..., n-1$. If n is the smallest length of a Boolean series, then $\mathbf{L}$ (respectively, $\mathbf{S}$) is *solvable of length n*.

Centrality in Mal'cev varieties is defined in [13]. We apply this definition on sloops similarly as in SQS-skeins [1]. A congruence of a sloop $\mathbf{L}$ (an SQS-skein $\mathbf{S}$) is called *central*, if it contains the diagonal relation

$$\Delta_L = \{(a, a) : a \in L\} \quad (\Delta_S = \{(a, a) : a \in S\})$$

as a normal subsloop of $\mathbf{L}$ (respectively, sub-SQS-skein of $\mathbf{S}$). A central congruence of the sloop $\mathbf{L}$ (SQS-skein $\mathbf{S}$) is denoted by $\xi(\mathbf{L})$ (respectively, by $\xi(\mathbf{S})$). If there is a series of congruences on $\mathbf{L}$ (of $\mathbf{S}$)

$$1 := \theta_0 \supseteq \theta_1 \supseteq \theta_2 \supseteq \ldots \supseteq \theta_n := 0$$

such that $\theta_i/\theta_{i+1} \subseteq \xi(\mathbf{L}/\theta_{i+1})$ (respectively, $\theta_i/\theta_{i+1} \subseteq \xi(\mathbf{S}/\theta_{i+1})$) for all $i = 0, 1, ..., n-1$, then this series is called central series of $\mathbf{L}$ (of $\mathbf{S}$). Also, $\mathbf{L}$ (respectively, $\mathbf{S}$) is called nilpotent of class n, if n is the smallest length of central series in $\mathbf{L}$ (in $\mathbf{S}$). A construction of nilpotent sloops (SQS-skeins) of class n for each positive integer n is given in [3] and [4].

It is routine matter to see that the class of all solvable sloops (SQS-skeins) and the class of all nilpotent sloops (SQS-skeins) are varieties. It is easy to show that each central series of $\mathbf{L}$ (of $\mathbf{S}$) is a Boolean series (cf.[1]). Then we may say that the variety of nilpotent sloops (SQS-skeins) is a subvariety of the variety of solvable sloops (SQS-skeins) [1]. Notice that not every solvable sloop (SQS-skein) is nilpotent (examples of a solvable sloop $\mathbf{L}$ (SQS-skein $\mathbf{S}$), which is not nilpotent, will be given in Lemma 2 for $n = 1$ and 2).

By the definition of solvability, we may say that the cardinality $|L|$ ($|S|$) of a solvable sloop $\mathbf{L}$ (SQS-skeins $\mathbf{S}$) is equal to 2^n for a positive integer n. The class of solvable sloops (SQS-skeins) of order 1 and the nilpotent sloops (SQS-skeins) of class 1 are exactly the Boolean sloops (SQS-skeins). Notice that all sloops (SQS-skeins) of cardinality 2, 2^2 and 2^3 are Boolean and for any positive integer n, there is exactly one Boolean sloop (SQS-skein) (up to isomorphism) with cardinality 2^n that is the direct power of the 2-element group.

To determine the different classes of sloops of cardinality 16, let $\mathbf{L}$ (respectively, $\mathbf{S}$) be a non-simple sloop (SQS-skein) with $|L| = 16$ ($|S| = 16$) and $C(\mathbf{L})$ ($C(\mathbf{S})$) be its congruence lattice. If $C(\mathbf{L})$ ($C(\mathbf{S})$) has more than one atom, then $\mathbf{L}$ (respectively, $\mathbf{S}$) is Boolean. If $C(\mathbf{L})$ ($C(\mathbf{S})$) has exactly one atom θ, then $C(\mathbf{L}/\theta)$ (respectivelt, $C(\mathbf{S}/\theta)$) is isomorphic to the lattice of subgroups $Sub(\mathbb{Z}_2^n)$ for $n = 1, 2$ or 3, where $\mathbb{Z}_2$ is the 2-element group. This leads directly to a similar classification of the class of SQS-skeins of cardinality 16 (cf. [1], [2]).

Lemma 2. *Let $\mathbf{L}(\mathbf{S})$ be a sloop (an SQS-skein) of cardinality 16 and θ be an atom of the congruence lattice $C(\mathbf{L})$ ($C(\mathbf{S})$). Then $\mathbf{L}(\mathbf{S})$ is simple or $C(\mathbf{L}/\theta) \cong C(\mathbf{S}/\theta) \cong Sub(\mathbb{Z}_2^n)$ for $n = 1, 2, 3$ or $C(\mathbf{L}) \cong C(\mathbf{S}) \cong Sub(\mathbb{Z}_2^4)$. Moreover, $\mathbf{L}(\mathbf{S})$ is solvable of length 2 for $n = 1$ or 2, nilpotent of length 2 for $n = 3$ and Boolean for the last case.*

Proof. The proof for SQS-skeins is given in [1]. Similarly, one can easily prove the lemma for sloops. □

Any subsloop (sub-SQS-skein) of cardinality $\frac{1}{2}|L|$ ($\frac{1}{2}|S|$) corresponds to a maximal congruence in $C(\mathbf{L})$ ($C(\mathbf{S})$). The converse is true specially for sloops (SQS-skeins) of cardinality 16, which means that a maximum congruence in $C(\mathbf{L})$ ($C(\mathbf{S})$) corresponds to a subsloop (2 sub-SQS-skeins) of cardinality 8. This leads us to reformulate the classification given in Lemma 2 into classification depending on the number of subsloops (sub-SQS-skeins) of cardinality 8, as in the following lemma.

Lemma 3. *Let $\mathbf{L}(\mathbf{S})$ be a sloop of cardinality 16, then $\mathbf{L}(\mathbf{S})$ has n subsloops ($2n$ sub-SQS-skeins) of cardinality 8 for $n = 0, 1, 3, 7$ or 15.*

In fact, these classes associate with the same well-known classes of triple systems of cardinality 15. In [5], [6] and [7] all possible triple systems of order 15 were given. This means that structures of sloops of cardinality 16 with any possible congruence lattice (equivalently with any possible number of subsloops of cardinality 8) are well known. Also, examples of SQS-skeins of cardinality 16 with each possible congruence lattice (equivalently with any possible number of sub-SQS-skeins of cardinality 8) are well known (cf. [1] and [2]).

3. Extending a sloop L(16) to an SQS-skein S(16)

Cole, White and Cummtings [7] first determined that there are exactly 80 non-isomorphic triple systems of order 15. A listing of all 80 triple systems can be found in Bussemark and Seidel [5]. A triple system is called derived, if it can be extended to a quadruple system. There are 23 triple systems of order 15 having subsystems of order 7. All are derived [11].

Let $\mathbf{L} = (L; \cdot, 1)$ be a derived sloop of an SQS-skein $\mathbf{S} = (S; q)$, so the fundamental operations of $\mathbf{L}$ are polynomial functions of the operation q, which means in general that the congruence lattice $C(\mathbf{S})$ is a sublattice of $C(\mathbf{L})$. Namely, if $C(\mathbf{L}/\theta) \cong Sub(\mathbb{Z}_2^m)$ and $C(\mathbf{S}/\theta) \cong Sub(\mathbb{Z}_2^n)$ for an atom θ, then $n \leqslant m$. As a special case, if $\mathbf{L}$ is simple derived sloop from the SQS-skein $\mathbf{S}$, then $\mathbf{S}$ must be simple. Notice that each triple system having no subsystems of order 7 associates with a simple sloop.

This paper is a generalization of the result of Phelps in [11] that every non-simple sloop of order 16 can be extended to a SQS-skein of order 16. The question that the following two sections nearly answers is therefore: Given a non-simple sloop $\mathbf{L}$ (Steiner loop) with any congruence lattice $C(\mathbf{L})$, does there exist an SQS-skein $\mathbf{S}$ of order 16 such that $\mathbf{L}$ is derived from $\mathbf{S}$ for all possible $C(\mathbf{S})$? The only situation not answered in this paper is: $\mathbf{L}$ any sloop and $\mathbf{S}$ simple. Otherwise, the answer is yes.

3.1. Extending a sloop L(16) to an SQS-skein S(16) with $C(\mathbf{S})$=$C(\mathbf{L})$

In this section, we will show that: A non-simple sloop $\mathbf{L}$ with a certain congruence lattice $C(\mathbf{L})$ can be extended to a non-simple SQS-skein $\mathbf{S}$ having the same congruence lattice $C(\mathbf{S})$; i. e., $C(\mathbf{L}) = C(\mathbf{S})$. In other words, an $STS(15)$ with a non-zero number n of sub-$STS(7)$s can be extended to an $SQS(16)$ having $2n$ sub-$SQS(8)$, for each possible number n; i.e., $n = 1, 3, 7$ or 15.

Now, let $\mathbf{L}_1 = (L_1; \cdot, 1)$ be the Boolean sloop of cardinality 8 and $(L_1 - \{1\}; B_1)$ be the corresponding triple system of $\mathbf{L}_1$. It is known that $(L_1 - \{1\}; B_1)$ and the projective plane $PG(2, 2)$ are isomorphic, so we can index the element of $L_1 - \{1\}$ as follows:

$\{a_0, a_1, ..., a_6\}$ where $\{0, 1, ..., 6\}$ is the set of points of $PG(2,2)$ such that $\{i, j, k\}$ is a line in $PG(2,2)$ if and only if $\{a_i, a_j, a_k\}$ is a block in B_1. Moreover, we denote the set of lines of $PG(2,2)$ by the set $\{i, i+1, i+3\}$ (mod 7).

Let $\mathbf{F} = \{F_0, F_1, ..., F_6\}$ be a 1-factorization of the complete graph with the vertices L_1, where $F_i = \{a_j a_k : a_j \cdot a_k = a_i \text{ in } \mathbf{L}_1\}$. We observe that $1a_i$ is an edge in F_i for each i. Also, we consider the sets $L_2 = \{b, b_0, b_1, ..., b_6\}$ and $L = L_1 \cup L_2$ such that $L_1 \cap L_2 = \emptyset$. We define the 1-factorization $\mathbf{G}$ of the complete graph K_8 with the set of vertices L_2 similarly as $\mathbf{F}$ by writing b instead of 1 and b_i instead of a_i in each factor of $\mathbf{F}$. Now we are ready to formulate the following well-known constructions for sloops and SQS-skeins of cardinality 16 [10].

Construction 1. *Let α be a permutation on the set $\{0, 1, ..., 6\}$. By taking $B := B_1 \cup \{\{a_i, b_j, b_k\} : b_j b_k \in G_{\alpha(i)}\}$, then $(L - \{1\}; B)$ is a triple system containing $(L_1 - \{1\}; B_1)$ as a subsystem* [10].

Let $\mathbf{L} = (L; \cdot, 1)$ be the given associated sloop with the triple system $(L - \{1\}; B)$ and $\mathbf{L}_1 = (L_1; \cdot, 1)$ be the associated subsloop, where the binary operation "." is defined by:

$$x \cdot y := \begin{cases} z & if \quad \{x, y, z\} \in B \\ 1 & if \quad x = y \end{cases}$$

By Theorem 1, we may say that $\mathbf{L}$ has at least one maximal congruence θ_0 determined by the normal subsloop $\mathbf{L}_1$.

Theorem 4. *Construction* 1 *yields precisely all non-simple sloops of cardinality* 16.

Proof. Without loss of generality, we may call the elements of L, L_1 and $L_2 = L - L_1$, the sloop $\mathbf{L} = (L; \cdot, 1)$, the subsloop $\mathbf{L}_1 = (L_1; \cdot, 1)$ and the 1-factorization $\mathbf{F}$ on L_1 exactly as the preceding definitions. Since $b \cdot a_i \in L_2$ for each $a_i \in L_1$, we may define the permutation α on the set $\{0, 1, 2, ..., 6\}$ by $b_{\alpha(i)} = b \cdot a_i$.

Moreover, we define a 1-factor $G_{\alpha(i)}$ on L_2 by the rule: $xy \in G_{\alpha(i)}$ if and only if $x \cdot y = a_i$ in $\mathbf{L}$. This supplies us with a 1-factorization $\mathbf{G} = \{G_0, G_1, ..., G_6\}$ on the set of points L_2.

Let $(L - \{1\}; B)$ be the triple system constructed by construction 1. If $\{a_i, a_j, a_k\}$ is a block in B_1, then $a_i \cdot a_j = a_k$ in $\mathbf{L}_1$ and if $\{a_i, b_j, b_k\}$ is a block in B, then $a_i = b_j \cdot b_k$ in $\mathbf{L}$. This means that the triple system $(L - \{1\}; B)$ coincides with the associated triple system with the sloop $\mathbf{L}$. This completes the proof of the theorem. □

Construction 2. *Let*

$Q_1 = \{\{x_1, x_2, y_1, y_2\} : 0 \leqslant i \leqslant 6, x_1x_2 \in F_i \& y_1y_2 \in F_i\}$,

$Q_2 = \{\{x_1, x_2, y_1, y_2\} : 0 \leqslant i \leqslant 6, x_1x_2 \in G_i \& y_1y_2 \in G_i\}$,

$Q = Q_1 \cup Q_2 \cup \{\{x_1, x_2, y_1, y_2\} : 0 \leqslant i \leq 6, x_1x_2 \in F_i \ \& y_1y_2 \in G_{\alpha(i)}\}$.

Since $\mathbf{L}_1$ is a Boolean sloop, so for all $x, y, z, w \in L_1$ if $x \cdot y = z \cdot w$, then $x \cdot z = y \cdot w$ and $y \cdot z = x \cdot w$. Then if $xy, zw \in F_i$, hence $xz, yw \in F_j$ and $xw, zy \in F_k$ for some j and k. This means that $\{x, y, z, w\}$ is the unique block *in* $\mathbf{Q}_1$ *containing any 3-element subset of it. Accordingly,* $\mathbf{Q}_1 = (L_1; Q_1)$ *and* $\mathbf{Q}_2 = (L_2; Q_2)$ *are* $SQS(8)$*s. Hence* $\mathbf{Q} = (L; Q)$ *is a quadruple system in which* $\mathbf{Q}_1$ *and* $\mathbf{Q}_2$ *are subsystems.*

The associated SQS-skein $\mathbf{S} = (L; q)$ with the quadruple system $\mathbf{Q} = (L; Q)$ has at least one maximum congruence θ_0 determined by the two classes L_1 and L_2 (cf. [1], [9], where the operation q is defined by:

$$q(x, y, z) = \begin{cases} w & if \quad \{x, y, z, w\} \in Q \\ z & if \quad x = y \end{cases}$$

By the definition of F_i, if $\{a_i, a_j, a_k, 1\} \in Q_1$, then $1a_i, a_ja_k \in F_i$, which means that $\mathbf{L}_1$ is a derived sloop of $\mathbf{Q}_1$. Moreover, if $\{x, y, z\} \in B$, then $\{x, y, z\} \in B_1$ or $\{x, y, z\} \in \{\{a_i, b_j, b_k\} : b_jb_k \in G_{\alpha(i)}\}$.

Hence $\{x, y, z\} = \{a_i, a_j, a_k\}$ or $\{x, y, z\} = \{a_i, b_j, b_k\}$ for $b_jb_k \in G_{\alpha(i)}$, which means that $1a_i, a_ja_k \in F_i$ or $1a_i \in F_i$ and $b_jb_k \in G_{\alpha(i)}$. This implies that $\{1, x, y, z\} \in Q$. Therefore, $(L - \{1\}; B)$ is a derived triple system of the quadruple system $\mathbf{Q} = (L; Q)$.

Now, consider two sets:

$$S`_1 = \{1, a_i, a_{i+1}, a_{i+3}, b, b_{\alpha(i)}, b_{\alpha(i+1)}, b_{\alpha(i+3)}\}$$

and

$$S`_2 = \{1, a_i, a_{i+1}, a_{i+3}, b_{\alpha(i+2)}, b_{\alpha(i+4)}, b_{\alpha(i+5)}, b_{\alpha(i+6)}\}.$$

By choosing a suitable permutation α, we will show in the following that there is a derived sloop $\mathbf{L}$ from an SQS-skein $\mathbf{S}$ of cardinality 16 in which both $\mathbf{L}$ and $\mathbf{S}$ have the same congruence lattice.

Lemma 5. $\mathbf{S}`_1$ *is a subsloop of* $\mathbf{L}$ *a sub-SQS-skein of* $\mathbf{S}$ *if and only if* $\{\alpha(i), \alpha(i+1), \alpha(i+3)\}$ *is a line in* $PG(2, 2)$.

Proof. Let $\mathbf{S`_1}$ be a subsloop of $\mathbf{L}$, then we have:

$$b \cdot b_{\alpha(i)} = a_i = b_{\alpha(i+1)} \cdot b_{\alpha(i+3)} \iff b\, b_{\alpha(i)}, b_{\alpha(i+1)} b_{\alpha(i+3)} \in G_{\alpha(i)}$$
$$\iff \{\alpha(i), \alpha(i+1), \alpha(i+3)\} \text{ is a line in } PG(2, 2).$$

Also,

$$b \cdot b_{\alpha(i+1)} = a_{i+1} = b_{\alpha(i)} \cdot b_{\alpha(i+3)} \iff \{\alpha(i), \alpha(i+1), \alpha(i+3)\}$$
$$\text{is a line in } PG(2, 2) \iff b \cdot b_{\alpha(i+3)} = a_{i+3} = b_{\alpha(i)} \cdot b_{\alpha(i+1)}.$$

Similarly, one can prove the other direction. The proof of this lemma for the SQS-skeins is given in [1]. □

Lemma 6. *If* $\mathbf{S`_1}$ *is a subsloop of* $\mathbf{L}$ (*a sub-SQS-skein of* $\mathbf{S}$), *then* $\mathbf{S`_2}$ *is also a subsloop of* $\mathbf{L}$ (*a sub-SQS-skein of* $\mathbf{S}$).

Proof. The 1-factorization of the complete graph K_4 with the set of vertices $\{b_{\alpha(i+2)}, b_{\alpha(i+4)}, b_{\alpha(i+5)}, b_{\alpha(i+6)}\}$ is included in the factors $G_{\alpha(i)}$, $G_{\alpha(i+1)}$, $G_{\alpha(i+3)}$. This shows directly that $\mathbf{S}`_2$ is a subsloop of $\mathbf{L}$ (an sub-SQS-skein of $\mathbf{S}$). □

Lemma 7. *For each line transformed into a line by the permutation α in $PG(2,2)$, two maximum congruences are formed in the lattice $C(\mathbf{L})$ ($C(\mathbf{S})$) in addition to θ_0.*

Proof. We have $|S`_1| = |S`_2| = \frac{1}{2}\,|L|$, so $\mathbf{S}`_1$ and $\mathbf{S}`_2$ are two distinct normal subsloops of $\mathbf{L}$ (sub-SQS-skeins of $\mathbf{S}$). Let θ_1 and θ_2 be the associated congruences with $\mathbf{S}`_1$ and $\mathbf{S}`_2$, respectively. Then $\theta_1 \cap \theta_2$ is a congruence with 4 congruence classes, which implies that there are exactly three covers of $\theta_1 \cap \theta_2$, namely θ_0, θ_1, θ_2. This completes the proof. $\square$

In fact, this similarity between properties of sloops and SQS-skeins leads directly to the following result.

Theorem 8. *Let $\mathbf{L}$ ($\mathbf{S}$) be a sloop (an SQS-skein) of cardinality 16 and assume that its congruence lattice $C(\mathbf{L})$ ($C(\mathbf{S})$) has an atom θ. If the permutation α transforms $2^{n-2}-1$ lines into lines in $PG(2,2)$ for $n = 2,3,4$, or 5, then $C(\mathbf{L}/\theta) \cong C(\mathbf{S}/\theta) \cong Sub(\mathbb{Z}_2^{n-1})$ for $n = 2,3,4$ and $C(\mathbf{L}) \cong C(\mathbf{S}) \cong Sub(\mathbb{Z}_2^4)$ for $n = 5$.*

Proof. According to the Lemmas 4, 5 and 6, we get directly the required. $\square$

Consequently, we may say that any sloop of cardinality 16 with n subsloops of cardinality 8 is a derived sloop from an SQS-skein of cardinality 16 having $2n$ sub-SQS-skeins for each possible non-zero number n; i.e. for $n = 1,3,7$ and 15.

3.2. Extending a sloop L(16) to an SQS-skein S(16) with arbitrary $C(\mathbf{S}) \leqslant C(\mathbf{L})$

In this section, we will show that: A non-simple sloop $\mathbf{L}$ with any possible congruence lattice $C(\mathbf{L})$ can be extended to a non-simple SQS-skein $\mathbf{S}$ with all possible congruence lattice $C(\mathbf{S})$; i.e., for all possible sublattice $C(\mathbf{S})$ of $C(\mathbf{L})$.

Without loss of generality and according to the definition of the 1-factorization $\mathbf{F}$ given in constructions 1 and 2, we may choose the sub-1-factors:

1− $f_0 = \{a_1a_3, a_4a_5\} \subseteq F_0$ and $f_2 = \{a_1a_4, a_3a_5\} \subseteq F_2$ on the set $\{a_1, a_3, a_4, a_5\}$.

2− $f_1 = \{a_2a_4, a_5a_6\} \subseteq F_1$ and $f_3 = \{a_2a_5, a_4a_6\} \subseteq F_3$ on the set $\{a_2, a_4, a_5, a_6\}$.

3− $f_4 = \{a_1a_2, a_0a_5\} \subseteq F_4$ and $f_6 = \{a_0a_2, a_1a_5\} \subseteq F_6$ on the set $\{a_0, a_1, a_2, a_5\}$.

By interchanging the sub-1-factors f_0 and f_2 in the 1-factors F_0 and F_2 we get new 1-factors $F_0^`$ and $F_2^`$, where $F_0^` = \{1a_0, a_1a_4, a_3a_5, a_2a_6\}$ and $F_2^` = \{1a_2, a_1a_3, a_4a_5, a_0a_6\}$. Similarly, we interchange the sub-1-factors f_1 and f_3 in the 1-factors F_1 and F_3 to get new 1-factors $F_1^`$ *and* $F_3^`$ and the sub-1-factors f_4 and f_6 in the 1-factors F_4 and F_6 to get new 1-factors $F_4^`$ and $F_6^`$.

Now, we consider three new 1-factorizations on the set L_1 :

$$
\begin{aligned}
{}_1\mathbf{F}^{`} &= \{F_0^{`}, F_1, F_2^{`}, F_3, F_4, F_5, F_6\},\\
{}_2\mathbf{F}^{`} &= \{F_0^{`}, F_1^{`}, F_2^{`}, F_3^{`}, F_4, F_5, F_6\},\\
{}_3\mathbf{F}^{`} &= \{F_0^{`}, F_1^{`}, F_2^{`}, F_3^{`}, F_4^{`}, F_5, F_6^{`}\}.
\end{aligned}
$$

Let Q_1 and Q_2 be the same as in construction 2, and let

$$
{}_jQ^{`} = Q_1 \cup Q_2 \cup \overline{Q},
$$

where

$\overline{Q} = \{\{x_1, x_2, y_1, y_2\} : x_1x_2 \in F_i^{`} \in_j \mathbf{F}^{`}$ and $y_1y_2 \in G_{\alpha(i)}$ for some $0 \leqslant i \leqslant 6\}$.

Indeed, the changes occurs only in the quadruple systems, so we will denote the new quadruple systems by $(L; {}_jQ^{`})$ for $j = 1, 2, 3$. Notice that the triple system $(L - \{1\}; B)$ is still as a derived triple system of $(L; {}_jQ^{`})$ for each $j = 1, 2, 3$.

The 1-factorization ${}_1\mathbf{F}^{`}$ contains exactly the three sub-1-factorizations $\{F_0^{`}, F_2^{`}, F_6\}$, $\{F_1, F_5, F_6\}$, $\{F_3, F_4, F_6\}$ in which each of them contains two disjoint sub-1-factorizations of the complete graph $K_4$. Similarly, the 1-factorization ${}_2\mathbf{F}^{`}$ contains exactly one sub-1-factorization$\{F_0^{`}, F_2^{`}, F_6\}$ containing two disjoint sub-1-factorizations of the complete graph K_4 and the 1-factorization ${}_3\mathbf{F}^{`}$ does not contain any sub-1-factorization of the complete graph K_4.

We observe that α may transform $2^{n-2} - 1$ lines into lines in $PG(2, 2)$ for $n = 2, 3, 4, 5$. Thus:

If $n = 2$, then α does not transform any line into a line.

If $n = 3$, then α transforms at most one line into a line among the lines of the subset $R = \{\{0, 2, 6\}, \{1, 5, 6\}, \{3, 4, 6\}\}$.

If $n \geqslant 4$, then α transforms 1 or 3 lines into lines among the lines of R.

Now, let $(L; {}_jq^{`})$ be the associated SQS-skein with $(L; {}_jQ^{`})$ for $j = 1, 2, 3$. Analogously, we may deduce the following result.

Theorem 9. *The constructed sloop $\mathbf{L} = (L; \cdot, 1)$ is a derived sloop from the constructed SQS-skein ${}_j\mathbf{S} = (L; {}_jq^{`})$ for each $j = 1, 2$ and 3 and for any permutation α. Moreover, each non-simple sloop L can be extended to a non-simple SQS-skein ${}_j\mathbf{S}$ with all possible congruence lattices for $C(\mathbf{L})$ and $C({}_j\mathbf{S})$.*

Proof. Any permutation α transforms $2^{n-2} - 1$ lines into lines in $PG(2, 2)$ for $n = 2, 3, 4, 5$. Notice in all cases that θ_0 is a congruence of each of $\mathbf{L}$ and ${}_j\mathbf{S}$ for $j = 1, 2$ and 3, where θ_0 is determined by the two classes L_1 and L_2.

In the following, we consider θ to be the unique atom of the lattices $C(\mathbf{L})$ and $C({}_j\mathbf{S})$ for $j = 1, 2$ and 3, except in the case for $n = 5$, when θ is considered to be any atom of $C(\mathbf{L})$. Now, we have the following result:

When $n = 2$, then α does not transform any line to a line, hence $C(\mathbf{L}/\theta) \cong C({}_j\mathbf{S}/\theta) \cong Sub(\mathbb{Z}_2)$ for $j = 1$, 2 and 3, where the atom θ is equal to θ_0.

When $n = 3$, then α transforms one line into line in $PG(2,2)$, by Lemma 3 hence $C(\mathbf{L}/\theta \cong Sub(\mathbb{Z}_2^2)$. Also, α transforms nothing or one line into a line in $PG(2,2)$ among the lines of the subset R, so $C({}_3\mathbf{S}/\theta) \cong C({}_2\mathbf{S}/\theta) \cong C({}_1\mathbf{S}/\theta) \cong Sub(\mathbb{Z}_2)$, where the atom θ is equal to θ_0 , or $C({}_2\mathbf{S}/\theta) \cong C({}_1\mathbf{S}/\theta) \cong Sub(\mathbb{Z}_2^2)$.

When $n = 4$, then α transforms 3 lines into 3 lines in $PG(2,2)$, by Lemma 3 hence $C(\mathbf{L}/\theta) \cong Sub(\mathbb{Z}_2^3)$. Also, α transforms 1 or 3 lines into lines in $PG(2,2)$ among the lines of the subset $R = \{\{0,2,6\},\{1,5,6\},\{3,4,6\}\}$, so $C({}_3\mathbf{S}/\theta) = C({}_3\mathbf{S}/\theta_0) \cong Sub(\mathbb{Z}_2)$ and $C({}_2\mathbf{S}/\theta) \cong C({}_1\mathbf{S}/\ \theta) \cong Sub(\mathbb{Z}_2^2)$ or $C({}_1\mathbf{S}/\theta) \cong Sub(\mathbb{Z}_2^3)$.

When $n\ = 5$, then α transforms 7 lines into 7 lines in $PG(2,2)$, by Lemma 3 and since $C(\mathbf{L})$ contains in this case more than one atom, hence $C(\mathbf{L}/\theta) \cong Sub(\mathbb{Z}_2^3)$ for each atom θ of $C(\mathbf{L})$ or $C(\mathbf{L}) \cong Sub(\mathbb{Z}_2^4)$. This means that α transforms the three lines of R into 3 lines in $PG(2,2)$, so $C({}_3\mathbf{S}/\theta) = C({}_3\mathbf{S}/\theta_0\ ) \cong Sub(\mathbb{Z}_2)$, $C({}_2\mathbf{S}/\theta) \cong Sub(\mathbb{Z}_2^2)$ and $C({}_1\mathbf{S}/\theta) \cong Sub(\mathbb{Z}_2^3)$, where θ is still the unique atom of $C({}_j\mathbf{S})$ for $j = 1, 2$ and 3.

For the case $C(\mathbf{L}) \cong C(\mathbf{S}) \cong Sub(\mathbb{Z}_2^4)$, we may choose the Boolean SQS-skein $\mathbf{S}$ of cardinality 16 and $\mathbf{L}$ any of its derived sloops. This completes the proof. $\square$

Consequently, we may say that any sloop with a non-zero number n of subsloops of cardinality 8 can be extended to an SQS-skein having $2m$ sub-SQS-skeins of cardinality 8 for each possible positive numbers n and m; i.e., for each n and $m = 1, 3, 7$ or 15 with $m \leqslant n$.

Examples. Example for each case can be determined by choosing the permutation α as follows:

• For $n = 2$ take $\alpha = (12)(345)$, hence α does not transform any line into a line in $PG(2,2)$, which means that the congruence lattices $C(\mathbf{L})$ and $C({}_j\mathbf{S})$ for $j = 1, 2$ and 3 have exactly one co-atom θ_0.

• For $n = 3$ take $\alpha = (012)(345)$ or $\alpha = (345)$. In both cases α transforms one line into a line in $PG(2,2)$. This implies that $\mathbf{L}$ has three maximum congruences, so $C(\mathbf{L}/\theta) \cong Sub(\mathbb{Z}_2^2)$. The permutation $\alpha = (012)(345)$ transforms the line $\{0,1,3\}$ into the line $\{1,2,4\}$, this means that $C({}_j\mathbf{S})$ for $j = 1, 2$ and 3 have only one co-atom θ_0.

But the permutation $\alpha = (345)$ transforms the line $\{0,2,6\}$ into itself, hence $C({}_j\mathbf{S})$ has exactly three co-atoms for $j = 1$ and 2 and $C({}_3\mathbf{S})$ has only one co-atom θ_0.

• For $n = 4$ take $\alpha = (012345)$ or $\alpha = (4321)(650)$, both cases α transforms three lines into three lines in $PG(2,2)$, then $\mathbf{L}$ has exactly 7 maximum congruences. $\alpha = (012345)$ transforms the three lines of the set $R = \{\{0,2,6\},\{1,5,6\},\{3,4,6\}\}$ into three lines in $PG(2,2)$, which implies that $C({}_1\mathbf{S})$ has exactly 7 co-atoms, $C({}_2\mathbf{S})$ has exactly three co-atoms and $C({}_3\mathbf{S})$ has only one co-atom θ_0.

$\alpha = (4321)(650)$ transforms only the line $\{0,2,6\}$ of R into a line of R, which means that the congruence lattices $C({}_j\mathbf{S})$ has exactly three co-atoms for $j = 1$ and 2 and $C({}_3\mathbf{S})$ has only the co-atom θ_0.

• For $n = 5$ take $\alpha = identity$ on $\{0, 1, ..., 6\}$, so α transforms all lines into lines in $PG(2, 2)$, which means that $C(\mathbf{L})$ has 15 co-atoms, $C({}_1\mathbf{S})$ has 7 co-atoms, $C({}_2\mathbf{S})$ has 3 co-atoms and $C({}_3\mathbf{S})$ has only the co-atom θ_0. □

Consequently, we may say that any $STS(15)$ with a non-zero number n of sub-$STS(7)$s can be extended to an $SQS(16)$ having $2m$ sub-$SQS(8)$s for all possible non-zero positive numbers n and m; i.e., for any n and $m \in \{1, 3, 7, 15\}$ with $m \leqslant n$.

Among the $DTS(15)$s determined in [11], there are 57 systems having no subsystems of order 7. The sloops associated with these 57 systems are simple. We therefore see that the sloops associated with these 57 systems must be derived from simple SQS-skeins. But it is not necessary for a sloop derived from a simple SQS-skein to be simple.

We finish this work with a natural question:

Question. *Is whether or not a sloop of cardinality* 16 *with each possible congruence lattice can be extended to a simple SQS-skein ?*

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Received June 24, 2003
Revised May 17, 2004