Quasigroups and Related Systems **14** (2004), 75 – 78

Embedding of an AG**-groupoid in a commutative monoid

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Abstract

In this paper we have proved that if an AG^{**}-groupoid contains a left cancellative AG^{**}-subgroupoid then it can be embedded in a commutative monoid whose cancellative elements form a commutative group and the identity of this group coincides with the identity of the commutative monoid.

An AG-groupoid is an algebraic structure midway between a groupoid and a commutative semigroup. In [3] it has been defined as a groupoid S in which the left invertive law holds, that is,

$$(ab)\,c = (cb)\,a \quad \text{for all } a, b, c \in S. \tag{1}$$

It is known [4] that S is medial, that is,

$$(ab)\,(cd) = (ac)\,(bd) \quad \text{for all } a, b, c, d \in S. \tag{2}$$

An AG-groupoid with the condition that,

$$a\,(bc) = b\,(ac) \quad \text{for all } a, b, c \in S, \tag{3}$$

is called an *AG^{**}-groupoid* [6]. It has been proved in the same paper that the following law holds in an AG^{**}-groupoid S.

$$(ab)\,(cd) = (db)\,(ca) \text{ for all } a, b, c \in S. \tag{4}$$

An element $a \in S$ is called *left cancellative*, if $ab = ac$ implies $b = c$, and *right cancellative*, if $ac = bc$ implies $a = b$.

In this paper we shall consider an AG^{**}-groupoid S containing a left cancellative AG^{**}-subgroupoid T, such that the elements of T commute with the elements of $S \setminus T$.

2000 Mathematics Subject Classification: 20M10, 20N99
Keywords: Invertive law, AG-groupoid, AG^{**}-groupoid, medial law

Example 1. The set $S = \{a, b, c, d, e\}$ with the binary operation defined by table:

$\cdot$	a	b	c	d	e
a	b	a	a	a	a
b	a	b	b	b	b
c	a	b	d	e	c
d	a	b	c	d	e
e	a	b	e	c	d

is an example of a groupoid in which the subset $T = \{c, d, e\}$ is an AG^{**}-subgroupoid such that elements from T commute with the elements from $S \setminus T$.

We introduce in $S \times T$ an equivalence relation ρ such that an AG^{**}-groupoid $S \times T/\rho$ will be with right identity. By using number of results from [4] and [5], we shall show that $S \times T/\rho$ is a commutative monoidin which the cancellative elements of $S \times T/\rho$ form a commutative group.

Theorem 1. *If S is an AG-groupoid, T is an AG-subgroupoid of S and elements of T commute with elements of $S \setminus T$, then the elements of T^2 commute with $S^2 \setminus T^2$.*

Proof. Indeed, by (2), we have $(s_1 s_2)(t_1 t_2) = (s_1 t_1)(s_2 t_2) = (t_1 s_1)(t_2 s_2) = (t_1 t_2)(s_1 s_2)$. □

Theorem 2. *If S is an AG^{**}-groupoid, and T is a left cancellative AG^{**}-subgroupoid of S such that elements of T commute with the elements of $S \setminus T$, then S can be embedded in a commutative monoid.*

Proof. Let $s_1, s_2, ... \in S$, $t_1, t_2, ... \in T$ and $(s_i, t_j)(s_k, t_l) = (s_i s_k, t_j t_l)$ for all $(s_i, t_j), (s_k, t_l) \in S \times T$. Then $S \times T$ with this operation is a groupoid.

Define on $S \times T$ the relation ρ by

$$(s_i, t_j)\rho(s_k, t_l) \Longleftrightarrow t_l s_i = t_j s_k.$$

It is obvious that ρ is reflexive and symmetric. To prove that it is transitive, let $(s_i, t_j)\rho(s_k, t_l)$ and $(s_k, t_l)\rho(s_m, t_n)$. Then $t_l s_i = t_j s_k$ and $t_n s_k = t_l s_m$. Multiplying from left the first equality by t_n we obtain $t_n(t_l s_i) = t_n(t_j s_k)$, which, by (3), gives $t_l(t_n s_i) = t_j(t_n s_k)$. But $t_n s_k = t_l s_m$, so, by (3), we obtain, $t_l(t_n s_i) = t_j(t_l s_m) = t_l(t_j s_m)$. Hence, using the left cancellativity, we get $t_n s_i = t_j s_m$. Thus $(s_i, t_j)\rho(s_m, t_n)$, means that ρ is transitive.

If $(s_i, t_j)\rho(s_k, t_l)$, then $t_l s_i = t_j s_k$. Multiplying this equality by $t_n s_m$, we get $(t_l s_i)(t_n s_m) = (t_j s_k)(t_n s_m)$. Whence $(t_l t_n)(s_i s_m) = (t_j t_n)(s_k s_m)$. Thus $(s_i s_m, t_j t_n)\rho(s_k s_m, t_l t_n)$, i.e. $(s_i, t_j)(s_m, t_n)\rho(s_k, t_l)(s_m, t_n)$. This shows that ρ is right compatible. Similarly, it can be shown that ρ is left compatible. Hence ρ is a congruence on $S \times T$.

Let $M = S \times T/\rho = \{[(s_i, t_j)] \mid s_i \in S$ and $t_j \in T\}$. It is straight forward to see that M is an AG^{**}-groupoid. It can easily be shown that $[(t_o, t_o)]$ is the right identity in M, where t_o is an arbitrary element of T. If $[(s_i, t_j)]$ is an arbitrary element in M, then, by (3) and the fact that the elements from T and $S \setminus T$ commute, we have $t_j(s_i t_o) = s_i(t_j t_o) = (t_j t_o)s_i$. So, $(s_i t_o, t_j t_o)\rho(s_i, t_j)$, i.e. $(s_i, t_j)(t_o, t_o)\rho(s_i, t_j)$. Hence $[(s_i, t_j)][(t_o, t_o)] = [(s_i, t_j)]$. This shows that $[(t_o, t_o)]$ is the right identity in M. The uniqueness follows from Theorem 2.2 of [2]. Since M is an AG^{**}-groupoid with right identity, it becomes a commutative monoid.

Let t_x be a fixed element of T. We define $\Phi : S \longrightarrow M$ by

$$(s_i)\Phi = [(s_i t_x, t_x)] \quad \text{for all } s_i \in S \text{ and } t_x \in T.$$

Suppose $[(s_i t_x, t_x)] \neq [(s_j t_x, t_x)]$. This implies that $(s_i t_x, t_x)$ is not ρ equivalent to $(s_j t_x, t_x)$, i.e. $t_x(s_i t_x) \neq t_x(s_j t_x)$. Thus $(s_i t_x) \neq (s_j t_x)$ and $s_i \neq s_j$, so, Φ is well defined.

Next we show that $(s_i s_j)\Phi = (s_i)\Phi(s_j)\Phi$.

Indeed, $(s_i)\Phi(s_j)\Phi = [(s_i t_x, t_x)][(s_j t_x, t_x)] = [((s_i t_x)(s_j t_x), t_x t_x)] = [((s_i s_j)(t_x t_x), t_x t_x)]$. But $((s_i s_j)(t_x t_x), t_x t_x)\rho((s_i s_j)t_x, t_x)$, because we have $t_x((s_i s_j)(t_x t_x)) = (t_x t_x)((s_i s_j)t_x)$, as $(t_x t_x)((s_i s_j)t_x) = (t_x t_x)(t_x(s_i s_j))$, due to the assumption that T commutes with $S \setminus T$. Also, by (3), we obtain $(t_x t_x)((s_i s_j)t_x) = t_x((s_i s_j)(t_x t_x))$. Hence $((s_i s_j)(t_x t_x), t_x t_x)\rho((s_i s_j)t_x, t_x)$ and so

$$[((s_i s_j)(t_x t_x), t_x t_x)] = [((s_i s_j)t_x, t_x)] = (s_i s_j)\Phi,$$

shows that $(s_i)\Phi(s_j)\Phi = (s_i s_j)\Phi$. Thus Φ is a homomorphism.

It is one-to-one, because $(s_i)\Phi = (s_j)\Phi$ implies $[(s_i t_x, t_x)] = [(s_j t_x, t_x)]$, i.e. $(s_i t_x, t_x)\rho(s_j t_x, t_x)$. Thus $t_x(s_i t_x) = t_x(s_j t_x)$ and $s_i t_x = s_j t_x$. So $s_i = s_j$, because the elements of T and $S \setminus T$ commute.

If $A = \{[(s_i t_x, t_x)] s_i \in S \textit{ and } t_x \in T\}$ then $A \subset M$ and monomorphism $\Phi : S \longrightarrow A$ is onto. Thus for every $[(s_i t_x, t_x)]$ in A, there exists s_i such that $(s_i)\Phi = [(s_i t_x, t_x)]$. Thus S can be embedded into M. □

Theorem 3. *The elements $[(t_i, t_j)]$ of M are cancellative in M and form a commutative group G such that the identity of G is the identity of M.*

Proof. Suppose $[(t_i, t_j)]\,[(s_k, t_l)] = [(t_i, t_j)]\,[(s_m, t_l)]$. Then $[(t_i s_k, t_j t_l)] = [(t_i s_m, t_j t_n)]$ or $(t_i s_k, t_j t_l)\,\rho\,(t_i s_m, t_j t_n)$, which further implies $(t_j t_n)\,(t_i s_k) = (t_j t_l)\,(t_i s_m)$. So, by (4), we get $(t_i s_k)\,(t_j t_n) = (t_i s_m)\,(t_j t_l)$. From $(t_i s_k)(t_j t_n) = (t_i s_m)(t_j t_l)$, the application of medial law yields $(t_i t_j)(s_k t_n) = (t_i t_j)(s_m t_l)$. Hence by the left cancellation it follows that $s_k t_n = s_m t_l$. Now, $t_n s_k = t_l s_m$, because these elements commute.

Hence $(s_k, t_l)\,\rho\,(s_m, t_n)$, that is, $[(s_k, t_l)] = [(s_m, t_n)]$, imply that for all i and j, $[(t_i, t_j)]$ are left cancellative in M. Similarly, it can be shown that elements of the form $[(t_i, t_j)]$ are right cancellative. These elements of M are the only cancellative elements in M. For, it is enough to show that the elements of the form $[(s_i, t_j)]$ are non-cancellative in M. Therefore there exist elements s_k and s_l with $s_k \neq s_l$ and $s_k s_i = s_l s_i$, or there exist elements s_m and s_n with $s_m \neq s_n$ and $s_i s_m = s_i s_n$.

In the first case $(s_k s_i)\,(t_x t_y) = (s_l s_i)\,(t_x t_y)$ or $(s_k s_i, t_x t_y)\,\rho\,(s_l s_i, t_x t_y)$ or $[(s_k, t_x)]\,[(s_i, t_y)] = [(s_l, t_x)]\,[(s_i, t_y)]$, with $[(s_k, t_x)]$ not ρ equivalent to $[(s_l, t_x)]$. In the second case the result follows similarly. Hence all the cancellative elements of M are of the form $[(t_i, t_j)]$ and these are the only cancellative elements in M, which form a group G in M.

We have proved in Theorem 1, that $[(t_o, t_o)]$ is the identity element of M. Since G contains elements of the form $[(t_x, t_y)]$, therefore $[(t_o, t_o)]$ is in G which is unique because G is a group. □

We can sum up Theorem 1 and Theorem 2 as follows.

If S is an AG^{**}-groupoid and T is left cancellative AG^{**}-subgroupoid of S then S can be embedded in a commutative monoid provided elements of T commute with elements of $S \setminus T$.

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Received November 11, 2003
Revised September 4, 2004