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Permutation representations of triangle group $\Delta(2,4,5)$

Qaiser Mushtaq and Muhammad S. Saeed

Abstract

Let $G(2,4,Z)$ be a linear-fractional group generated by the transformations $x : z \longmapsto \frac{-1}{2z}$ and $y : z \longmapsto \frac{-1}{2(z+1)}$, satisfying the relations $x^2 = y^4 = 1$. In this paper, corresponding to each θ in F_p we shall determine the coset diagrams $D(\theta,p)$ depicting the actions of $G(2,4,Z)$ on $PL(F_p)$ and find also the values of p for which there exist vertices on the vertical line of symmetry in $D(\theta,p)$. Also, we find conditions for the existence of certain useful fragments of coset diagrams in $D(\theta,p)$.

1. Introduction

The group $G(2,4,Z)$ is defined as a linear-fractional group generated by the transformations $x : z \longmapsto \frac{-1}{2z}$ and $y : z \longmapsto \frac{-1}{2(z+1)}$, satisfying the relations $x^2 = y^4 = 1$. The group $G(2,4,Z)$ can be extended by adjoining an involution $t : z \longmapsto \frac{1}{2z}$ such that $(xt)^2 = (yt)^2 = 1$. We denote the extended group by $G^*(2,4,Z)$.

Let $PL(F_p)$ denote the projective line over the Galois field F_p , where p is a prime. The points of $PL(F_p)$ are the elements of F_p together with the additional point ∞.

The group $G^*(2,4,p)$ has its customary meanings, as the group of all transformations $z \rightarrow \frac{az+b}{cz+d}$ where a,b,c,d are in F_p and $ad-bc \neq 0$.

The homomorphism $\alpha : G^*(2,4,Z) \longrightarrow G^*(2,4,p)$ give rise to an action of $G^*(2,4,Z)$ on $PL(F_p)$. We denote the generators $x\alpha$ and $y\alpha$ of $G^*(2,4,p)$ by $\overline{x}$ and $\overline{y}$ respectively. A homomorphism $\alpha : G^*(2,4,Z) \longrightarrow G^*(2,4,p)$ is called a *non-degenerate homomorphism* if neither x nor y lies in the kernel

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of α, so that $\overline{x} = x\alpha$ and $\overline{y} = y\alpha$ are of orders 2 and 4 respectively. As always, two non-degenerate homomorphisms α and β are called *conjugate* if there exists an inner automorphism ρ of $G^*(2,4,p)$ such that $\beta = \alpha\rho$. These conjugacy classes will contain homomorphisms from $G^*(2,4,Z)$ to $G^*(2,4,p)$.

The *triangle groups* $\Delta(l,m,k) =< x,y : x^l = y^m = (xy)^k = 1 >$, where $l,m,k > 1$, are described explicitly in [1, 2, 3, 4]. The triangle groups $\Delta(2,4,k) =< x,y : x^2 = y^4 = (xy)^k = 1 >$ can be obtained as subgroups of S_{q+1} through actions of the group $G(2,4,Z)$ on $PL(F_q)$ where q is a power of a prime p. According to [2], the triangle groups $\Delta(2,4,k)$ are known as infinite groups if and only if $k \geqslant 4$. The group $\Delta(2,4,k)$ is C_2, D_8, and S_4, for $k = 1, 2, 3$, respectively. When $k = 4$, the triangle group $\Delta(2,4,4)$ is Abelian-by-cyclic [6].

2. Coset diagrams

The coset diagrams depict an action of

$$G^*(2,4,Z) =< x,y,t : x^2 = y^4 = t^2 = (xt)^2 = (yt)^2 = 1 >$$

on a finite set (or space).

These coset diagrams may be used to provide diagrammatic interpretations of several aspects of combinatorial group theory, such as the proof of the Ree-Singerman theorem (on the cycle structures of generating-permutations for a transitive group). They can be used also as an equivalent to the Abelianized form of the Reidemeister-Schreier process. The same sort of method is also useful for the construction of infinite families of finite quotients of a given finitely-presented group. Use of coset diagrams to find torsion-free subgroups of certain finitely-presented groups has been instrumental in the construction of small volume hyperbolic 3-orbifolds and other hyperbolic 3-manifolds with interesting properties. They are also applied to the construction of arc-transitive graphs and maximal automorphism groups of Riemann surfaces. Coset diagrams can often be used to prove certain groups are infinite, by joining diagrams together to construct permutation representations (of a given group) of arbitrarily large degree.

The coset diagrams for the action of $G^*(2,4,Z)$ on a finite set (or space) are defined as follows.

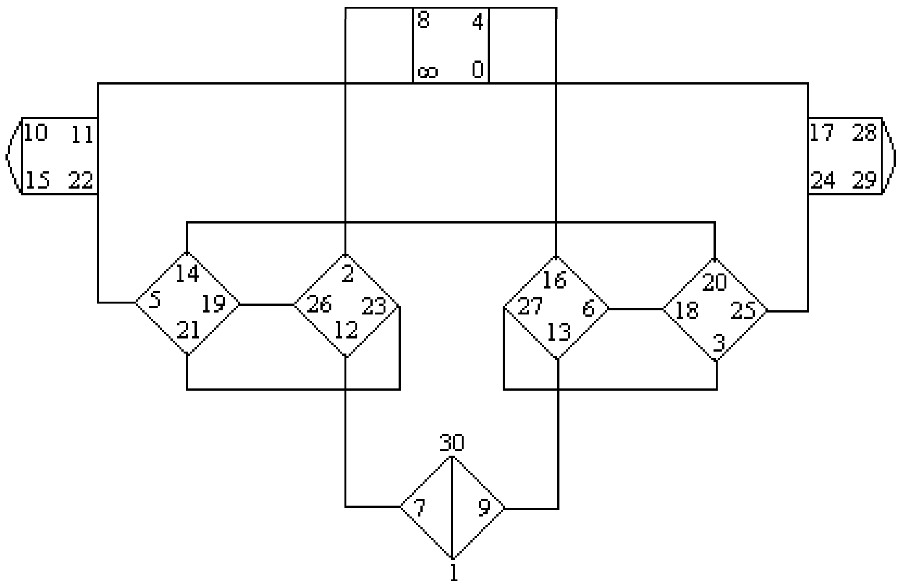
The four cycles of y are represented by small squares whose vertices are permuted counter-clockwise by y. Any two vertices which are interchanged by the involution x, is represented by an edge. The action of t is represented by reflection about a vertical axis of symmetry. The fixed points of x and y, if they exist, are denoted by heavy dots.

For instance, the action of $G^*(2,4,Z)$ on $PL\,(F_{31})$ yields the following permutation representations

$$\overline{x}\ :\ (\infty,11)\,(0,17)\,(1,30)\,(2,8)\,(3,27)\,(4,16)\,(5,22)\,(6,18)\,(7,12)\,(9,13)\,(10,15)(14,20)(19,26)(21,23)(24,25)(28,29)$$

$$\overline{y}\ :\ (0,4,8,\infty)\,(1,9,30,7)\,(2,26,12,23)\,(3,25,20,18)\,(5,21,19,14)\,(6,16,27,13)(10,15,22,11)(17,24,29,28)$$

and the coset diagram depicting this action is:



We shall determine coset diagrams, denoted by $D(\theta,p)$, depicting the actions of $G(2,4,Z)$ on $PL(F_p)$ and find also the values of p for which there exist vertices on the vertical line of symmetry in $D(\theta,p)$. Also, we find conditions for the existence of certain fragments of coset diagrams in $D(\theta,p)$.

The conjugacy classes of non-degenerate homomorphisms α of $G^*(2,4,Z)$ into $G^*(2,4,p)$ correspond in a one-to-one fashion with the conjugacy classes of non-trivial elements of $G^*(2,4,p)$, under a correspondence which assigns to the non-degenerate homomorphism α the class containing the element $(xy)\alpha$. This, of course, means that we can actually parametrize the conjugacy classes of non-degenerate homomorphisms except for a few uninteresting ones, by the elements of F_p. That is, we can in fact parametrize the actions of $G^*(2,4,Z)$ on $PL(F_p)$.

Let X, Y and T denote matrices corresponding to the elements $\overline{x}, \overline{y}$ and $\overline{t}$ in $G^*(2,4,p)$, where as described earlier, $\overline{x} = x\alpha$, $\overline{y} = y\alpha$ and $\overline{t} = t\alpha$, for some non-degenerate homomorphism α from the group $G^*(2,4,Z)$ into $G^*(2,4,p)$. Then X, Y and T will satisfy the relations

$$X^2 = Y^4 = T^2 = (XT)^2 = (YT)^2 = \lambda I$$

for some scalar λ. Since X, Y and T are of orders 2,4, and 2 respectively therefore we can choose

$$X = \begin{bmatrix} a & kc \\ c & -a \end{bmatrix}, \quad Y = \begin{bmatrix} d & kf \\ f & m-d \end{bmatrix} \text{ and } T = \begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix},$$

where $m = trace(Y)$ and $a, c, d, f, k \in F_p$ with $k \neq 0$. Also $m \equiv \theta (mod\, p)$ for some θ in F_p.

To find m, the trace of Y, we adopt the following method. Since $y^4 = 1$, we have $Y^4 = \lambda I$. As in Theorem 3.3.1 in [5], some scalar multiple of Y is conjugate to the matrix $\begin{bmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{bmatrix}$, where ρ is $8th$ root of unity, so that $\rho^8 = 1$ or $(\rho^4 - 1)(\rho^4 + 1) = 0$. But $\rho^4 \neq 1$, therefore $(\rho^4 + 1) = 0$. This implies that $(\rho^2 + \sqrt{2}\rho + 1)(\rho^2 - \sqrt{2}\rho + 1) = 0$. That is,

$$\begin{aligned} (\rho^2 + \sqrt{2}\rho + 1) &= 0 \quad (2.1) \\ \text{or} \quad (\rho^2 - \sqrt{2}\rho + 1) &= 0 \end{aligned}$$

But $m = \rho + \rho^{-1}$ implies that $m\rho = \rho^2 + 1$, that is, $\rho^2 - m\rho + 1 = 0$. Thus comparing this equation with the characteristic equation of Y, we obtain $m = \pm\sqrt{2}$. Let $m = \sqrt{2}$, so that $trace(Y) = \sqrt{2}$ where Y satisfy the relation $Y^4 = \lambda I$ for some scalar λ.

So $X = \begin{bmatrix} a & kc \\ c & -a \end{bmatrix}$, and $Y = \begin{bmatrix} e & kf \\ f & \sqrt{2} - e \end{bmatrix}$, and the characteristic equations of X, Y and XY are:

$$X^2 + \Delta I = 0, \quad (2.2)$$

$$Y^2 - \sqrt{2}Y + I = 0, \quad (2.3)$$

and

$$(XY)^2 - rXY + \Delta I = 0. \quad (2.4)$$

In the following we see that any element g (not of order 1, 2 or 5) of $G^*(2,4,p)$ is the image of xy under some non-degenerate homomorphism of $G^*(2,4,Z)$ into $G^*(2,4,p)$.

By Lemma 3.2 [5], it is sufficient to show that every element of $G^*(2,4,p)$ is a product of an element of order 2 and an element of order 4. So we shall look for elements $\overline{x}, \overline{y}, \overline{t}$ of $G^*(2,4,p)$ satisfying the relations

$$\overline{x}^2 = \overline{y}^4 = \overline{t}^2 = (\overline{x}\overline{t})^2 = (\overline{y}\overline{t})^2 = 1 \tag{2.5}$$

with $\overline{x}\,\overline{y}$ in a given conjugacy class.

We shall take $\overline{x}$, $\overline{y}$ and $\overline{t}$ to be represented by

$$X = \begin{bmatrix} a & kc \\ c & -a \end{bmatrix}, \quad Y = \begin{bmatrix} d & kf \\ f & m-d \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix},$$

where $a, c, d, f, k \in F_p$.

Since X is non-singular, we shall write

$$a^2 + kc^2 = -\Delta \tag{2.6}$$

and require that $\det Y = 1$ so that

$$d^2 - \sqrt{2}d + kf^2 + 1 = 0 \tag{2.7}$$

This certainly yields the elements satisfying the relations (2.5). So we only have to check on the conjugacy class of $\overline{x}\,\overline{y}$.

Now the matrix XY is

$$\begin{bmatrix} ad + kfc & akf + \sqrt{2}kc - kcd \\ cd - af & kfc - \sqrt{2}a + ad \end{bmatrix}$$

and therefore the matrix representing $\overline{x}\,\overline{y}$ has the trace

$$r = a(2d - \sqrt{2}) + 2kfc \tag{2.8}$$

and the determinant $\Delta = -(a^2 + kc^2)$, because $\det Y = 1$.

The matrix XYT is given by

$$\begin{bmatrix} akf + \sqrt{2}kc - kcd & -akd - k^2fc \\ kfc - \sqrt{2}a + ad & akf - kcd \end{bmatrix}$$

and if $sk = trace(XYT)$ then

$$s = 2af + c(\sqrt{2} - 2d), \tag{2.9}$$

and so

$$r^2 + ks^2 = 2\Delta\,. \tag{2.10}$$

Thus, corresponding to each θ in F_p, by using equations (2.6) to (2.10), we can find a triplet $(\overline{x}, \overline{y}, \overline{t})$ such that $\overline{x}^2 = \overline{y}^4 = \overline{t}^2 = (\overline{x}\overline{t})^2 = (\overline{y}\overline{t})^2 = 1$. Therefore, we can draw the coset diagram depicting an action of $G^*(2, 4, Z)$ on $PL(F_p)$.

Example 1. If $p = 89$ and $\theta = 11$, then by using equations (2.6) to (2.10), we obtain $\Delta = 1$, $k = -1$, $r = 10$, $s = 3$, $f = 20$, $d = 2$, $a = -13$, $c = -9$ and so

$$x(z) = \frac{-13z + 9}{-9z + 13}\,, \qquad y(z) = \frac{2z - 20}{20z + 23}\,, \qquad t(z) = \frac{1}{z}\,.$$

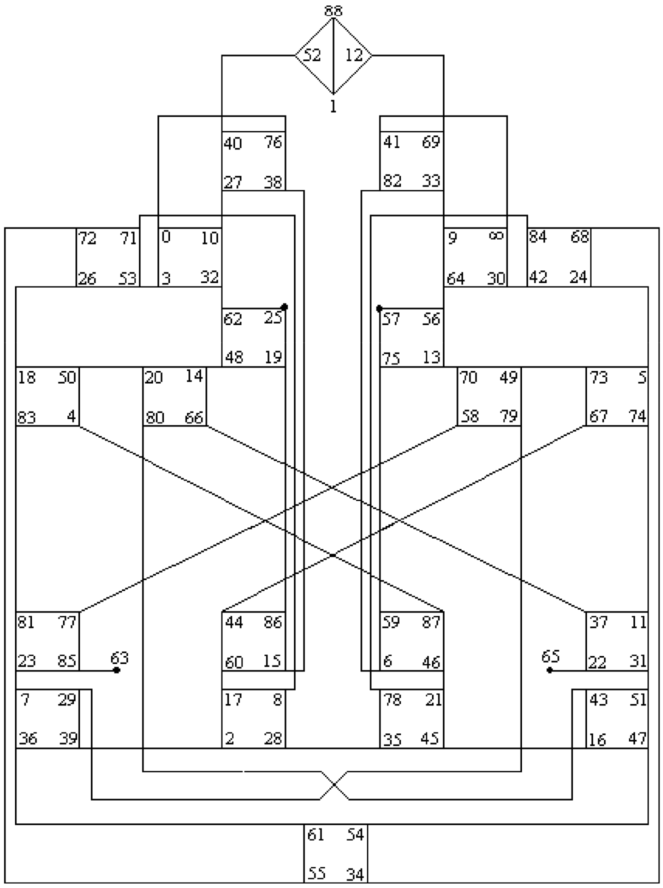
Thus our $\overline{x}, \overline{y}, \overline{t}$ act as

$\overline{x}$: $(\infty, 41)(0, 76)(1, 88)(2, 39)(3, 53)(4, 87)(5, 24)(6, 82)(7, 23)(8, 71)(9, 33)$
$(10, 27)(11, 74)(12, 69)(13, 70)(14, 48)(15, 38)(16, 45)(17, 60)(18, 26)$
$(19, 86)(20, 50)(21, 46)(22, 65)(25)(28, 35)(29, 79)(30, 42)(31, 51)(32, 62)$
$(34, 68)(36, 61)(37, 66)(40, 52)(43, 80)(44, 67)(47, 54)(49, 73)(55, 72)$
$(56, 64)(57)(58, 77)(63, 85)(59, 75)(78, 84)(81, 83)$

$\overline{y}$: $(0, 3, 32, 10)(1, 12, 88, 52)(2, 28, 8, 17)(4, 50, 18, 83)(5, 73, 67, 74)$
$(6, 46, 87, 59)(7, 36, 39, 29)(9, 64, 30, \infty)(11, 37, 22, 31)(13, 56, 57, 75)$
$(14, 20, 80, 66)(15, 86, 44, 60)(16, 47, 51, 43)(19, 25, 62, 48)(21, 78, 35, 45)$
$(23, 85, 77, 81)(24, 68, 84, 42)(26, 53, 71, 72)(27, 38, 76, 40)(33, 69, 41, 82)$
$(34, 54, 61, 55)(49, 70, 58, 79)(63)(65)$

$\overline{t}$: $(0, \infty)(1)(2, 45)(3, 30)(4, 67)(5, 18)(6, 15)(7, 51)(8, 78)(9, 10)(11, 81)$
$(12, 52)(13, 48)(14, 70)(16, 39)(17, 21)(19, 75)(20, 49)(22, 85)(23, 31)$
$(24, 26)(25, 75)(27, 33)(28, 35)(29, 43)(32, 64)(34, 55)(36, 47)(37, 77)$
$(38, 82)(40, 69)(41, 76)(42, 53)(44, 87)(46, 60)(50, 73)(54, 61)(56, 62)$
$(58, 66)(59, 86)(63, 65)(68, 72)(71, 84)(74, 83)(79, 80)(88)$

and yield the coset diagram $D(11, 89)$



D(11,89)

The coset diagrams for the actions of $G^*(2,4,Z)$ on $PL\,(F_p)$ contain fixed points of t, which lie on the vertical line of symmetry. Here we have determined the condition under which these fixed vertices exist in $D(\theta,p)$.

Theorem 1. *The transformation* $\bar{t}$ *has fixed vertices in* $D\,(\theta,p)$ *if and only if* $\theta\,(\theta-2)$ *is a square in* F_p.

Proof. First we show that the fixed points of $\overline{x}$ exist in $D\,(\theta,p)$ if $p \equiv 1\,(mod\,4)$ and there do not exist fixed points of $\overline{x}$ if $p \equiv 3\,(mod\,4)$.

Since $\overline{y}$ and $\overline{x}\ \overline{y}$ have even orders, they lie in $G^*(2,4,p)$ and hence so does $\overline{x}$. This implies that the permutation $\overline{x}$ is even. Since $r^2 = \Delta\theta$, Δ is a square if and only if θ is. This means that $\overline{x}$ is in $G^*(2,4,p)$ if and only if -2 is not a square in F_p and $p \equiv 1\,(mod\,4)$. Thus $\overline{x}$ has fixed vertices in $D\,(\theta,p)$ if and only if -1 and θ are either both squares or both non-squares in F_p. That is, $\overline{x}$ has fixed vertices in $D\,(\theta,p)$ if $p \equiv 1\,(mod\,4)$

and it does not have fixed vertices if $p \equiv 3\,(mod\,4)$. This means that for the non-degenerate homomorphism with parameters θ, $\overline{x}$ is an element of $G^*(2,4,p)$ if and only if $-\theta$ is a square in F_p.

Let δ be the automorphism of $G^*(2,4,p)$ defined by $x\delta = \overline{x}\overline{t}$, $y\delta = \overline{y}$ and $t\delta = \overline{t}$. Then if $\alpha : G^*(2,4,Z) \longrightarrow G^*(2,4,p)$ maps x, y, t to $\overline{x}, \overline{y}, \overline{t}$, the homomorphism $\alpha' = \delta\alpha$ maps x, y, t to $\overline{x}, \overline{y}, \overline{t}$. If we let X, Y and T denote elements of $GL(2,p)$ which yield the elements $\overline{x}$, $\overline{y}$ and $\overline{t}$ in $G^*(2,4,p)$, then obviously X, Y and T can be taken as follows

$$X = \begin{bmatrix} a & kc \\ c & -a \end{bmatrix}, \quad Y = \begin{bmatrix} d & kf \\ f & \sqrt{2}-d \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix}$$

where $k \neq 0$ and $a, c, d, k, f \in F_p$ such that they satisfy the equations (2.6) to (2.10). We recall that, $\overline{x}\,\overline{y}$ will be of order 2 if and only if $tr\,(XY) = r = 0$ and similarly $\overline{x}\;\overline{y}\overline{t}$ will be of order 2 if and only if $tr\,(XYT) = ks = 0$. Since the determinant of XY is Δ, therefore the parameter of $\overline{x}\,\overline{y}$ is r^2/Δ, which we have denoted by θ. Also ks is the trace of XYT and $k\Delta$ is its determinant. If we let $\varphi = \frac{ks^2}{\Delta}$ we get $\theta + \varphi = r^2 + ks^2/\Delta$. Substituting the values of r and s from the equations (2.8) and (2.9), in $\theta + \varphi = r^2 + ks^2/\,\Delta$ and then making the substitution of the equation (2.7) and $\Delta = -(a^2 + kc^2)$ we obtain $\theta + \varphi = 2$. That is if θ is the parameter of α then $2 - \theta$ is the parameter of α'.

Since change from α to α' interchanges both $\overline{x}$ and $\overline{x}\overline{t}$ and θ and $2-\theta$, it follows that $\overline{x}\overline{t}$ maps to an element of $G^*(2,4,p)$ if and only if $\theta\,(2-\theta)$ is a square in F_p. Since $\overline{t}$ is in $G^*(2,4,p)$ if both of $\overline{x}$ and $\overline{x}\overline{t}$ is, but not if just one of them is, $\overline{t}$ is in $G^*(2,4,p)$ if and only if $\theta\,(2-\theta)$ is a square in F_p. Now $\overline{t}$ has fixed points in $PL\,(F_p)$ if either $\overline{t}$ belongs to $G^*(2,4,p)$ and $p \equiv -1\,(mod\,4)$ or $\overline{t}$ dose not belong to $G^*(2,4,p)$ and $p = 1\,(mod\,4)$ is equivalent to saying that -1 is a square in F_p, we conclude that $\overline{t}$ has fixed vertices in $D\,(\theta,p)$ if and only if $-\theta\,(2-\theta) = \theta\,(\theta-2)$ is a square in F_p. Hence the result. □

We can see in Example 1 that the coset diagram depicting actions of $G(2,4,Z)$ on $PL\,(F_{89})$ contain fixed points of $\overline{t}$ on the line of symmetry.

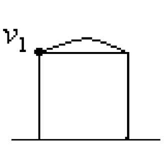
The fact that $\overline{t}$ has fixed vertices on the line of symmetry in $D(\theta,p)$ or not helps us to determine the structure of the group $< \overline{x}, \overline{y}, \overline{t} >$. It also enables us to show that for infinitely many values of p, the group $G^*(2,4,p)$ has minimal genus.

Corollary 1. *If $p \equiv \pm 1\,(mod\,5)$ then the transformation $\overline{t}$ has fixed vertices in $D\,(\theta,p)$ if and only if $\theta - 2$ is a square in F_p and $(\overline{x}\;\overline{y})^5 = 1$.*

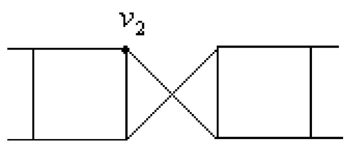
3. Fragments of coset diagrams

By joining graphs representing groups of smaller degree we can obtain a bigger graph representing a group of larger degree. Then it is easy to study the properties of the new group just by studying its graph. We have different methods of joining graphs together, to give representations of the group of larger degree. We need not have to study the entire group of a smaller degree, we can achieve this just by studying its fragment and find a condition for the existence of the fragment in the coset diagram, so that if the fragment exists in a coset diagram of larger degree, we can study the properties of the diagram for the related group of larger degree.

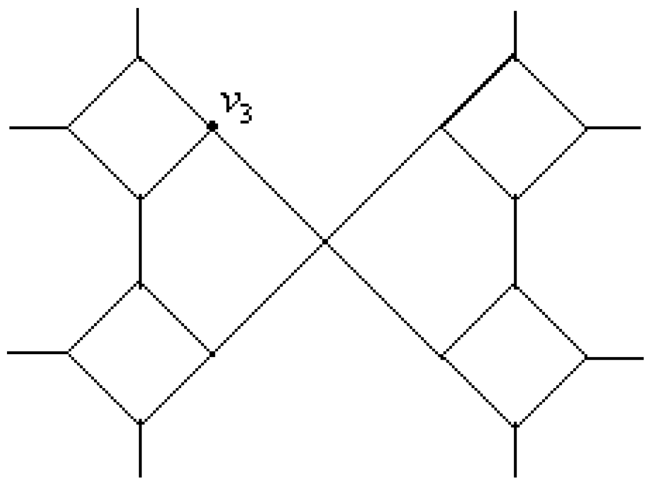
The coset diagrams, depicting actions of $G^*(2,4,p)$ on $PL(F_p)$, frequently contain some special fragments, namely γ_1, γ_2 and γ_3 respectively



γ_1



γ_2



γ_3

We determine conditions on θ and p for the existence of these fragments in the coset diagrams $D(\theta,p)$.

Theorem 2.

(*i*) *The fragment* γ_1 *will occur in* $D(\theta,p)$ *if* 5 *is a square in* F_p.

(*ii*) *The fragment* γ_2 *will occur in* $D(\theta,p)$ *if* -11 *is a square in* F_p.

(*iii*) *The fragment* γ_3 *will occur in* $D(\theta,p)$ *if* -19 *is a square in* F_p.

Proof. The vertices v_1, v_2 and v_3 are fixed by the elements $\overline{x}\,\overline{y}$, $\overline{x}\,\overline{y}^3$ and $\overline{x}\,\overline{y}^3\overline{x}\overline{y}^3\overline{x}\overline{y}\overline{x}\overline{y}$ respectively. Recall that $\det X = \Delta$, $trace(Y) = \sqrt{2}$, $\det(XY) = \Delta$, and $trace(XY) = r$. After suitable manipulations, the equations

$$Y^3 = Y - \sqrt{2}I \qquad (3.1)$$

$$Y^4 = -I \qquad (3.2)$$

$$XYX = rX + \Delta Y - \sqrt{2}\Delta I \qquad (3.3)$$

can be obtained from the equations (2.2), (2.3) and (2.4).

In fragment γ_1 the vertex v_1 is fixed by $\overline{x}\,\overline{y}$. The matrix corresponding to $\overline{x}\,\overline{y}$ will be $M_1 = XY$. The determinant of M_1 will be $\det(XY) = \det(X)\det(Y) = \Delta$, and the trace of M_1 will be equal to $trace(XY) = r$. So the discriminant of the characteristic equation of M_1 will be $r^2 - 4\Delta$. But $r^2 = \theta\Delta$. This means that the discriminant is, in fact, $r^2 - 4\Delta = \theta\Delta - 4\Delta = (\theta - 4)\Delta$. Since Δ is a square if and only if θ is, we can eliminate Δ, as we are in field F_p. So the discriminant of the characteristic equation of the matrix corresponding to the element $\overline{x}\,\overline{y}$ of $G^*(2,4,p)$ will be $d_1(\theta) = \theta - 4$.

In fragment γ_2 the vertex v_2 is fixed by $\overline{x}\,\overline{y}^3\overline{x}\,\overline{y}$. The matrix corresponding to $\overline{x}\,\overline{y}^3\overline{x}\,\overline{y}$ will be $M_2 = XY^3XY$. Now $\det M_2 = \Delta^2$. If we substitute the value of Y^3 from equation (3.1) in equation $M_2 = XY^3XY$, we get $M_2 = X(Y - \sqrt{2}I)XY = (XY)^2 - \sqrt{2}X^2Y$. If we now substitute values of $(XY)^2$ and X^2 (from equations (2.4) and (2.2)) in equation $M_2 = (XY)^2 - \sqrt{2}X^2Y$ the result will be an equation $M_2 = rXY - \Delta I + \sqrt{2}\Delta M$. So the trace of M_2 will be $trace(rXY) - trace(\Delta I) + \sqrt{2}\,trace(\Delta Y)$. That is, $trace(M_2) = r^2 - 2\Delta + 2\Delta = r^2$. This implies that the discriminant of the characteristic equation of M_2 will be $r^4 - 4\Delta^2$. But $r^2 = \theta\Delta$. This means that the discriminant is, in fact, $\theta^2\Delta^2 - 4\Delta^2 = (\theta^2 - 4)\Delta^2$. Since Δ is a square if and only if θ is, we can eliminate Δ so the discriminant of the characteristic equation of the matrix corresponding to the element $\overline{x}\,\overline{y}^3\overline{x}\,\overline{y}$ of $G^*(2,4,p)$ will be $d_2(\theta) = \theta^2 - 4 = (\theta - 2)(\theta + 2)$.

In fragment γ_3 the vertex v_3 is fixed by $\overline{x}\,\overline{y}^3\overline{x}\,\overline{y}^3\overline{x}\,\overline{y}\,\overline{x}\,\overline{y}$. The matrix corresponding to $\overline{x}\,\overline{y}^3\overline{x}\,\overline{y}^3\overline{x}\,\overline{y}\,\overline{x}\,\overline{y}$ will be $M_3 = XY^3XY^3XYXY$. So $\det M_3 = \Delta^4$. If we substitute the value of Y^3 from equation (3.1) in equation $M_3 = XY^3XY^3XYXY$, we get

$$\begin{aligned} M_3 &= X(Y-\sqrt{2}I)X(Y-\sqrt{2}I)(XY)^2 \\ &= (XY-\sqrt{2}X)(XY-\sqrt{2}X)(XY)^2 \\ &= [(XY)^2+2X^2-\sqrt{2}XYX-\sqrt{2}X^2Y](XY)^2. \end{aligned} \tag{3.4}$$

If we now substitute values of $(XY)^2$, X^2 and XYX (from equations (2.4), (2.2) and (3.3)) in equation (3.4) the result will be an equation

$$M_3 = r^3XY - r^2\Delta I - 2r\Delta XY + \Delta^2 I + \sqrt{2}r^2\Delta Y + \sqrt{2}rX. \tag{3.5}$$

So the trace of M_3 will be $r^4 - 2r^2\Delta - 2r^2\Delta + 2\Delta^2 + 2r^2\Delta$. That is, $trace(M_3) = r^4 - 2r^2\Delta + 2\Delta^2$. This implies that the discriminant of the characteristic equation of M_3 will be

$$(r^4 - 2r^2\Delta + 2\Delta^2)^2 - 4\Delta^4 = r^8 + 8r^4\Delta^2 - 4r^6\Delta - 8r^2\Delta^3.$$

This means that the discriminant is, in fact,

$$\theta^4\Delta^4 + 8\theta^2\Delta^4 - 4\theta^3\Delta^4 - 8\theta\Delta^4 = (\theta^4 + 8\theta^2 - 4\theta^3 - 8\theta)\Delta^4.$$

Since Δ is a square if and only if θ is, we can eliminate Δ, so the discriminant of the characteristic equation of the matrix corresponding to $\overline{x}\ \overline{y}^3\overline{x}\ \overline{y}^3\overline{x}\ \overline{y}\ \overline{x}\ \overline{y}$ of $G^*(2,4,p)$ will be

$$d_3(\theta) = \theta^4 - 4\theta^3 + 8\theta^2 - 8\theta = \theta(\theta-2)[\theta-(1+\sqrt{-3})][(\theta-(1-\sqrt{-3})]$$

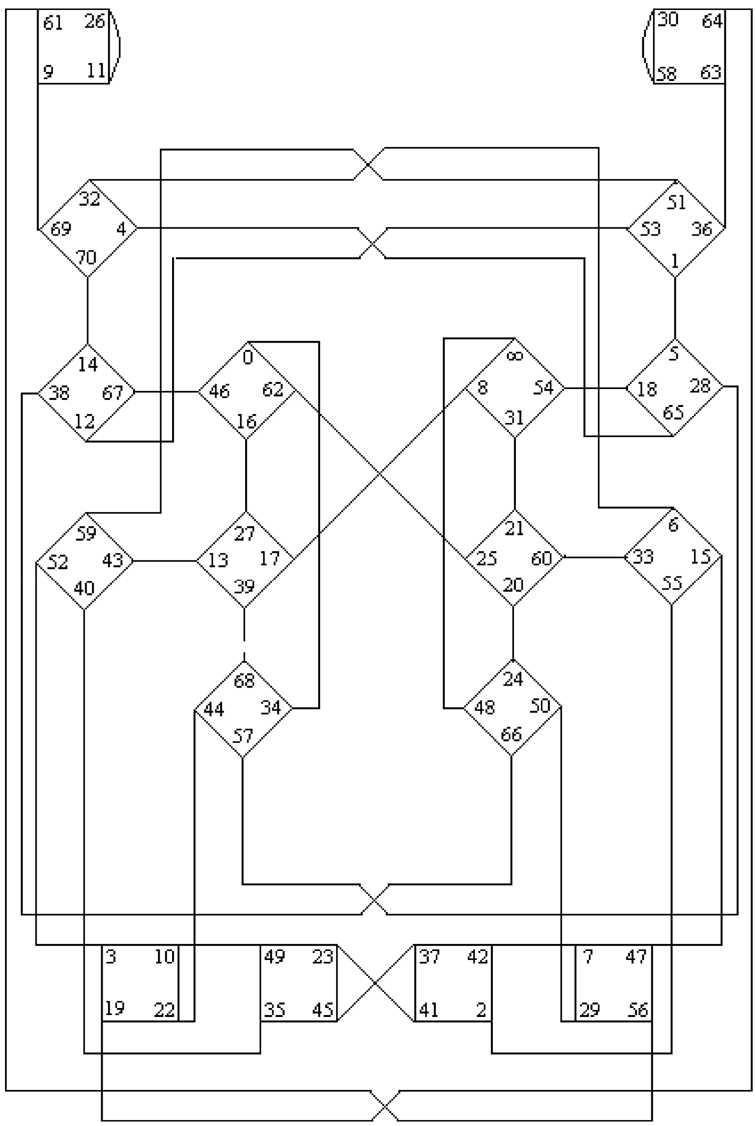
Thus,

(i) the fragment γ_1 will occur in $D(\theta,p)$ if and only if $d_1(\theta) = \theta - 4$ is a square in F_p. If θ_1 and θ_2 are the roots of $f(z) = z^2 - 3z + 1$ then $\prod_{i=1}^{2} d_1(\theta_i) = f(4) = 5$. Thus γ_1 will exist in some $D(\theta_i,p)$ if 5 is a square in F_p.

(ii) the fragment γ_2 will occur in $D(\theta,p)$ if and only if $d_2(\theta) = (\theta-2)(\theta+2)$ is a square in F_p. If θ_1 and θ_2 are the roots of $f(z) = z^2-3z+1$ then $\prod_{i=1}^{2} d_2(\theta_i) = f(2)f(-2) = -11$. Thus γ_2 will exist in some $D(\theta_i,p)$ if -11 is a square in F_p.

(iii) the fragment γ_3 will occur in $D(\theta,p)$ if and only if $d_3(\theta) = \theta(\theta-2)(\theta-(1+\sqrt{-3}))((\theta-(1-\sqrt{-3}))$ is a square in F_p. If θ_1 and θ_2 are the roots of $f(z) = z^2 - 3z + 1$ then $\prod_{i=1}^{2} d_3(\theta_i) = f(0)f(2)f(1+\sqrt{-3})$ $f(1-\sqrt{-3}) = -19$. Thus γ_3 will occur in some $D(\theta_i,p)$ if -19 is a square in F_p. Hence the result. □

Example 2. In the coset diagram given below, we can see that all the three fragments are present.



D(64,71)

In the following we give a hand-calculated list summarizing the situation for all primes $p \leqslant 241$. We let p denote the primes congruent to ± 1 or $\pm 9 (mod\, 40)$ and θ is a root of the polynomial $\theta^2 - 3\theta + 1$.

p	θ	$\overline{x}$	$\overline{y}$	$\bar{t}$
31	14	$z \longrightarrow \frac{z+2}{z-1}$	$z \longrightarrow \frac{-z+6}{3z+5}$	$z \longrightarrow \frac{-2}{z}$
	20	$z \longrightarrow \frac{2z-3}{3z-2}$	$z \longrightarrow \frac{1}{-z+8}$	$z \longrightarrow \frac{1}{z}$
41	8	$z \longrightarrow \frac{3z-7}{16z-3}$	$z \longrightarrow \frac{-z-18}{6z+25}$	$z \longrightarrow \frac{3}{z}$
	36	$z \longrightarrow \frac{4z+21}{7z-4}$	$z \longrightarrow \frac{z-5}{12z+13}$	$z \longrightarrow \frac{-3}{z}$
71	10	$z \longrightarrow \frac{7z-10}{5z-7}$	$z \longrightarrow \frac{-2}{z+2}$	$z \longrightarrow \frac{2}{z}$
	64	$z \longrightarrow \frac{13z-16}{-16z-13}$	$z \longrightarrow \frac{z+9}{9z+11}$	$z \longrightarrow \frac{-1}{z}$
79	51	$z \longrightarrow \frac{-1}{z}$	$z \longrightarrow \frac{z-25}{25z+8}$	$z \longrightarrow \frac{1}{z}$
	31	$z \longrightarrow \frac{23z-21}{29z-23}$	$z \longrightarrow \frac{2z+42}{21z+7}$	$z \longrightarrow \frac{-2}{z}$
89	11	$z \longrightarrow \frac{-13z+9}{-9z+13}$	$z \longrightarrow \frac{2z-20}{20z+23}$	$z \longrightarrow \frac{1}{z}$
	81	$z \longrightarrow \frac{20z-34}{-17z-20}$	$z \longrightarrow \frac{8}{4z+5}$	$z \longrightarrow \frac{-2}{z}$
151	29	$z \longrightarrow \frac{-29z+67}{28z+29}$	$z \longrightarrow \frac{z+64}{29z+45}$	$z \longrightarrow \frac{3}{z}$
	125	$z \longrightarrow \frac{4z-35}{35z-4}$	$z \longrightarrow \frac{z-4}{4z+22}$	$z \longrightarrow \frac{1}{z}$
191	90	$z \longrightarrow \frac{-60z+4}{65z+60}$	$z \longrightarrow \frac{3z+30}{10z+54}$	$z \longrightarrow \frac{-3}{z}$
	104	$z \longrightarrow \frac{-31z-46}{46z+31}$	$z \longrightarrow \frac{-z-21}{21z+58}$	$z \longrightarrow \frac{1}{z}$
199	63	$z \longrightarrow \frac{41z-45}{-45z-41}$	$z \longrightarrow \frac{z+60}{60z+19}$	$z \longrightarrow \frac{-1}{z}$
	139	$z \longrightarrow \frac{-41z+63}{-63z+41}$	$z \longrightarrow \frac{-2z-29}{29z+22}$	$z \longrightarrow \frac{1}{z}$
239	17	$z \longrightarrow \frac{-21z+43}{-43z+21}$	$z \longrightarrow \frac{z-57}{57z+98}$	$z \longrightarrow \frac{1}{z}$
	225	$z \longrightarrow \frac{-38z+25}{25z+38}$	$z \longrightarrow \frac{z-57}{57z+98}$	$z \longrightarrow \frac{-1}{z}$
241	53	$z \longrightarrow \frac{33z+119}{119z-33}$	$z \longrightarrow \frac{32}{32z+11}$	$z \longrightarrow \frac{-1}{z}$
	191	$z \longrightarrow \frac{97z-101}{101z-97}$	$z \longrightarrow \frac{32}{32z+11}$	$z \longrightarrow \frac{-1}{z}$

References

[1] **G. Baumslag, J. W. Morgan, and P. B. Shalen**: *Generalized triangle groups*, Math. Proc. Camb. Phil. Soc. **102** (1987), 25 – 31.

[2] **H. S. M. Coxeter and W. O. J Moser**: *Generators and relations for discrete groups*, Springer-verlag, Berlin 1980.

[3] **J. Howie, V. Metaftsis and R. M. Thomas**: *Finite generalized triangle groups*, Trans. Amer. Math. Soc. **347** (1995), 3613 – 3623.

[4] **L. Levai, G. Rosenberger and B. Souvignier**: *All finite generalized triangle groups*, Trans. Amer. Math. Soc. **347** (1995), 3625 – 3627.

[5] **Q. Mushtaq**: *Parametrization of all homomorphisms from* $PGL(2, Z)$ *into* $PSL(2, q)$, Comm. Algebra **20** (1992), 1023 – 1040.

[6] **Q. Mushtaq and F. Shaheen**: *Finite presentation of alternating groups*, Acta Math. Sinica, New Ser. **11** (1995), 221 – 224.

Department of Mathematics
Quaid-i-Azam University
Islamabad
Pakistan
e-mail: qmushtaq@apollo.net.pk

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