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Quotient hyper BCK-algebras

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Abstract

In this note first we use the equivalence relation $\sim_I$ which has been introduced in [1] and construct a quotient hyper BCK-algebra H/I from a hyper BCK-algebra H via a reflexive hyper BCK-ideal I of H. Then we study the properties of this algebra, in particular we give some examples of this algebra. Finally we obtain some relationships between H/I and H.

1. Introduction

The hyperalgebraic structure theory was introduced by F. Marty [7] in 1934. Imai and Iséki [4] in 1966 introduced the notion of a BCK-algebra. Recently [6] Jun, Borzooei and Zahedi et.al. applied the hyperstructure to BCK-algebras and introduced the concept of hyper BCK-algebra which is a generalization of BCK-algebra. Now, in this note we use the equivalence relation given in [1] and construct a quotient hyper BCK-algebra H/I via a hyper BCK-ideal I, then we obtain some related results which have been mentioned in the abstract.

2. Preliminaries

Definition 2.1. Let H be a nonempty set and "$\circ$" be a *hyperoperation* on H, that is "$\circ$" is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H)\backslash\{\emptyset\}$. Then H is called a *hyper BCK-algebra* if it contains a constant 0 and satisfies the following axioms:

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(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HK3) $x \circ H \ll \{x\}$,

(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$.

Proposition 2.2. [6] *In any hyper* BCK*-algebra* H*, for all* $x, y, z \in H$*, the following statements hold*:

(i)	$0 \circ 0 = \{0\}$,	(iv)	$0 \circ x = \{0\}$,
(ii)	$0 \ll x$,	(v)	$x \circ y \ll x$,
(iii)	$x \ll x$,	(vi)	$x \circ 0 = \{x\}$.

Definition 2.3. Let I be a nonempty subset of a hyper BCK-algebra $(H, \circ, 0)$ and $0 \in I$. Then, I is called a *hyper* BCK*-ideal* of H if $x \circ y \ll I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$. If additionally $x \circ x \subseteq I$ for all $x \in H$, then I is called a *reflexive hyper* BCK*-ideal*.

Lemma 2.4. [5] *Let* A*,* B *and* I *be subsets of* H*.*

(i) *If* $A \subseteq B \ll C$*, then* $A \ll C$*.*

(ii) *If* $A \circ x \ll I$ *for* $x \in H$*, then* $a \circ x \ll I$ *for all* $a \in A$*.*

(iii) *If* I *is a hyper* BCK*-ideal of* H *and if* $A \circ x \ll I$ *for* $x \in I$*, then* $A \ll I$*.*

(iv) *If* I *be a reflexive hyper* BCK*-ideal of* H *and let* A *be a subset of* H*. If* $A \ll I$*, then* $A \subseteq I$*.*

Definition 2.5. [3] A hyper BCK-algebra H is said to be

– *weak positive implicative* if $(x \circ z) \circ (y \circ z) \ll (x \circ y) \circ z)$,

– *positive implicative* if $(x \circ z) \circ (y \circ z) = (x \circ y) \circ z$,

– *implicative* if $x \ll x \circ (y \circ x)$

holds for all $x, y, z \in H$.

Definition 2.6. [3] A nonempty subset I of a hyper BCK-algebra H containing 0 is called

– a *weak implicative hyper* BCK*-ideal* if for all $x, y, z \in H$

$$(x \circ z) \circ (y \circ x) \subseteq I \text{ and } z \in I \text{ imply } x \in I,$$

– an *implicative hyper* BCK*-ideal* if for all $x, y, z \in H$

$$(x \circ z) \circ (y \circ x) \ll I \text{ and } z \in I \text{ imply } x \in I.$$

Definition 2.7. [6] Let H be a hyper BCK-algebra. Define the set $\nabla(a, b) := \{x \in H \mid 0 \in (x \circ a) \circ b\}$. If for any $a, b \in H$, the set $\nabla(a, b)$ has the greatest element, then we say that H satisfies the hyper condition.

Proposition 2.8. [1] *Let I be a reflexive hyper BCK-ideal of H and let*

$$x \sim_I y \ \text{ if and only if } \ x \circ y \subseteq I \ \text{ and } \ y \circ x \subseteq I.$$

Then $\sim_I$ is an equivalence relation on H.

Proposition 2.9. [1] *Let A, B are subsets of H, and I a reflexive hyper BCK-ideal of H. Then we define $A \sim_I B$ if and only if $\forall a \in A$, $\exists b \in B$ in which $a \sim_I b$, and $\forall b \in B$, $\exists a \in A$ in which $a \sim_I b$. Then relation $\sim_I$ is an equivalence relation on $\mathcal{P}^*(H)$.*

3. Quotient hyper BCK-algebras

From now on H is a hyper BCK-algebra and I is a reflexive hyper BCK-ideal of H, unless otherwise is stated.

Lemma 3.1. *Let $A, B \in \mathcal{P}^*(H)$, and I be a hyper BCK-ideal of H. Then $A \circ B \ll I$ and $B \circ A \ll I$ imply that $A \sim_I B$.*

Proof. For all $a \in A$ and $b \in B$ we have $b \circ a \subseteq B \circ A$ and $a \circ b \subseteq A \circ B$. Since $A \circ B \ll I$, and $B \circ A \ll I$, then we have $b \circ a \ll I$, and $a \circ b \ll I$. Since I is reflexive then $a \sim_I b$, which implies that $A \sim_I B$. □

Theorem 3.2. *The relation $\sim_I$ is a congruence relation on H.*

Proof. By considering Proposition 2.8, it is enough to show that If $x \sim_I y$ and $u \sim_I v$, then $x \circ u \sim_I y \circ v$. Since $x \sim_I y$, we have $x \circ y \ll I$ and $y \circ x \ll I$. So $(x \circ v) \circ (y \circ v) \ll x \circ y$ and $x \circ y \ll I$ imply that $(x \circ v) \circ (y \circ v) \ll I$. Similarly $(y \circ v) \circ (x \circ v) \ll I$. Therefore by Lemma 3.1 $x \circ v \sim_I y \circ v$.

Also we have $(x \circ u) \circ (v \circ u) \ll x \circ v$. Then for all $t \in x \circ u$ and $r \in v \circ u$ we have $t \circ r \subseteq (x \circ u) \circ (v \circ u)$. Therefore for all $s \in t \circ r$ there exists $a \in x \circ v$ such that $s \ll a$, hence $(s \circ a) \cap I \neq \emptyset$. Since $s \circ a \subseteq (t \circ r) \circ a$, then $((t \circ r) \circ a) \cap I \neq \emptyset$. By Lemma 2.4 we have $(t \circ r) \circ a \ll I$. Thus $(t \circ a) \circ r \ll I$ and $r \in I$, which implies that $t \circ a \ll I$. Since $t \in x \circ u$ and $r \in v \circ u$ we can get that $(x \circ u) \circ (x \circ v) \ll I$. Similarly $(x \circ v) \circ (x \circ u) \ll I$. Then by Lemma 3.1 we can see that $x \circ v \sim_I x \circ u$.

Since $\sim_I$ is an equivalence relation on $\mathcal{P}^*(H)$, then $x \circ v \sim_I y \circ v$ and $x \circ v \sim_I x \circ u$ imply that $x \circ u \sim_I y \circ v$. □

Suppose I is a reflexive hyper BCK-ideal of $(H, \circ, 0)$. Denote the equivalence classes of x by C_x.

Lemma 3.3. *In any hyper BCK-algebra H we have $I = C_0$.*

Proof. Let $x \in I$. Since $x \in x \circ 0$, we have $(x \circ 0) \cap I \neq \emptyset$. Then $x \circ 0 \subseteq I$ and since $0 \circ x = 0$ hence $0 \circ x \subseteq I$. Then $0 \sim_I x$ therefore $x \in C_0$. Conversely let $x \in C_0$ hence $x \sim_I 0$ which means that $x \circ 0 \subseteq I$. Since $x \in x \circ 0$ then we have $x \in I$. □

Denote $H/I = \{C_x : x \in H\}$ and define $C_x * C_y = \{C_t \mid t \in x \circ y\}$. If $C_x = C_{x'}$ and $C_y = C_{y'}$, then $C_x * C_y = C_{x'} * C_{y'}$. Indeed, if $C_x = C_{x'}$ and $C_y = C_{y'}$ then $x \sim_I x'$ and $y \sim_I y'$, we can conclude that $x \circ y \sim_I x' \circ y'$ since $\sim_I$ is a congruence relation. Now let $C_t \in C_x * C_y$ then $t \in x \circ y$. Then there exist $r \in x' \circ y'$ such that $t \sim_I r$ hence $C_t = C_r$. Therefore $C_x * C_y \subseteq C_{x'} * C_{y'}$, and similarly $C_{x'} * C_{y'} \subseteq C_x * C_y$. Hence $*$ is well-defined.

On H/I we define $\ll$ putting: $C_x \ll C_y$ if and only if $C_0 \in C_x * C_y$. Observe that: $x \ll y \Longrightarrow 0 \in x \circ y \Longrightarrow C_0 \in C_x * C_y \Longrightarrow C_x \ll C_y$.

Theorem 3.4. *Let $(H, \circ, 0)$ be a hyper BCK-algebra and let I be a reflexive hyper BCK-ideal of H. Then $(H/I, *, C_0)$ is a hyper BCK-algebra.*

Proof. (HK1): Since H is a hyper BCK-algebra, we have $(x \circ z) \circ (y \circ z) \ll (x \circ y)$. So for all $t \in a \circ b \subseteq (x \circ z) \circ (y \circ z)$ there exists $s \in (x \circ y)$ such that $t \ll s$. Therefore $C_t \ll C_s$, where $C_t \in C_a * C_b \subseteq (C_x * C_z) * (C_y * C_z)$ and $C_s \in C_x * C_y$, hence $(C_x * C_z) * (C_y * C_z) \ll C_x * C_y$.

(HK2): We must show that $(C_x * C_y) * C_z = (C_x * C_z) * C_y$. Let $C_t \in (C_x * C_y) * C_z$. Then $t \in a \circ z \subseteq (x \circ y) \circ z = (x \circ z) \circ y$, which means that $C_t \in (C_x * C_z) * C_y$. Hence $(C_x * C_y) * C_z \subseteq (C_x * C_z) * C_y$. Similarly $(C_x * C_z) * C_y \subseteq (C_x * C_y) * C_z$.

(HK3): $C_x * \{C_t \mid t \in H\} = \{C_x * C_t \mid t \in H\} = \bigcup_{t \in H} \{C_y \mid y \in x \circ t\}$.

By Proposition 2.2 for all $y \in x \circ t$ we have $y \ll x$. So $C_y \ll C_x$, therefore $\{C_y \mid y \in x \circ t\} \ll C_x$. Thus $\bigcup_{t \in H} \{C_y \mid y \in x \circ t\} \ll C_x$. Therefore $C_x * H/I \ll C_x$.

(HK4): Let $C_x \ll C_y$ and $C_y \ll C_x$. We must show that $C_x = C_y$. Since $C_x \ll C_y$ then $C_0 \in C_x * C_y$. So there exists a $t \in x \circ y$ such that $t \sim_I 0$. Therefore $t \circ 0 \ll I$, thus $t \in I$. Hence $(x \circ y) \cap I \neq \emptyset$. Now, since I is a reflexive hyper BCK-ideal we conclude that $x \circ y \subseteq I$. Similarly $y \circ x \subseteq I$. Thus $x \sim_I y$ which means that $C_x = C_y$. □

Theorem 3.5. *If H is a bounded hyper BCK-algebra with the greatest element 1, then $(H/I, *, C_0)$ is also a bounded hyper BCK-algebra with the greatest element C_1.*

Proof. It is enough to prove that C_1 is the greatest element of H/I. For any $x \in H$, since $0 \in x \circ 1$ then $C_0 \in C_x * C_1$. This means that C_1 is the greatest element of H/I. □

The inverse of the above theorem does not hold.

Example 3.6. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper BCK-algebra structure on H, which is not bounded.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$

Then $I = \{0, 2\}$ is a reflexive hyper BCK-ideal of H. Now construct the quotient hyper BCK-algebra H/I via I. Because

$$C_0 = I = \{0, 2\} = C_2 = \{y \mid y \sim_I 2\}, \qquad C_1 = \{y \mid y \sim_I 1\} = \{1\},$$

then $H/I = \{C_0,\ C_1\}$ and

$*$	C_0	C_1
C_0	C_0	C_0
C_1	C_1	C_0

We can check that $(H/I, *, C_0)$ is a bounded hyper BCK-algebra. □

Theorem 3.7. *If J is a reflexive hyper BCK-ideal of H and $I \subseteq J$, then:*

(a) *I is a hyper BCK-ideal of the hyper BCK-subalgebra J of H,*

(b) *the quotient hyper BCK-algebra J/I is a hyper BCK-ideal of H/I.*

Proof. (a) At first we show that J is a hyper BCK-subalgebra of H. To show this let $x, y \in J$ we must show that $x \circ y \subseteq J$. Since $x \circ y \ll x$, then for all $a \in x \circ y$ we have $a \ll x$. Hence $0 \in a \circ x$. Thus $(a \circ x) \cap I \neq \emptyset$, since I is reflexive then $a \circ x \subseteq I$ and therefore $a \circ x \subseteq J$. Now $x \in J$ implies that $a \in J$, thus $x \circ y \subseteq J$. Hence J is a hyper BCK-subalgebra of H. It is clear that I is hyper BCK-ideal of the hyper BCK-subalgebra of J.

(b) We can check that $J/I \subseteq H/I$. If $C_x * C_y \ll J/I$ and $C_y \in J/I$, then for any $t \in x \circ y$, there exists $s \in J$ such that $C_t \ll C_s$. Thus $C_0 \in C_t * C_s$. So $C_0 = C_r$ for some $r \in t \circ s$. Therefore $0 \sim_I r$ and this implies that $0 \circ r \subseteq I$ and $r \circ 0 \subseteq I$. Hence $r \in I$, which means that $(t \circ s) \cap I \neq \emptyset$. Since I is reflexive, then $t \circ s \subseteq I$. Now $t \circ s \subseteq J$, and $s \in J$ implies that $t \in J$. Thus $x \circ y \ll J$. Since $y \in J$, so $x \in J$, thus $C_x \in J/I$. Hence J/I is a hyper BCK-ideal of H/I. □

Theorem 3.8. *If L is a hyper BCK-ideal of H/I, then $J = \{x \mid C_x \in L\}$ is a hyper BCK-ideal of H and moreover $I \subseteq J$. Furthermore $L = J/I$.*

Proof. Since $I = C_0 \in L$, then $0 \in J$. Let $x \circ y \ll J$ and $y \in J$. Then for any $t \in x \circ y$ there exists $s \in J$ such that $t \ll s$. Hence $C_t \ll C_s$, which implies that $C_x * C_y \ll L$. Since $y \in J$, we get that $C_y \in L$, thus $C_x \in L$. Therefore $x \in J$, hence J is a hyper BCK-ideal of H. Let $x \in I = C_0$. Then $x \sim_I 0$, thus $C_x = C_0$ and hence $C_x \in L$. Therefore $x \in J$, that is $I \subseteq J$. Clearly $L = J/I$. □

Theorem 3.9. *If I is a hyper BCK-ideal of H, then there is a bijection from the set $\mathcal{I}(H, I)$ of all hyper BCK-ideals of H containing I to the set $\mathcal{I}(H/I)$ of all hyper BCK-ideals of H/I.*

Proof. Define $f : \mathcal{I}(H, I) \to \mathcal{I}(H/I)$ by $f(J) = J/I$. By Theorem 3.7(b) f is well-defined, also Theorems 3.8 implies that f is onto. Let $A, B \in \mathcal{I}(H, I)$ and $A \neq B$. Without loss of generality, we may assume that there is an $x \in (B \backslash A)$. If $f(A) = f(B)$, then $C_x \in f(B) = B/I$ and $C_x \in f(A) = A/I$. Thus there exists $y \in A$ such that $C_x = C_y$ so $x \sim_I y$, that is $x \circ y \ll I$ and $y \circ x \ll I$. Since $I \subseteq A$ we have $x \circ y \ll A$. Thus $y \in A$ implies that $x \in A$, which is a contradiction. So f is one-to-one. □

Theorem 3.10. *Let I be a hyper BCK-ideal of H. Then there exists a canonical surjective homomorphism $\varphi : H \longrightarrow H/I$ by $\varphi(x) = C_x$, and $ker\varphi = I$, where $ker\varphi = \varphi^{-1}(C_0)$.*

Proof. It is clear that φ is well-defined. Let $x, y \in H$. Then $\varphi(x \circ y) = \{\varphi(t) \mid t \in x \circ y\} = \{C_t \mid t \in x \circ y\} = C_x * C_y = \varphi(x) * \varphi(y)$. Hence φ is homomorphism. Clearly φ is onto. We have $ker\varphi = \{x \in H \mid \varphi(x) = C_0\} = \{x \in H \mid C_x = C_0 = I\} = \{x \in H \mid x \in I\} = I$. □

Theorem 3.11. *Let $f : H_1 \longrightarrow H_2$ be a homomorphism of hyper BCK-algebras, and let I be a hyper BCK-ideal of H_1 such that $I \subseteq kerf$. Then there exists a unique homomorphism $\bar{f} : H_1/I \longrightarrow H_2$ such that $\bar{f}(C_x) = f(x)$ for all $x \in H_1$, $Im(\bar{f}) = Im(f)$ and $ker\bar{f} = kerf/I$. Moreover $\bar{f}$ is an isomorphism if and only if f is surjective and $I = kerf$.*

Proof. Let $C_x = C_{x'}$. Then $x \sim_I x'$, which implies that $x \circ x' \subseteq I$ and $x' \circ x \subseteq I$. Thus there exists $t \in (x \circ x') \bigcap I$. Then $0 = f(t) \in f(x \circ x') = f(x) \circ f(x')$, hence $f(x) \ll f(x')$. Similarly $f(x') \ll f(x)$, therefore $\bar{f}$ is well-defined.

We have $\bar{f}(C_x * C_y) = \bar{f}(\{C_t \mid t \in x \circ y\}) = \{\bar{f}(C_t) \mid t \in x \circ y\} = \{f(t) \mid t \in x \circ y\} = f(x \circ y) = f(x) \circ f(y) = \bar{f}(C_x) * \bar{f}(C_y)$. Then $\bar{f}$ is a homomorphism. On the other hand

$$C_x \in ker\bar{f} \Longleftrightarrow \bar{f}(C_x) = 0 \Longleftrightarrow f(x) = 0 \Longleftrightarrow x \in kerf.$$

Note that $\bar{f}$ is unique, since it is completely determined by f. Finally it is clear that $\bar{f}$ is surjective if and only if f is surjective. □

Theorem 3.12. *Let $f : H_1 \longrightarrow H_2$ be a homomorphism of hyper BCK-algebras. Then $H_1/kerf \cong Im(f)$.* □

Theorem 3.13. *Let I, J be hyper BCK-ideals of H. Then there is a (natural) homomorphism of hyper BCK-algebras between $I/(I \cap J)$ and $< I \cup J > /J$, where $< I \cup J >$ is the hyper BCK-ideal generated by $I \cup J$.*

Proof. Define $\varphi : I \to < I \cup J > /J$ by $\varphi(x) = C_x^J$, where C_x^J is the equivalence classes C_x via the hyper BCK-ideal J. If $x_1 = x_2$, then it is clear that $C_{x_1}^J = C_{x_2}^J$, which means that φ is well-defined. Also we have

$$\varphi(x \circ y) = \{\varphi(t) \mid t \in x \circ y\} = \{C_t^J \mid t \in x \circ y\} = C_x^J * C_y^J = \varphi(x) * \varphi(y).$$

So that φ is a homomorphism. Moreover

$$\begin{aligned} ker\varphi &= \{x \in I \mid \varphi(x) = C_0^J\} = \{x \in I \mid C_x^J = C_0^J = J\} \\ &= \{x \in I \mid x \in J\} = I \cap J. \end{aligned}$$

Thus by Theorem 3.12 the proof is completed. □

Open Problem 1. *Under what condition(s) is the defined homomorphism in Theorem* 3.11 *an isomorphism* ?

Theorem 3.14. *Let I, J be hyper BCK-ideals of H such that $I \subseteq J$. Then $(H/I)/(J/I) \cong H/J$.*

Proof. It is clear that $J/I \subseteq H/I$. Define $f : H/I \longrightarrow H/J$ by $C_x^I \mapsto C_x^J$, where $C_x^I \in H/I$ and $C_x^J \in H/J$.

If $C_x^I = C_y^I$, then $x \sim_I y$ which implies that $x \circ y \subseteq I$ and $y \circ x \subseteq I$. Since $I \subseteq J$ hence $x \circ y \subseteq J$ and $y \circ x \subseteq J$. Thus $x \sim_J y$ then $C_x^J = C_y^J$ which means that f is well-defined.

$$f(C_x^I * C_y^I) = f(\{C_t^I \mid t \in x \circ y\}) = \{C_t^J \mid t \in x \circ y\} = C_x^J * C_y^J = f(C_x^I) * f(C_y^I).$$

Clearly f is onto and

$$\begin{aligned} kerf &= \{C_x^I \in H/I \mid f(C_x^I) = C_0^J\} = \{C_x^I \in H/I \mid C_x^J = C_0^J\} \\ &= \{C_x^I \in H/I \mid x \in J\} = J/I. \end{aligned}$$

Now by Theorem 3.12 the proof is completed. □

4. Some result in quotient hyper BCK-algebras

Let $C_a, C_b \in H/I$. Then according to Definition 2.7 we have

$$\nabla(C_a, C_b) := \{C_x \in H/I \mid C_0 \in (C_x * C_a) * C_b\}.$$

Obviously $C_0, C_a, C_b \in \nabla(C_a, C_b)$, $\nabla(C_0, C_0) = \{C_0\}$ and $\nabla(C_a, C_b) = \nabla(C_b, C_a)$ for all $C_a, C_b \in H/I$.

Theorem 4.1. *If H satisfies the hyper condition, then H/I so is.*

Proof. If $x \in \nabla(a, b)$, then we have $x \circ a \ll b$. Thus for all $t \in x \circ a$, $t \ll b$. Therefore $C_t \ll C_b$, thus $C_x * C_a \ll C_b$. Hence $C_x \in \nabla(C_a, C_b)$. Since $\nabla(a, b)$ has the greatest element, then by Theorem 3.5, $\nabla(C_a, C_b)$ has the greatest element too. □

Remark 4.2. The converse of the above theorem is not correct in general. Let $H = \{0, 1, 2\}$ and

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

Then $I = \{0, 1\}$ is a reflexive hyper BCK-ideal of a hyper BCK-algebra $(H, \circ, 0)$ and the elements of the quotient hyper BCK-algebra H/I are as follows: $C_0 = I = \{0, 1\} = C_1 = \{y \mid y \sim_I 1\}$, $C_2 = \{y \mid y \sim_I 2\} = \{2\}$. Hence $H/I = \{C_0, C_2\}$ and

$*$	C_0	C_2
C_0	C_0	C_0
C_2	C_2	C_0

It can be checked that the quotient hyper BCK-algebra H/I satisfies the hyper condition, but H does not satisfy the hyper condition, since $\nabla(1, 2) = \{0, 1, 2\}$, $1 \not\ll 2$ and $2 \not\ll 1$. □

Theorem 4.3. *If H is an implicative hyper BCK-algebra, then so is H/I.*

Proof. The proof is easy. □

Note that the converse of the above theorem is not correct in general.

Example 4.4. The set $H = \{0, 1, 2\}$ with the operation

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1\}$	$\{0,1\}$

is a hyper BCK-algebra. $I = \{0, 1\}$ is a reflexive hyper BCK-ideal such that $C_0 = I = \{0, 1\} = C_1 = \{y \mid y \sim_I 1\}$, $C_2 = \{y \mid y \sim_I 2\} = \{2\}$ and

$*$	C_0	C_2
C_0	C_0	C_0
C_2	C_2	C_0

We can check that $H/I = \{C_0, C_2\}$ is an implicative hyper BCK-algebra, while the hyper BCK-algebra H is not, since $1 \not\ll 1 \circ (2 \circ 1)$. □

Theorem 4.5. *If H is a (weak) positive implicative hyper BCK algebra, then so is H/I.*

Proof. Let H be a positive implicative hyper BCK-algebra. Then we have

$$\begin{aligned} C_t \in (C_x * C_z) * (C_y * C_z) &\iff C_t = C_s \ \text{ for some } \ s \in (x \circ z) \circ (y \circ x), \quad t \sim_I s \\ &\iff C_t = C_s \ \text{ for some } \ s \in (x \circ y) \circ z, \quad s \sim_I t \\ &\iff C_t \in (C_x * C_y) * C_z. \end{aligned}$$

The other case is similar. □

Note that Example 4.4 shows that the converse of the above theorem is not correct in general. Since H/I is positive implicative while H is not, since $(2 \circ 2) \circ (2 \circ 2) = \{0, 1\} \neq \{0\} = (2 \circ 2) \circ 2$.

Theorem 4.6. *Let I and J be reflexive hyper BCK-ideals of H and $I \subseteq J$. If J is a weak implicative hyper BCK-ideal of H, then J/I is a weak implicative hyper BCK-ideal of H/I.*

Proof. Let J be a weak implicative hyper BCK-ideal of H and $(C_x * C_z) * (C_y * C_x) \subseteq J/I$ and $C_z \in J/I$. Then for all $C_s \in (C_x * C_z) * (C_y * C_x)$ where $s \in (x \circ z) \circ (y \circ x)$, we have $C_s \in J/I$. Thus $s \sim_I r$, for some $r \in J$. So $s \circ r \subseteq I$, hence $s \circ r \subseteq J$. Consequently $r \in J$ implies that $s \in J$. Thus $(x \circ z) \circ (y \circ x) \subseteq J$, and from $C_z \in J/I$ we can conclude that $z \in J$. Since J is a weak implicative hyper BCK-ideal, then we get that $x \in J$. Hence $C_x \in J/I$, which means that J/I is a weak implicative hyper BCK-ideal of H/I. □

Open Problem 2. *Does the converse of the above theorem true ?*

Theorem 4.7. *Let $I \subseteq J$ be reflexive hyper BCK-ideals of H. Then J/I is an implicative hyper BCK-ideal of H/I if and only if J is an implicative hyper BCK-ideal of H.*

Proof. Let J be an implicative hyper BCK-ideal and $C_x * (C_y * C_x) \ll J/I$. Then for all $C_t \in C_x * (C_y * C_x)$ there exists $C_r \in J/I$ such that $C_t \ll C_r$, where $t \sim_I s$, $s \in x \circ (y \circ x)$ and $r \in J$. Since $C_t \ll C_r$ then $C_0 \in C_t * C_r$, hence there exists $u \in t \circ r$ such that $0 \sim_I u$. Thus $u \circ 0 \subseteq I$, therefore $u \in I$. Then $(t \circ r) \cap I \neq \emptyset$ which means that $t \circ r \cap J \neq \emptyset$. Therefore $r \in J$ implies that $t \in J$. Since $t \sim_I s$ thus $s \circ t \subseteq I$ and hence $s \circ t \subseteq J$. Thus $t \in J$

implies that $s \in J$, hence $x \circ (y \circ x) \ll J$. Since J is an implicative hyper BCK-ideal by Theorem 3.6 of [3] we can get that $x \in J$. Hence $C_x \in J/I$. Now Theorem 3.6 [3] implies that J/I is an implicative hyper BCK-ideal of H/I.

Conversely, let J/I be an implicative hyper BCK-ideal of H/I and $x \circ (y \circ x) \ll J$. Then for all $t \in x \circ (y \circ x)$ there exists $r \in J$ such that $t \ll r$. Thus $C_t \ll C_r$, and we can conclude that $C_x * (C_y * C_x) \ll J/I$. Since J/I is an implicative hyper BCK-ideal of H, then $C_x \in J/I$, we can get that $x \in J$. Therefore J is an implicative hyper BCK-ideal of H, by Theorem 3.6 of [3]. □

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