Quasigroups and Related Systems **13** (2005), 185 – 191

On graded weakly primary ideals

Shahabaddin Ebrahimi Atani

Abstract

Let G be an arbitrary monoid with identity e. Weakly prime ideals in a commutative ring with non-zero identity have been introduced and studied in [1]. Here we study the graded weakly primary ideals of a G-graded commutative ring. Various properties of graded weakly primary ideals are considered. For example, we show that an intersection of a family of graded weakly primary ideals such that their homogeneous components are not primary is graded weakly primary.

1. Introduction

Weakly prime ideals in a commutative ring with non-zero identity have been introduced and studied by D. D. Anderson and E. Smith in [1]. Also, weakly primary ideals in a commutative ring with non-zero identity have been introduced and studied in [2]. Here we study the graded weakly primary ideals of a G-graded commutative ring. In this paper we introduce the concepts of graded weakly primary ideals and the structures of their homogeneous components. A number of results concerning graded weakly primary ideals are given. In section 2, we introduce the concepts primary and weakly primary subgroups (resp. submodules) of homogeneous components of a G-graded commutative ring. Also, we first show that if P is a graded weakly primary ideal of a G-graded commutative ring, then for each $g \in G$, either P_g is a primary subgroup of R_g or $P_g^2 = 0$. Next, we show that if P and Q are graded weakly primary ideals such that P_g and Q_h are not primary for all $g, h \in G$ respectively, then $\mathrm{Grad}(P) = \mathrm{Grad}(Q) = \mathrm{Grad}(0)$ and $P + Q$ is a graded weakly primary ideal of $G(R)$. Moreover, we give two other characterizations of homogeneous components of graded ideals.

2000 Mathematics Subject Classification:Primary: 13A02, 16W50
Keywords:Graded ring, graded weakly primary ideals.

Before we state some results let us introduce some notation and terminology. Let G be an arbitrary monoid with identity e. By a *G-graded commutative ring* we mean a commutative ring R with non-zero identity together with a direct sum decomposition (as an additive group) $R = \underset{g\in G}{\oplus} R_g$ with the property that $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$; here $R_g R_h$ denotes the additive subgroup of R consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in R_h$. We consider supp$R = \{g \in G : R_g \neq 0\}$. The summands R_g are called *homogeneous components* and elements of these summands are called *homogeneous elements*. If $a \in R$, then a can be written uniquely as $\sum_{g\in G} a_g$ where a_g is the component of a in R_g. Also, we write $h(R) = \cup_{g\in G} R_g$. Moreover, if $R = \underset{g\in G}{\oplus} R_g$ is a graded ring, then R_e is a subring of R, $1_R \in R_e$ and R_g is an R_e-module for all $g \in G$.

Let I be an ideal of R. For $g \in G$, let $I_g = I \cap R_g$. Then I is a graded ideal of R if $I = \underset{g\in G}{\oplus} I_g$. In this case, I_g is called the *g-component* of I for $g \in G$. Moreover, R/I becomes a G-graded ring with g-component $(R/I)_g = (R_g + I)/I \cong R_g/I_g$ for $g \in G$. Clearly, 0 is a graded ideal of R. A graded ideal I of R is said to be *graded prime ideal* if $I \neq R$; and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. The *graded radical* of a graded ideal I of R, denoted by Grad(I), is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element of R, then $r \in$ Grad(I) if and only if $r^n \in I$ for some positive integer n. We say that a graded ideal I of R is a *graded primary ideal* of R if $I \neq R$; and whenever $a, b \in h(R)$ with $ab \in I$ then $a \in I$ or $b \in$ Grad(I) (see [4]).

2. Weakly primary subgroups

Let I be a graded ideal of R and $x \in G$. The set

$$\{a \in R_x : a^n \in I \text{ for some positive integer } n\}$$

is a subgroup of R_x and is called the *x-radical* of I, denoted by xrad(I). Clearly, $I_x \subseteq$ xrad(I) and If $r \in R_x$ with $r \in$ Grad(I), then $r \in$ xrad(I). Our starting point is the following definitions:

Definition 2.1. Let P be a graded ideal of R and $g \in G$.

(i) We say that P_g is a *primary subgroup* of R_g if $P_g \neq R_g$; and whenever $a, b \in R_g$ with $ab \in P_g$, then either $a \in P_g$ or $b \in$ grad(P).

(ii) We say that P_g is a *weakly primary subgroup* of R_g if $P_g \neq R_g$; and whenever $a, b \in R_g$ with $0 \neq ab \in P_g$, then either $a \in P_g$ or $b \in \mathrm{grad}(P)$.

(iii) We say that P_g is a *primary submodule* of the R_e-module R_g if $P_g \neq R_g$; and whenever $a \in R_g$, $b \in R_e$ with $ab \in P_g$, then either $a \in P_g$ or $b^n \in (P_g :_{R_e} R_g)$ for some positive integer n (that is, $b \in \mathrm{erad}(P_g :_{R_e} R_g)$).

(iv) We say that P_g is a *weakly primary submodule* of the R_e-module R_g if $P_g \neq R_g$; and whenever $a \in R_g$, $b \in R_e$ with $0 \neq ab \in P_g$, then either $a \in P_g$ or $b^n \in (P_g :_{R_e} R_g)$ for some positive integer n (that is, $b \in \mathrm{erad}(P_g :_{R_e} R_g)$).

(v) We say that P is a *graded weakly primary ideal* of R if $P \neq R$; and whenever $a, b \in h(R)$ with $0 \neq ab \in P$, then either $a \in P$ or $b \in \mathrm{Grad}(P)$.

Clearly, a graded primary ideal of R is a graded weakly primary ideal of R. However, since 0 is always a graded weakly primary (by definition), a graded weakly primary ideal need not be graded primary.

Lemma 2.2. *Let* $P = \bigoplus_{g \in G} P_g$ *be a graded weakly primary ideal of* R. *Then the following hold:*

(i) P_g *is a weakly primary subgroup of* R_g *for every* $g \in G$.

(ii) P_g *is a weakly primary submodule of* R_g *for every* $g \in G$.

Proof. (i) For $g \in G$, assume that $0 \neq ab \in P_g \subseteq P$ where $a, b \in R_g$, so either $a \in P$ or $b^n \in P$ for some positive integer n since P is graded weakly primary. It follows that either $a \in P_g$ or $b \in P_{g^n}$ for some n; hence either $a \in P_g$ or $b \in \mathrm{grad}(P)$.

(ii) Suppose that P is a graded weakly primary ideal of R. For $g \in G$, assume that $0 \neq ab \in P_g \subseteq P$ where $a \in R_g$ and $b \in R_e$, so P graded weakly primary gives either $a \in P$ or $b \in \mathrm{Grad}(P)$. As b is a homogeneous element, either $a \in P$ or $b^m \in P$ for some m. If $a \in P$, then $a \in P_g$. If $b^m \in P$, then $b^m R_g \subseteq P_g$. So P_g is weakly primary. □

Proposition 2.3. *Let* P *be a graded weakly primary ideal of* R *and* $g \in G$. *Then the following hold:*

(i) *For* $a \in R_g - \mathrm{grad}(P)$, *either* $\mathrm{erad}(P_g :_{R_e} a) = \mathrm{erad}(P)$ *or* $\mathrm{erad}(P_g :_{R_e} a) = (0 :_{R_e} a)$.

(ii) *For* $a \in h(R) - P$, $\mathrm{Grad}(P :_R a) = \mathrm{Grad}P + (0 :_R a)$.

Proof. (i) It is well-known that if an ideal (resp. a subgroup) is the union of two ideals (resp. two subgroups), then it is equal to one of them, so for $a \in R_g - \mathrm{grad}(P)$, it is enough to show that

$$\mathrm{erad}(P_g :_{R_e} a) = \mathrm{erad}(P) \cup (0 :_{R_e} a) = H.$$

If $b \in \mathrm{erad}(P)$, then $ab^n \in R_g \cap P = P_g$, so $b^n \in (P_g :_{R_e} a)$; hence $b \in \mathrm{erad}(P_g :_{R_e} a)$. Clearly, $(0 :_{R_e} a) \subseteq \mathrm{erad}(P_g :_{R_e} a)$. Thus, $H \subseteq \mathrm{erad}(P_g :_{R_e} a)$. For the reverse inclusion, assume that $c \in \mathrm{erad}(P_g :_{R_e} a)$. Then $ac^m \in P_g$ for some m. If $0 \neq ac^m \in P_g \subseteq P$, then P graded weakly primary gives $c^m \in P$; hence $c \in \mathrm{erad}(P) \subseteq H$. If $ac^m = 0$, then assume that k is the smallest integer with $ac^k = 0$. If $k = 1$, then $c \in (0 :_{R_e} a) \subseteq H$. Otherwise, $c \in \mathrm{erad}(P) \subseteq H$, we have equality.

(ii) Clearly, for $a \in h(R) - P$, $\mathrm{Grad}(P) + (0 :_R a) \subseteq \mathrm{Grad}(P :_R a)$. For the other containment, assume that $b \in \mathrm{Grad}(P :_R a)$ where $a \in h(R) - P$.

Without loss of generality assume $b = \sum_{i=1}^{n} b_{g_i}$ where $b_{g_i} \neq 0$ for all $i = 1, \ldots, n$ and $b_g = 0$ for all $g \notin \{g_1, \ldots, g_n\}$. As $b \in \mathrm{Grad}(P :_R a)$, for each i, there exists a positive integer m_{g_i} such that $b_{g_i}^{m_{g_i}} a \in P$. If $b_{g_i}^{m_{g_i}} a \neq 0$, then $b_{g_i}^{m_{g_i}} \in \mathrm{Grad}(P)$ since P is graded weakly primary. Therefore, $b_{g_i} \in \mathrm{Grad}(\mathrm{Grad}(P)) = \mathrm{Grad}(P)$ by [4, Proposition 1.2]. So suppose that $b_{g_i}^{m_{g_i}} a = 0$ for some i. Then assume that s_{g_i} is the smallest integer with $b_{g_i}^{s_{g_i}} a = 0$. If $s_{g_i} = 1$, then $b_{g_i} \in (0 :_R a)$. Otherwise, $b_{g_i} \in \mathrm{Grad}(P)$, so $b \in \mathrm{Grad}(P) + (0 :_R a)$, as required. $\square$

Proposition 2.4. *Let $P = \bigoplus_{g \in G} P_g$ be a graded weakly primary ideal of R. Then for each $g \in G$, either $(P_g :_{R_e} R_g)P_g = 0$ or P_g is a primary submodule of R_g.*

Proof. By Lemma 2.2, P_g is a weakly primary submodule of R_g for every $g \in G$. It is enough to show that if $(P_g :_{R_e} R_g)P_g \neq 0$ for some $g \in G$, then P_g is primary. Let $ab \in P_g$ where $a \in R_g$ and $b \in R_e$. If $ab \neq 0$, then either $a \in P_g$ or $b^n \in (P_g :_{R_e} R_g)$ for some n since P_g is weakly primary. So suppose that $ab = 0$. First suppose that $bP_g \neq 0$, say $bc \neq 0$ where $c \in P_g$. Then $0 \neq bc = b(c + a) \in P_g$, so either $b^m \in (P_g :_{R_e} R_g)$ for some m or $(a + c) \in P_g$. As $c \in P_g$ we have either $b^m \in (P_g :_{R_e} R_g)$ or $a \in P_g$. So we can assume that $bP_g = 0$. Suppose that $a(P_g :_{R_e} R_g) \neq 0$, say $ad \neq 0$ where $d \in (P_g :_{R_e} R_g)$. Then $0 \neq ad = a(d + b) \in P_g$, so either $a \in P_g$ or $(d + b)^s \in (P_g :_{R_e} R_g)$ for some s. It follows that either $a \in P_g$

or $b^s + r \in (P_g :_{R_e} R_g)$ where $r \in (P_g :_{R_e} R_g)$. Thus, either $a \in P_g$ or $b^s \in (P_g :_{R_e} R_g)$. So we can assume that $a(P_g :_{R_e} R_g) = 0$.

Since $(P_g :_{R_e} R_g)P_g \neq 0$, there exist $u \in (P_g :_{R_e} R_g)$ and $v \in P_g$ such that $uv \neq 0$. Then $(b+u)(a+v) = uv \in P_g$, so either $(b+u)^n \in (P_g :_{R_e} R_g)$ for some n or $a+v \in P_g$, and hence either $b^n \in (P_g :_{R_e} R_g)$ or $a \in P_g$. Thus P_g is primary. $\square$

We next give two other characterizations of homogeneous components of graded ideals.

Theorem 2.5. *Let P be a graded ideal of R and $g \in G$. Then the following assertion are equivalent.*

(*i*) *P_g is a weakly primary submodule of R_g.*

(*ii*) *For $a \in R_g - \mathrm{grad}(P)$, $\mathrm{erad}(P_g :_{R_e} a) = \mathrm{erad}(P_g :_{R_e} R_g) \cup (0 :_{R_e} a)$.*

(*iii*) *For $a \in R_g - \mathrm{erad}(P)$, $\mathrm{erad}(P_g :_{R_e} a) = \mathrm{erad}(P_g :_{R_e} R_g)$ or $\mathrm{erad}(P_g :_{R_e} a) = (0 :_{R_e} a)$.*

Proof. $(i) \Rightarrow (ii)$ Suppose first that P_g is a weakly primary submodule of R_g. Clearly, if $a \in R_g - \mathrm{grad}(P)$, then $H = \mathrm{erad}(P_g :_{R_e} R_g) \cup (0 :_{R_e} a) \subseteq \mathrm{erad}(P_g :_{R_e} a)$. Let $b \in \mathrm{erad}(P_g :_{R_e} a)$ where $a \in R_g - \mathrm{grad}(P)$. Then $ab \in P_g$. If $ab \neq 0$, then $b^n \in (P_g :_{R_e} R_g)$ for some n since $a \notin \mathrm{grad}(P) \supseteq P_g$ and P_g is weakly primary, so $b \in \mathrm{erad}(P_g :_{R_e} R_g) \subseteq H$. If $ab = 0$, then $b \in (0 :_{R_e} a) \subseteq H$, and hence we have equality.

$(ii) \Rightarrow (iii)$ Is obvious.

$(iii) \Rightarrow (i)$ Suppose that $0 \neq ab \in P_g$ with $b \in R_e$ and $a \in R_g - P_g$. Then $b \in (P_g :_{R_e} a) \subseteq \mathrm{erad}(P_g :_{R_e} a)$ and $b \notin (0 :_{R_e} a)$. It follows from (ii) that $b \in \mathrm{erad}(P_g :_{R_e} R_g)$, as required. $\square$

Theorem 2.6. *Let $P = \bigoplus_{g \in G} P_g$ be a graded weakly primary ideal of R. Then for each $g \in G$, either P_g is a primary subgroup of R_g or $P_g^2 = 0$.*

Proof. By Lemma 2.2, P_g is a weakly primary subgroup of R_g for every $g \in G$. It is enough to show that if $P_g^2 \neq 0$ for some $g \in G$, then P_g is a primary subgroup of R_g. Let $ab \in P_g$ where $a, b \in R_g$. If $ab \neq 0$, then P_g weakly primary gives either $a \in P_g$ or $b \in \mathrm{grad}(P)$. So suppose that $ab = 0$. If $aP_g \neq 0$, then there is an element c of P_g such that $ac \neq 0$, so $0 \neq ac = a(c+b) \in P$; hence either $a \in P$ or $(c+b) \in \mathrm{Grad}(P)$. As $c \in P \subseteq \mathrm{Grad}(P)$ (by [4, Proposition 1.2]), we have either $a \in P_g$ or

$b \in \text{Grad}(P)$; hence either $p \in P_g$ or $b \in \text{grad}(P)$. So we can assume that $aP_g = 0$. Similarly, we can assume that $bP_g = 0$. Since $P_g^2 \neq 0$, there exist $p, q \in P_g \subseteq P \subseteq \text{Grad}(P)$ such that $pq \neq 0$. Then $(a+p)(b+q) = pq \in P$, so either $a+p \in P$ or $b+q \in \text{Grad}(P)$, and hence either $a \in P_g$ or $b \in \text{grad}(P)$. Thus P_g is primary. □

Proposition 2.7. *Let* $P = \bigoplus_{g\in G} P_g$ *be a graded weakly primary ideal of* R *such that* P_g *is not a primary subgroup of* R_g *for every* $g \in G$. *Then* $\text{Grad}(P) = \text{Grad}(0)$.

Proof. Clearly, $\text{Grad}(0) \subseteq \text{Grad}(P)$. For the other containment, assume that $a \in P$. By Theorem 2.6, $a_g^2 = 0 \in (0)$ for every $g \in G$, so $a \in \text{Grad}(0)$; hence $P \subseteq \text{Grad}(0)$. It follows that $\text{Grad}(P) \subseteq \text{Grad}(0)$ by [4, Proposition 1.2], as needed. □

Theorem 2.8. *Let* $\{P_i\}_{i\in I}$ *be a family of graded weakly primary ideals of* R *such that for each* $i \in I$, $(P_i)_g$ *is not a primary subgroup of* R_g *for every* $g \in G$. *Then* $P = \bigcap_{i\in I} P_i$ *is a graded weakly primary ideal of* R.

Proof. First, we show that $\text{Grad}(P) = \bigcap_{i\in I} \text{Grad}(P_i)$. Clearly, $\text{Grad}(P) \subseteq \bigcap_{i\in I} \text{Grad}(P_i)$. For the reverse inclusion, suppose that $a \in \bigcap_{i\in I} \text{Grad}(P_i)$, so for each $g \in G$, $a_g^{m_g} = 0$ for some m_g since $\bigcap_{i\in I} \text{Grad}(P_i) = \text{Grad}(0)$ by Proposition 2.7. It follows that for each $i \in I$, $a_g^{m_g} \in P_i$ for every $g \in G$, and hence $a \in \text{Grad}(P)$.

As $\text{Grad}(P) = \text{Grad}(0) \neq R$, we have P is a proper ideal of R. Suppose that $a, b \in h(R)$ are such that $0 \neq ab \in P$ but $b \notin P$. Then there is an element $j \in I$ such that $b \notin P_j$ and $ab \in P_j$. It follows that $a \in \text{Grad}(P_j) = \text{Grad}(P)$ since P_i is graded weakly primary, as needed. □

Proposition 2.9. *Let* $I \subseteq P$ *be graded ideals of* R *with* $P \neq R$. *Then the following hold:*

(i) *If* P *is graded weakly primary, then* P/I *is graded weakly primary.*

(ii) *If* I *and* P/I *are graded weakly primary, then* P *is graded weakly primary.*

Proof. (i) Let $0 \neq (a+I)(b+I) = ab + I \in P/I$ where $a, b \in h(R)$, so $ab \in P$. If $ab = 0 \in I$, then $(a+I)(b+I) = 0$, a contradiction. If $ab \neq 0$,

P graded weakly primary gives either $a \in P$ or $b \in \mathrm{Grad}(P)$; hence either $a+I \in P/I$ or $b^n + I = (b+I)^n \in P/I$ for some integer n. It follows that either $a+I \in P/I$ or $b+I \in \mathrm{Grad}(P/I)$, as needed.

(ii) Let $0 \neq ab \in P$ where $a, b \in h(R)$, so $(a+I)(b+I) \in P/I$. If $ab \in I$, then I graded weakly primary gives either $a \in I \subseteq P$ or $b \in \mathrm{Grad}(I) \subseteq \mathrm{Grad}(P)$. So we may assume that $ab \notin I$. Then either $a+I \in P/I$ or $b^m + I \in P/I$ for some integer m since P/I is graded weakly primary. It follows that either $a \in P$ or $b \in \mathrm{Grad}(P)$, as required. □

Theorem 2.10. *Let P and Q be graded weakly primary ideals of R such that P_g and Q_h are not primary subgroups of R_g and R_h respectively for all $g, h \in G$. Then $P+Q$ is a graded weakly primary ideal of R.*

Proof. By Proposition 2.7, we have

$$\mathrm{Grad}(P) + \mathrm{Grad}(Q) = \mathrm{Grad}(0) \neq R,$$

so $P+Q$ is a proper ideal of R. Since $(P+Q)/Q \cong Q/(P \cap Q)$, we get $(P+Q)/Q$ is graded weakly primary by Propositin 2.9 (i). Now the assertion follows from Proposition 2.9 (ii). □

Acknowledgement. The author thanks the referee for valuable comments.

References

[1] **D. D. Anderson, R. Smith:** *Weakly prime ideals*, Houston J. Math. **29** (2003), 831 – 840.

[2] **S. Ebrahimi Atani, F. Farzalipour:** *On weakly primary ideals*, Georgian Math. J. (to appear).

[3] **C. Nastasescu, F. Van Oystaeyen:** *Graded Ring Theory*, Mathematical Library 28, North Holand, Amsterdam, 1982.

[4] **M. Refai, K. Al-Zoubi:** *On graded primary ideals*, Turkish J. Math. **28** (2004), 217 – 229.

Received February 17, 2005

Department of Mathematics
University of Guilan
P.O. Box 1914
Rasht
Iran
Email: ebrahimi@guilan.ac.ir