On decomposable hyper BCK-algebras

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Abstract

In this manuscript, we introduce the concept of decomposable hyper BCK-algebras and we give a condition for a hyper BCK-algebra to be a decomposable hyper BCK-algebra. Moreover, we state and prove some theorems about (weak, implicative) strong hyper BCK-ideal of a decomposable hyper BCK-algebra. Finally, we give a characterization of some decomposable hyper BCK-algebras.

1. Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iséki [5] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK-algebras. The hyperstructure theory (called also multialgebras) was introduced in 1934 by F. Marty [9] at the 8th congress of Scandinavian Mathematiciens. In [8], Y.B. Jun et al. applied the hyperstructures to BCK-algebras, and introduced the notion of a hyper BCK-algebra which is a generalization of BCK-algebra, and investigated some related properties. Now we follow [7] and [8] and introduce the concept of decomposable hyper BCK-algebra and give a condition for a hyper BCK-algebra to be a decomposable hyper BCK-algebra. Moreover, we state and prove some theorems about (weak, implicative) strong hyper BCK-ideal a of decomposable hyper BCK-algebra.

2. Preliminaries

Definition 2.1. [8] By a hyper BCK-algebra we mean a non-empty set \( H \) endowed with a hyperoperation "\( \circ \)" and a constant 0 satisfying the following
axioms:
  (HK1) \((x \circ z) \circ (y \circ z) \preceq x \circ y\),
  (HK2) \((x \circ y) \circ z = (x \circ z) \circ y\),
  (HK3) \(x \circ H \preceq \{x\}\),
  (HK4) \(x \preceq y\) and \(y \preceq x\) imply \(x = y\)
for all \(x, y, z \in H\), where \(x \preceq y\) is defined by \(0 \in x \circ y\) and for every \(A, B \subseteq H\), \(A \preceq B\) is defined by \(\forall a \in A, \exists b \in B\) such that \(a \preceq b\). In such case, we call \(\preceq\) the hyperorder in \(H\).

Theorem 2.2. [8] In any hyper BCK-algebra \(H\), the following hold:
  (i) \(0 \circ 0 = \{0\}\),
  (ii) \(0 \preceq x\),
  (iii) \(x \preceq x\),
  (iv) \(A \preceq A\),
  (v) \(A \subseteq B\) implies \(A \preceq B\),
  (vi) \(0 \circ x = \{0\}\),
  (vii) \(x \circ y \preceq x\),
  (viii) \(x \circ 0 = \{x\}\),
  (ix) \(x \preceq z\) implies \(x \circ z \preceq x \circ y\)
for all \(x, y, z \in H\) and for all non-empty subsets \(A\) and \(B\) of \(H\).

Definition 2.3. Let \(I\) be a subset of a hyper BCK-algebra \(H\) and \(0 \in I\). Then \(I\) is said to be a weak hyper BCK-ideal of \(H\) if \(x \circ y \subseteq I\) and \(y \in I\) imply \(x \in I\) for all \(x, y \in H\), hyper BCK-ideal of \(H\) if \(x \circ y \preceq I\) and \(y \in I\) imply \(x \in I\) for all \(x, y \in H\), strong hyper BCK-ideal if \((x \circ y) \cap I \neq \emptyset\) and \(y \in I\) imply \(x \in I\) for all \(x, y \in H\), reflexive hyper BCK-ideal of \(H\) if \(I\) is a hyper BCK-ideal of \(H\) and \(x \circ x \subseteq I\) for all \(x \in H\).

Theorem 2.4. [6, 7, 8] Let \(H\) be a hyper BCK-algebra. Then,
  (i) any strong hyper BCK-ideal of \(H\) is a hyper BCK-ideal of \(H\),
  (ii) if \(I\) is a hyper BCK-ideal of \(H\) and \(A\) be a nonempty subset of \(H\), then \(A \subseteq I\) implies \(A \subseteq I\),
  (iii) \(H\) is a BCK-algebra if and only if \(H = \{x \in H : x \circ x = \{0\}\}\).

Definition 2.5. [3] Let \(H\) be a hyper BCK-algebra, \(\Theta\) be an equivalence relation on \(H\) and \(A, B \subseteq H\). Then,
  (i) we write \(A \Theta B\), if there exist \(a \in A\) and \(b \in B\) such that \(a \Theta b\),
  (ii) we write \(A \Theta B\), if for all \(a \in A\) there exists \(b \in B\) such that \(a \Theta b\)
and for all \(b \in B\) there exists \(a \in A\) such that \(a \Theta b\),
  (iii) \(\Theta\) is called a congruence relation on \(H\), if \(x \Theta y\) and \(x' \Theta y'\), then
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\[ x \circ x' \Theta y \circ y', \text{ for all } x, y \in H, \]

(iv) \( \Theta \) is called a regular relation on \( H \) if \( x \circ y \Theta \{0\} \) and \( y \circ x \Theta \{0\} \), then \( x \Theta y \) for all \( x, y \in H \).

**Theorem 2.6.** [3] Let \( \Theta \) and \( \Theta' \) are two regular congruence relations on \( H \) such that \( [0]_\Theta = [0]_{\Theta'} \). Then \( \Theta = \Theta' \).

**Theorem 2.7.** [3] Let \( \Theta \) be a regular congruence relation on \( H \) and \( H/\Theta = \{I_x : x \in H\} \), where \( I_x = [x]_\Theta \), for all \( x \in H \). Then \( H/\Theta \) with hyperoperation \( I_x \circ I_y = \{I_z : z \in x \circ y\} \) and hyper order \( I_x < I_y \iff I \in I_x \circ I_y \) is a hyper BCK-algebra which is called quotient hyper BCK-algebra.

**Theorem 2.8.** [3] (Isomorphism Theorem) Let \( \Theta \) be a regular congruence relation on hyper BCK-algebra \( H \). If \( f : H \longrightarrow H' \) is a homomorphism of hyper BCK-algebras such that \( \text{Ker} f = [0]_\Theta \), then \( H/\Theta \cong f(H) \).

### 3. Decomposable hyper BCK-algebras

**Definition 3.1.** A hyper BCK-algebra \( H \) is called decomposable if there exists a nontrivial family \( \{A_i\}_{i \in \Lambda} \) of hyper BCK-ideals of \( H \) such that

(i) \( H \neq A_i \neq \{0\} \) for all \( i \in \Lambda \),

(ii) \( H = \bigcup_{i \in \Lambda} A_i \),

(iii) \( A_i \cap A_j = \{0\} \) for all \( i \neq j \in \Lambda \).

In this case, we say that \( H = \bigcup_{i \in \Lambda} A_i \) is a decomposition of \( H \) and we write \( H = \bigoplus_{i \in \Lambda} A_i \).

**Example 3.2.** (i) Let \( H \) be a hyper BCK-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th>( \circ )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0,1}</td>
<td>{1}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{2}</td>
<td>{0,2}</td>
</tr>
</tbody>
</table>

It is easy to check that \( A_1 = \{0,1\} \) and \( A_2 = \{0,2\} \) are hyper BCK-ideals of \( H \) such that \( H = A_1 \cup A_2 \) and \( A_1 \cap A_2 = \{0\} \). Therefore, \( H \) is decomposable.

(ii) Let \( H = N \cup \{0\} \). Consider the hyperoperation

\[ x \circ y = \begin{cases} \{0\} & \text{if } x = 0 \text{ or } x = y, \\ \{x\} & \text{otherwise.} \end{cases} \]
It is easily verified that \((H, \circ, 0)\) is a hyper \(BCK\)-algebra and \(A_n = \{0, n\}\) is a hyper \(BCK\)-ideal of \(H\), for all \(n \in N\). Now, since \(H = \bigcup_{n \in N} A_n\) and \(A_n \cap A_m = \{0\}\), for each \(n \neq m \in N\). Therefore, \(H\) is decomposable.

(iii) Let \(N = \{0, 1, 2, 3, \ldots\}\) and hyper operation "\(\circ\)" on \(N\) is defined as follow:
\[
x \circ y = \begin{cases} 
\{0, x\} & \text{if } x \leq y, \\
\{x\} & \text{if } x > y
\end{cases}
\]
for all \(x, y \in H\). Then \((N, \circ, 0)\) is a hyper \(BCK\)-algebra but it is not a decomposable hyper \(BCK\)-algebra. Since every hyper \(BCK\)-ideal of \(H\) is equal to \(H\) or \(\{0, 1, 2, \ldots, n - 1\}\), for some \(n \in N\).

**Note.** From now on, we let \(H\) be a hyper \(BCK\)-algebra.

**Theorem 3.3.** Let \(H\) be decomposable with decomposition \(H = \bigoplus_{i \in \Lambda} A_i\). Then \(A_i\) is a strong hyper \(BCK\)-ideal of \(H\) for all \(i \in \Lambda\).

**Proof.** Let \(H = \bigoplus_{i \in \Lambda} A_i\) be a decomposition of \(H\) and let \((x \circ y) \cap A_i \neq \emptyset\) and \(y \in A_i\) for \(x \in H\) and \(i \in \Lambda\). Then there exists \(t \in x \circ y\) such that \(t \in A_i\). From \(x \in H = \bigcup_{i \in \Lambda} A_i\) we conclude that there exists \(j \in \Lambda\) such that \(x \in A_j\). Since \(x \circ y \leq x \in A_j\), then \(x \circ y \leq A_j\) and so by Theorem 2.4, \(x \circ y \subseteq A_j\). Therefore, \(t \in A_i \cap A_j\). Now, we consider the following two cases. If \(j = i\), then \(A_j = A_i\) and \(x \in A_i\). If \(j \neq i\), then \(t \in A_i \cap A_j = \{0\}\) that \(t = 0\) and so \(0 \in x \circ y\). This implies that \(x \leq y\). It follow from \(y \in A_i\) and Theorem 2.4 (ii) \(x \in A_i\). Therefore, \(A_i\) is a strong hyper \(BCK\)-ideal of \(H\).

**Theorem 3.4.** Let \(H\) be decomposable with decomposition \(H = \bigoplus_{i \in \Lambda} A_i\). Then \(A_i \cup A_j\) is a strong hyper \(BCK\)-ideal of \(H\) for all \(i, j \in \Lambda\).

**Proof.** Let \(i, j \in \Lambda\) and \(x, y \in H\) be such that \((x \circ y) \cap (A_i \cup A_j) \neq \emptyset\) and \(y \in A_i \cup A_j\). Without loss of generality, assume that \(y \in A_i\). Since \((x \circ y) \cap (A_i \cup A_j) \neq \emptyset\), then there exists \(t \in H\) such that \(t \in (x \circ y) \cap (A_i \cup A_j)\) and so \(t \in A_i\) or \(t \in A_j\). If \(t \in A_i\), then \(A_i\) is a strong hyper \(BCK\)-ideal of \(H\) and \(y \in A_i\), then \(x \in A_i \subseteq A_i \cup A_j\). If \(t \in A_j\), then \(x \in H = \bigcup_{i \in \Lambda} A_i\) there exists \(k \in \Lambda\) such that \(x \in A_k\). It follow from \(x \circ y \leq x \in A_k\) and Theorem 2.4 (ii) that \(x \circ y \leq A_k\) and so \(x \circ y \subseteq A_k\). Hence we have \(t \in A_j \cap A_k\). If \(j = k\) then \(A_j = A_k\) and so \(x \in A_j \subseteq A_i \cup A_j\). If \(j \neq k\), then \(t \in A_j \cap A_k = \{0\}\) and so \(t = 0\). Then \(0 \in x \circ y\) and so \(x \leq y\). Now, since \(y \in A_i\) and \(A_i\) is a hyper \(BCK\)-ideal of \(H\) then \(x \in A_i \subseteq A_i \cup A_j\). Therefore, \(A_i \cup A_j\) is a strong hyper \(BCK\)-ideal of \(H\).
Theorem 3.5. Let $H$ be decomposable with decomposition $H = \bigoplus_{i \in \Lambda} A_i$. Then $\bigcup_{i \in \Omega} A_i$ is a strong hyper $BCK$-ideal of $H$ for all $\emptyset \neq \Omega \subseteq \Lambda$.

Proof. We proceed by induction on $|\Omega|$. For $\Omega \subseteq \Lambda$ with $|\Omega| = 1$ the result holds by Theorem 3.3. Suppose that for $2 \leq m \in N$ and all $\Omega \subseteq \Lambda$ with $|\Omega| \leq m$ the result holds and let $\Omega \subseteq \Lambda$ be such that $|\Omega| = m + 1$. Let $i, j$ be arbitrary elements of $\Omega$. Taking $A_{ij} = A_i \cup A_j$ and by using Theorems 3.4 and 2.4(i), we conclude that $A_0$ is a hyper $BCK$-ideal of $H$. Taking $\Omega' = (\Omega - \{i, j\}) \cup \{ij\}$ and by using the hypothesis of induction, we conclude that $\bigcup_{i \in \Omega'} A_i$ is a strong hyper $BCK$-ideal of $H$. Now, since $\bigcup_{i \in \Omega} A_i = \bigcup_{i \in \Omega'} A_i$ then $\bigcup_{i \in \Omega} A_i$ is a strong hyper $BCK$-ideal of $H$. Therefore for all $\emptyset \neq \Omega \subseteq \Lambda$, $\bigcup_{i \in \Omega} A_i$ is a strong hyper $BCK$-ideal of $H$. \hfill $\Box$

Corollary 3.6. Let $H$ be decomposable. Then there exist nontrivial strong hyper $BCK$-ideals $A, B$ of $H$ such that $H = A \cup B$ and $A \cap B = \{0\}$, that is $H = A \bigoplus B$.

Proof. The proof come immediately from Theorem 3.5. \hfill $\square$

Theorem 3.7. Let $H$ be a hyper $BCK$-algebra. Then $H$ is decomposable if and only if there exists a nontrivial strong hyper $BCK$-ideal $A$ of $H$ such that $0 \notin (A' \circ B) \circ B$, where $A' = A - \{0\}$ and $B = H - A'$.

Proof. ($\Rightarrow$) Let $H$ be decomposable. Then by Corollary 3.6 there exist nontrivial strong hyper $BCK$-ideals $A$ and $B$ of $H$ such that $H = A \bigoplus B$. Let $0 \in (A' \circ B) \circ B$, by contrary. Since, $(A' \circ B) \circ B = \bigcup_{b \in B, t \in A' \circ B} t \circ b$, then there exist $t \in A' \circ B$ and $b \in B$ such that $0 \in t \circ b$. Now, since $b \in B$ and $B$ is a strong hyper $BCK$-ideal of $H$, then $t \in B$. But, $t \in A' \circ B$ implies that there exist $a \in A'$ and $b_1 \in B$ such that $t = a \circ b_1$ and so $a \circ b_1 \cap B \neq \emptyset$ and this implies that $a \in B$. Hence, $0 \neq a \in A \cap B = \{0\}$, which is impossible. Therefore, $0 \notin (A' \circ B) \circ B$.

($\Leftarrow$) It is enough to prove that $B$ is a hyper $BCK$-ideal of $H$. Let for $a, b \in H$, $a \circ b \ll B$ and $b \in B$ but $a \notin B$. Hence, $a \in A'$. Since $a \circ b \ll B$, then there exist $t \in a \circ b$ and $b_1 \in B$ such that $t \ll B_1$ and so $0 \in t \circ B_1$. Hence

$$0 \in t \circ b_1 \subseteq (a \circ b) \circ b' \subseteq (A' \circ B) \circ B$$

which is impossible. \hfill $\square$

Theorem 3.8. Let $H$ be decomposable with decomposition $H = A \bigoplus B$. Then $A$ and $B$ are implicatve hyper $BCK$-ideals of $H$ if and only if for all $x, y \in H$, $x \circ (y \circ x) = \{0\}$ imply $x = 0$.  

\hfill $\square$
Proof. Let $A$ and $B$ be implicative hyper $BCK$-ideals of $H$ and $x \circ (x \circ y) = \{0\}$ for $x, y \in H$. Then $x \circ (y \circ x) \ll A$ and $x \circ (y \circ x) \ll B$ and so by Theorem 2.4 (iii), $x \in A \cap B = \{0\}$.

Conversely, let for $x, y \in H$, $x \circ (y \circ x) \ll A$ but $x \notin A$, by contrary. Hence, $0 \neq x \in H$, $x \circ (y \circ x) \ll A$ but $x \notin A$, by contrary. Hence, $x \circ (y \circ x) \subseteq A \cap B = \{0\}$ and so $x \circ (y \circ x) = \{0\}$. Now, by hypothesis $x = 0$, which is a contradiction. Therefore, $x \in A$ and so by Theorem 2.4 (iii) $A$ is a implicative hyper $BCK$-ideal of $H$. The proof of case $B$ is similar. 

Proposition 3.9. Let $H$ be decomposable with decomposition $H = A \oplus B$. If $A$ and $B$ are reflexive, then $H$ is a $BCK$-algebra.

Proof. Let $A$ and $B$ be reflexive. Then we have $x \circ x \subseteq A$ and $x \circ x \subseteq B$ for all $x \in H$. Hence $x \circ x \subseteq A \cap B = \{0\}$ and so $x \circ x = 0$. It follows from Theorem 2.4 (iv) that $H$ is a $BCK$-algebra. 

Definition 3.10. Let $\emptyset \neq A \subset H$. Then subset $I$ of $H$ is called a weak hyper $BCK$-ideal of $H$ related to $A$ if

(r1) $0 \in I$,

(r2) $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$ for all $x \in A$.

Note that, for all nonempty subset $A$ of $H$ if $I$ is a weak hyper $BCK$-ideal of $H$, then $I$ is a weak hyper $BCK$-ideal of $H$ related to $A$. But the converse is not true in general.

Example 3.11. Consider a hyper $BCK$-algebra $H$ with the following Cayley table:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0}$</td>
<td>${0}$</td>
<td>${0}$</td>
<td>${0}$</td>
</tr>
<tr>
<td>1</td>
<td>${1}$</td>
<td>${0}$</td>
<td>${0}$</td>
<td>${0}$</td>
</tr>
<tr>
<td>2</td>
<td>${2}$</td>
<td>${1}$</td>
<td>${0}$</td>
<td>${0}$</td>
</tr>
<tr>
<td>3</td>
<td>${3}$</td>
<td>${3}$</td>
<td>${3}$</td>
<td>${0,3}$</td>
</tr>
</tbody>
</table>

Then $I = \{0,2\}$ is a weak hyper $BCK$-ideal of $H$ related to $A = \{0,2,3\}$. But, $I$ is not a weak hyper $BCK$-ideal of $H$. Since $1 \circ 2 \subseteq I$ and $2 \in I$ but $1 \notin I$.

Theorem 3.12. Let $H$ be decomposable with decomposition $H = A \oplus B$ and $I \subseteq A$. If $I$ is a weak hyper $BCK$-ideal of $H$ related to $A$, then $I$ is a weak hyper $BCK$-ideal of $H$. 

Proof. Let \( I \) be a weak hyper \( BCK \)-ideal of \( H \) related to \( A \) and \( x \circ y \subseteq I \) and \( y \in I \), for \( x, y \in H \). If \( x \in A \), then by hypothesis \( x \in I \). Now, let \( x \in B \). Then by Theorem 2.2 (vii), \( x \circ y \ll B \), which implies that \( x \circ y \subseteq B \) by Theorem 2.4 (i,ii). Hence \( x \circ y \subseteq A \cap B = \{0\} \), which implies that \( x \circ y = \{0\} \) and so \( x \ll y \). Since \( y \in I \subseteq A \), we have \( x \ll A \) and so by Theorem 2.4, we get \( x \in A \). Thus \( x \in A \cap B = \{0\} \). This implies that \( x = 0 \) and so \( x \ll y \). Since \( y \in I \subseteq A \), we have \( x \ll A \) and so by Theorem 2.4, we get \( x \in A \). Therefore, \( I \) is a weak hyper \( BCK \)-ideal of \( H \).

Definition 3.13. Let \( \emptyset \neq A \subset H \). Then subset \( I \) of \( H \) is called a hyper \( BCK \)-ideal of \( H \) related to \( A \) if

\[
\begin{align*}
(r1) & \quad 0 \in I, \\
(r3) & \quad x \circ y \ll I \text{ and } y \in I \text{ imply } x \in I \text{ for all } x \in A.
\end{align*}
\]

Note that, for all nonempty subset \( A \) of \( H \) if \( I \) is a hyper \( BCK \)-ideal of \( H \), then \( I \) is a hyper \( BCK \)-ideal of \( H \) related to \( A \). But the converse is not true in general.

Example 3.14. Let \( J = \{0,1\} \) and \( B = \{0,1,3\} \) in Example 3.12. It is easy to show that \( J \) is a hyper \( BCK \)-ideal of \( H \) related to \( B \), but \( J \) is not hyper \( BCK \)-ideal of \( H \). Since \( 2 \circ 1 \ll J \) and \( 1 \in J \), but \( 2 \not\in J \).

Theorem 3.15. Let \( H \) be decomposable with decomposition \( H = A \oplus B \) and \( I \subseteq A \). If \( I \) is a hyper \( BCK \)-ideal of \( H \) related to \( A \), then \( I \) is a hyper \( BCK \)-ideal of \( H \).

Proof. The proof is similar to the proof of Theorem 3.12 by some modification.

4. Quotient structure

Theorem 4.1. Let \( H \) be decomposable with decomposition \( H = A \oplus B \). Then there exists a regular congruence relation \( \Theta \) on \( H \) and a hyper \( BCK \)-algebra \( X \) of order 2 such that \( H/\Theta \cong X \).

Proof. Let relation \( \Theta \) on \( H \) is defined as follows:

\[
x \Theta y \iff x, y \in A \text{ or } x, y \in B - \{0\}.
\]

Since \( H = A \oplus B \) is a decomposition of \( H \), then it is easily verified that \( \Theta \) is an equivalence relation on \( H \). Now, let \( x, y \in H \) such that \( x \Theta y \). Then \( x, y \in A \) or \( x, y \in B - \{0\} \). Without loss of generality we can suppose that \( x, y \in A \). It follow from Theorem 2.2 (vii) and Theorem 2.4 we get that
\( x \circ a \subseteq A \ (y \circ a \subseteq A) \), which implies that \( x \circ a \Theta y \circ a \) for all \( a \in H \). On the other hand by using Theorem 2.2 (vii) and Theorem 2.4 (i,ii), we get \( a \circ x \subseteq A \ (a \circ y \subseteq A) \) if \( a \in A \), and \( a \circ x \subseteq B \ (a \circ y \subseteq B) \) if \( a \in B \), for all \( a \in H \) and so \( a \circ x \Theta a \circ y \). Hence \( \Theta \) is a congruence relation on \( H \). Now, let \( x, y \in H \) such that \( x \circ y \Theta \{0\} \) and \( y \circ x \Theta \{0\} \). Then there exist \( s \in x \circ y \) and \( t \in y \circ x \) such that \( s \Theta 0 \) and \( t \Theta 0 \), which imply that \( s, t \in A \). Hence, we have \( (x \circ y) \cap A \neq \emptyset \) and \( (y \circ x) \cap A \neq \emptyset \). Now, if \( x \in A \) since \( (y \circ x) \cap A \neq \emptyset \) and \( A \) is a strong hyper \( BCK \)-ideal of \( H \) then \( y \in A \) and so \( x \Theta y \).

Similarly, if \( y \in A \), then we get that \( x \in A \) and so \( x \Theta y \). Now, remind only the case \( x, y \in B - \{0\} \). But in this case by definition of \( \Theta \), we get that \( x \Theta y \). Hence, \( \Theta \) is a regular relation on \( H \). Therefore, \( \Theta \) is a regular congruence relation on \( H \) and so by Theorem 2.7, \( H/\Theta \) is a hyper \( BCK \)-algebra. Now, it is easy to prove that \( H/\Theta = \{(0)_\Theta = A, \ [b]_\Theta = B\} \), where \( b \in B - \{0\} \). Hence \( |H/\Theta| = 2 \). Now, since we have only to hyper \( BCK \)-algebra \( X = \{0, a\} \) of order 2 which are as follows:

\[
\begin{array}{c|cc|c|cc}
\circ_1 & 0 & a & 0 & a \\
\hline
0 & \{0\} & \{0\} & 0 & \{0\} \\
\{a\} & \{0\} & \{0\} & a & \{0, a\} \\
\end{array}
\]

Now, if \( b \circ b = \{0\} \) then \( [b]_\Theta \circ [b]_\Theta = \{(0)_\Theta \} \) and so \( H/\Theta \cong (X, \circ_1) \) and if \( b \circ b \neq \{0\} \) then \( [b]_\Theta \circ [b]_\Theta = \{(0)_\Theta, [b]_\Theta \} \) and so \( H/\Theta \cong (X, \circ_2) \). 

\textbf{Theorem 4.2.} Let \( H \) be decomposable with decomposition \( H = A \bigoplus B \) and let \( b \circ x = b \circ y \) for all \( b \in B \) and \( x, y \in A \). Then there exists a regular congruence relation \( \Gamma \) on \( H \) such that \( H/\Gamma \cong B \).

\textbf{Proof.} Define the relation \( \Gamma \) on \( H \) as follows:

\( x \Gamma y \iff x, y \in A \) or \( x = y \not\in A \).

It is easy to prove that \( \Gamma \) is an equivalence relation on \( H \). Let \( x, y \in H \) be such that \( x \Gamma y \). Then \( x, y \in A \) or \( x = y \not\in A \).

\textbf{Case 1.} Let \( x, y \in A \). Then by Theorem 2.2 (vii), \( x \circ a \leq x \ (y \circ a \leq y) \) and so by Theorem 2.4, we get that \( x \circ a \subseteq A \ (y \circ a \subseteq A) \), which implies that \( x \circ a \Gamma y \circ a \) for all \( a \in H \). Now, we prove that \( a \circ x \Gamma a \circ y \), for all \( a \in H \). If \( a \in A \), the by the similar way in the above proof, we can show that \( a \circ x \Gamma a \circ y \). If \( a \not\in A \), then \( a \in B \) and so by the hypothesis we have \( a \circ x = a \circ y \), which implies that \( a \circ x \Gamma a \circ y \).

\textbf{Case 2.} Let \( x = y \not\in A \). Then \( x \circ a = y \circ a \) and \( a \circ x = a \circ y \) for all \( a \in H \), which implies that \( x \circ a \Gamma y \circ a \) and \( a \circ x \Gamma a \circ y \) for all \( a \in H \).
Therefore, $\Gamma$ is a congruence relation on $H$. Now, let $x, y \in H$ such that $x \circ y \Gamma \{0\}$ and $y \circ x \Gamma \{0\}$. Then, there exist $s \in x \circ y$ and $t \in y \circ x$ such that $s \Gamma 0$ and $t \Gamma 0$ and so $s, t \in A$ and this implies that $(x \circ y) \cap A \neq \emptyset$ and $(y \circ x) \cap A \neq \emptyset$. Now, if $x \in A(y \in A)$, then since $A$ is a strong hyper $BCK$-ideal of $H$, then $y \in A(x \in A)$, which implies that $x \Gamma y$. If $x, y \notin A$, then $x, y \in B - \{0\}$. Hence, by Theorem 2.2 (vii), $x \circ y \ll x (y \circ x \ll y)$ and so by Theorem 2.4, $x \circ y \subseteq B (y \circ x \subseteq B)$. So, $t, s \in A \cap B = \{0\}$ and this implies that $s = t = 0$. Thus $x \ll y$ and $y \ll x$ and so $x = y$, which implies that $x \Gamma y$. Therefore, $\Gamma$ is a regular congruence relation on $H$. Now, we define the function $f : H \rightarrow H$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in A, \\ x & \text{if } x \in B. \end{cases}$$

It follows from $A \cap B = \{0\}$, that $f$ is well-defined. Now, let $x, y \in H$. We consider the following four cases:

**Case 1.** $x, y \in A$.

In this case, by Theorem 2.2 (vii), $x \circ y \ll x$ and so by Theorem 2.4, we get $x \circ y \subseteq A$. Hence,

$$f(x \circ y) = f(\bigcup_{t \in x \circ y} t) = \bigcup_{t \in x \circ y \subseteq A} \{f(t)\} = \{0\} = 0 \circ 0 = f(x) \circ f(y)$$

**Case 2.** $x, y \in B$.

Similar to the proof of Case 1, we get that $x \circ y \subseteq B$. Hence,

$$f(x \circ y) = f(\bigcup_{t \in x \circ y} t) = \bigcup_{t \in x \circ y \subseteq B} \{f(t)\} = \bigcup_{t \in x \circ y} \{t\} = x \circ y = f(x) \circ f(y)$$

**Case 3.** $x \in A$ and $y \in B - \{0\}$.

Similar to the proof of Case 1, we get that $x \circ y \subseteq A$ and so $f(x \circ y) = \{0\}$. On the other hand, since $f(x) = 0$, we have $f(x) \circ f(y) = 0 \circ y = \{0\}$. Hence

$$f(x \circ y) = f(x) \circ f(y)$$

**Case 4.** $x \in B - \{0\}$ and $y \in A$.

By hypothesis, we have $x \circ y = x \circ 0 = \{x\}$ and so

$$f(x \circ y) = \{f(x)\} = f(x) \circ 0 = f(x) \circ f(y)$$

Therefore, $f(x \circ y) = f(x) \circ f(y)$ for all $x, y \in H$ and so $f$ is a homomorphism. It is easy to check that $\text{Ker} f = A = [0]_\Gamma$ and $f(H) = B$. Hence by Theorem 2.8, we have $H/\Gamma \cong B$. □
Corollary 4.3. Let \( H \) be decomposable with decomposition \( H = A \bigoplus B \) and let \( b \circ x = b \circ y \) for all \( b \in B \) and \( x, y \in A \). Then \( |B| = 2 \).

Proof. Let regular congruence relations \( \Theta \) and \( \Gamma \) on \( H \) are as Theorems 4.1 and 4.2, respectively. Since \([0]_\Theta = A = [0]_\Gamma\), then by Theorem 2.6 that \( \Theta = \Gamma \) and so \( H/\Theta = H/\Gamma \). Now, by Theorem 4.1, \( H/\Theta \cong X \), where \( X \) is a hyper BCK-algebra of order 2 and by Theorem 4.2, \( H/\Gamma \cong B \). Hence,

\[
X \cong H/\Theta = H/\Gamma \cong B
\]

and so \( |B| = |X| = 2 \). \( \square \)

References


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