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Intuitionistic fuzzy approach to n-ary systems

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Abstract

We adopt the fundamental concepts of intuitionistic fuzzy subalgebras to n-ary groupoids, i.e., on algebras containing one n-ary operation. We describe some similarities and differences between the n-ary and binary case. In the case of n-ary quasigroups and groups we suggest the common method of investigations based on some methods used in the universal algebra.

1. Introduction

After the introduction of the concept of fuzzy sets by Zadeh, several researches were conducted on the generalizations of the notion of fuzzy set and application to many algebraic structures such as: groups, quasigroups, rings, semirings, BCK-algebras et cetera. All these applications are connected with binary operations.

But in many branches of mathematics (also in applications) one can find so-called *n-ary groupoids*, i.e., sets with one n-ary operation $f : G^n \to G$, where $n \geqslant 2$ is fixed. Such groupoid are called also *polyadic* or *n-ary systems* and are investigated by many authors, for example by Post [15] and Belousov [2]. Some special types of n-ary groupoids are used by Belousov in the theory of nets [2]. Mullen and Shcherbakov studied codes based on n-ary quasigroups [13]. Grzymała-Busse applied polyadic groupoids to the theory of automata [10]. Applications in modern physics are described by Kerner [11]. In such applications some role plays (intuitionistic) fuzzy subsets.

The main role in the theory of n-ary systems plays n-ary groups and quasigroups, which are a natural generalization of binary ($n = 2$) groups

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and quasigroups. It is clear that many classical results can be extended to the n-ary case. But for $n > 2$ we obtain the large set of theorems which are not true for $n = 2$. Moreover, the part of obtained results is true only for ternary ($n = 3$) groupoids.

2. Preliminaries

According to the general convention used in the theory of n-ary systems the sequence of elements $x_i, \ldots, x_j$ will be denoted by x_i^j (for $j < i$ it is empty symbol). This means that $f(x_1, x_2, \ldots, x_n)$ will be written as $f(x_1^n)$.

An n-ary groupoid (G, f) is called *unipotent* if it contains an element θ such that $f(x, x, \ldots, x) = \theta$ for all $x \in G$. Such groupoid is obviously an *n-ary semigroup*, i.e., for all $i, j \in \{1, 2, \ldots, n\}$ and $x_1^{2n-1} \in G$ it satisfies the *n-ary associativity*

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}),$$

which is a natural generalization on the classical associativity.

An *n-ary quasigroup* is defined as an n-ary groupoid (G, f) in which for all $1 \leqslant i \leqslant n$ and all $x_0^n \in G$ there exists a uniquely determined element $z_i \in G$ such that

$$f(x_1^{i-1}, z_i, x_{i+1}^n) = x_0\,. \tag{1}$$

An n-ary quasigroup (G, f) in which the operation f is associative in the above sense is called an n-ary group. For $n = 2$ we obtain an arbitrary group.

It is worthwhile to note that, under the assumption of the associativity of the operation f, it suffices only to postulate the existence of a solution of (1) at the place $i = 1$ and $i = n$ or at one place i other than 1 and n. Then one can prove uniqueness of the solution of (1) for all $1 \leqslant i \leqslant n$ (cf. [15], p.213[17]).

For any fixed n, the class of all n-ary groups is a variety. Very useful systems of identities defining this variety one can find in [9] and [7].

3. Intuitionistic fuzzy subgroupoids

Now generalize some classical results obtained for binary algebras such as BCC-algebras [8] and groups [16] to the case of n-ary groupoids.

Definition 3.1. A fuzzy set μ defined on G is called a *fuzzy subgroupoid* of an n-ary groupoid (G, f) if

$$\mu(f(x_1^n)) \geqslant \min\{\mu(x_1), \ldots, \mu(x_n)\}$$

for all $x_1^n \in G$.

Lemma 3.2. *If μ is a fuzzy subgroupoid of a unipotent groupoid (G, f), then $\mu(\theta) \geqslant \mu(x)$ for all $x \in G$ and $\theta = f(x, x, \ldots, x)$.*

Analogously as in a binary case we can prove

Theorem 3.3. *A fuzzy set μ of an n-ary groupoid (G, f) is a fuzzy subgroupoid if and only if for every $t \in [0, 1]$, the level*

$$L(\mu, t) = \{x \in G : \mu(x) \geqslant t\}$$

is either empty or a subgroupoid of (G, f).

This implies that (similarly as in a binary case) any subgroupoid of (G, f) can be realized as a level subgroupoid of some fuzzy subgroupoid μ defined on G.

The *complement* of μ, denoted by $\overline{\mu}$, is the fuzzy set in G given by $\overline{\mu}(x) = 1 - \mu(x)$ for all $x \in G$.

An *intuitionistic fuzzy set* (IFS for short) of a nonempty set X is defined by Atanassov (cf. [1]) in the following way.

Definition 3.4. An *intuitionistic fuzzy set* A of a nonempty set X is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\},$$

where $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leqslant \mu_A(x) + \gamma_A(x) \leqslant 1$ for all $x \in X$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$.

Definition 3.5. An IFS $A = (\mu_A, \gamma_A)$ of an n-ary groupoid (G, f) is an *intuitionistic fuzzy subgroupoid* (*IFS subgroupoid* for short) if

$$\mu_A(\, f(x_1^n)\,) \geqslant \min\{\mu_A(x_1), \ldots, \mu_A(x_n)\}$$
$$\gamma_A(\, f(x_1^n)\,) \leqslant \max\{\gamma_A(x_1), \ldots, \gamma_A(x_n)\}$$

hold for all $x_1^n \in G$.

It is not difficult to see that the following statements are true.

Proposition 3.6. *If $A = (\mu_A, \gamma_A)$ is an IFS intuitionistic fuzzy subgroupoid of (G, f), then so is $\Box A = (\mu_A, \overline{\mu_A})$ and $\Diamond A = (\overline{\gamma_A}, \gamma_A)$.*

Proposition 3.7. *If $A = (\mu_A, \gamma_A)$ is an IFS subgroupoid of a unipotent n-ary groupoid (G, f), then $\mu_A(\theta) \geqslant \mu_A(x)$ and $\gamma_A(\theta) \leqslant \gamma_A(x)$ for all $x \in G$ and $\theta = f(x, \dots, x)$.*

Proposition 3.8. *If $A = (\mu_A, \gamma_A)$ is an IFS subgroupoid of a unipotent n-ary groupoid (G, f), then*

$$G_\mu = \{x \in G : \mu_A(x) = \mu_A(\theta)\} \quad \text{and} \quad G_\gamma = \{x \in G : \gamma_A(x) = \gamma_A(\theta)\}$$

are subgroupoids of (G, f).

In some n-ary groupoids there exists an element e satisfying the identity

$$f(e, \dots, e, x, e, \dots, e) = x,$$

where x is at the place k. Such element (if it exists) is called a *k-identity*. There are n-ary groupoids containing two or three such elements. Moreover, there are groupoids containing only such elements. For example, in any n-ary group derived from a commutative group $(G, +)$, i.e., in an n-ary groupoid (G, f) with the operation $f(x_1^n) = x_1 + x_2 + \ldots + x_n$, all elements satisfying the identity $nx = x$ are k-identities (for every k). But the set of all k-identities is not an n-ary subgroupoid in general (cf. [6]).

Proposition 3.9. *If $A = (\mu_A, \gamma_A)$ is an IFS subgroupoid of an n-ary groupoid (G, f) with a k-identity e, then $\mu_A(e) \geqslant \mu_A(x)$ and $\gamma_A(e) \leqslant \gamma_A(x)$ for all $x \in G$ and*

$$G_\mu = \{x \in G : \mu_A(x) = \mu_A(e)\}$$
$$G_\gamma = \{x \in G : \gamma_A(x) = \gamma_A(e)\}$$

are subgroupoids of (G, f).

Obviously $\mu_A(e_1) = \mu_A(e_2)$ and $\gamma_A(e_1) = \gamma_A(e_2)$ for any k-identity e_1 and t-identity e_2. This means that in n-ary groupoids containing only k-identities all IFS subgroupoids are constant.

For any $\alpha \in [0,1]$ and fuzzy set μ of G, the set

$$U(\mu;\alpha) = \{x \in G \ : \mu(x) \geqslant \alpha\}$$
$$L(\mu;\alpha) = \{x \in G : \mu(x) \leqslant \alpha\}$$

is called an *upper* (respectively *lower*) *α-level cut* of μ.

Theorem 3.10. *If $A = (\mu_A, \gamma_A)$ is an IFS subgroupoid of an n-ary groupoid (G, f), then the sets $U(\mu_A;\alpha)$ and $L(\gamma_A;\alpha)$ are subgroupoids of (G, f) for every $\alpha \in \mathrm{Im}(\mu_A) \cap \mathrm{Im}(\gamma_A)$.*

Theorem 3.11. *If $A = (\mu_A, \gamma_A)$ is an IFS in an n-ary groupoid (G, f) such that the nonempty sets $U(\mu_A;\alpha)$ and $L(\gamma_A;\alpha)$ are subgroupoids of (G, f) for all $\alpha \in [0,1]$. Then $A = (\mu_A, \gamma_A)$ is an IFS subgroupoid of (G, f).*

The proof of the above two theorems is analogous to the proof of the corresponding theorems for binary groupoids (cf. [12]).

Also it is not difficult to verify that the following two statements are true.

Theorem 3.12. *Let B be a nonempty subset of an n-ary groupoid (G, f) and let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy set on G defined by*

$$\mu_A(x) = \begin{cases} s_0 & if\ x \in B, \\ s_1 & otherwise, \end{cases}$$

and

$$\gamma_A(x) = \begin{cases} t_0 & if\ x \in B, \\ t_1 & otherwise, \end{cases}$$

for all $x \in G$ and $s_i, t_i \in [0,1]$, where $s_0 > s_1$, $t_0 < t_1$ and $s_i + t_i \leqslant 1$ for $i = 0, 1$. Then $A = (\mu_A, \gamma_A)$ is an IFS subgroupoid of (G, f) if and only if B is an n-ary subgroupoid of (G, f). Moreover, $U(\mu_A; s_0) = B = L(\gamma_A; t_0)$.

Corollary 3.13. *Let χ_A be the characteristic function of an n-ary subgroupoid of an n-ary groupoid (G, f). Then the intuitionistic fuzzy set $A_\sim = (\chi_A, \overline{\chi_A})$ is an IFS subgroupoid of (G, f).*

A fuzzy set μ defined on G is said to be *normal* if there exists $x \in G$ such that $\mu(x) = 1$. A simple example of normal fuzzy sets are characteristic functions of subsets of G.

If an n-ary groupoid (G, f) is unipotent, then a fuzzy set μ defined on G is normal if and only if $\mu(\theta) = 1$, where $\theta = f(x, x, \ldots, x)$.

The set $\mathcal{N}(G,f)$ of all normal fuzzy subgroupoids on (G,f) is partially ordered by the relation

$$\mu \sqsubseteq \rho \Longleftrightarrow \mu(x) \leqslant \rho(x)$$

for all $x \in G$.

Moreover, similarly as in the binary case, for any fuzzy subgroupoid μ of (G,f) there exists $\rho \in \mathcal{N}(G,f)$ such that $\mu \sqsubseteq \rho$. If an n-ar groupoid (G,f) is unipotent, then the maximal element of $(\mathcal{N}(G,f),\sqsubseteq)$ is either constant or characteristic function of some subset of G.

4. Fuzzification of quasigroups

A groupoid $(G,\cdot)$ is called a *quasigroup* if each of the equations $ax = b$, $xa = b$ has a unique solution for any $a, b \in G$.

A fuzzification of quasigroups (binary and n-ary) is more complicated as a fuzzification of arbitrary groups (cf. for example [16]). The problem lies in the fact that a subset of a quasigroup $(G,\cdot)$ closed with respect to the quasigroup operation in general is not a quasigroup with respect to this operation.

A fuzzification of quasigroups (cf. [4, 12]) is based on the second equivalent definition of a quasigroup. Namely, (cf. [14]) a quasigroup $(G,\cdot)$ may be defined as an algebra $(G,\cdot,\backslash,/)$ with the three binary operations $\cdot$, $\backslash$, $/$ satisfying the identities

$$(xy)/y = x, \quad x\backslash(xy) = y,$$
$$(x/y)y = x, \quad x(x\backslash y) = y.$$

The quasigroup $(G,\cdot,\backslash,/)$ corresponds to quasigroup $(G,\cdot)$, where

$$x\backslash y = z \Longleftrightarrow xz = y \quad \text{and} \quad x/y = z \Longleftrightarrow zy = x.$$

A quasigroup is called *unipotent* if $xx = yy$ for all $x, y \in G$. These quasigroups are connected with Latin squares which have one fixed element in the diagonal. Such quasigroups may be defined as quasigroups $(G,\cdot)$ with the special element θ satisfying the identity $xx = \theta$. In this case also $x\backslash\theta = x$ and $\theta/x = x$ for all $x \in G$.

A nonempty subset S of a quasigroup $(G,\cdot,\backslash,/)$ is called a *subquasigroup* if it is closed under these three operations $\cdot$, $\backslash$, $/$, i.e., if $x * y \in S$ for all $* \in \{\cdot,\backslash,/\}$ and $x, y \in S$.

Thus a fuzzy set μ on a quasigroup $(G,\cdot)$ is a *fuzzy subquasigroup* if $\mu(x * y) \geqslant \min\{\mu(x), \mu(y)\}$ for all $* \in \{\cdot,\backslash,/\}$ and $x, y \in S$ (cf. [4]).

In the case of n-ary quasigroups the situation is more complicated. According to [2] in any n-ary quasigroup (G, f) for every $s = 1, 2, \ldots, n$ one can define the *s-th inverse n-ary operation* $f^{(s)}$ putting

$$f^{(s)}(x_1^n) = y \Longleftrightarrow f(x_1^{s-1}, y, x_{s+1}^n) = x_s\,.$$

Obviously, the operation $f^{(s)}$ is the s-inverse operation for the operation f if and only if

$$f^{(s)}(x_1^{s-1}, f(x_1^n), x_{s+1}^n) = x_s$$

for all $x_1^n \in G$ (cf. [2]). Therefore the class of all n-ary quasigroups may be treated as the variety of equationally definable algebras with $n+1$ fundamental operations $f, f^{(1)}, \ldots, f^{(n)}$.

A nonempty subset S of G is called a *subquasigroup* of (G, f) if it is an n-ary quasigroup with respect to f. This means that a nonempty subset S of an n-ary quasigroup (G, f) is an *n-ary subquasigroup* if and only if it is closed with respect to $n+1$ operations $f, f^{(1)}, \ldots, f^{(n)}$, i.e., if and only if $g(x_1^n) \in G$ for all $x_1^n \in G$ and all $g \in \mathcal{F} = \{f, f^{(1)}, f^{(2)}, \ldots, f^{(n)}\}$.

Basing on the Definition 3.1 we can define a fuzzy subquasigroup of an n-ary quasigroup in the following way.

Definition 4.1. A fuzzy set μ defined on G is called a *fuzzy subquasigroup* of an n-ary quasigroup (G, f) if

$$\mu(g(x_1^n)) \geqslant \min\{\mu(x_1),\, \ldots,\, \mu(x_n)\}$$

for all $g \in \mathcal{F}$ and $x_1^n \in G$.

For such defined fuzzy subquasigroups we can prove results analogous to the results from the previous part.

Definition 4.2. An IFS $A = (\mu_A, \gamma_A)$ of an n-ary quasigroup (G, f) is an *intuitionistic fuzzy subquasigroup* (*IFS subquasigroup* for short) if

$$\mu_A(\,g(x_1^n)\,) \geqslant \min\{\mu_A(x_1),\, \ldots,\, \mu_A(x_n)\}$$
$$\gamma_A(\,g(x_1^n)\,) \leqslant \max\{\gamma_A(x_1),\, \ldots,\, \gamma_A(x_n)\}$$

hold for all $g \in \mathcal{F}$ and $x_1^n \in G$.

It is not difficult to see that in an n-ary quasigroup an IFS subquasigroup is an IFS subgroupoid and the results of the previous part will be true for n-ary quasigroup if we replace "IFS subgroupoid" by "IFS subquasigroup".

Moreover, the following characterization of IFS subquasigroups is valid.

Lemma 4.3. *$A = (\mu_A, \gamma_A)$ is an IFS subquasigroup of an n-ary quasigroup (G, f) if and only if μ_A and $\overline{\gamma_A}$ are fuzzy subquasigroups of (G, f).*

Proof. Straightforward. □

Theorem 4.4. *If $A = (\mu_A, \gamma_A)$ is an IFS in an n-ary quasigroup (G, f) such that the nonempty sets $U(\mu_A; \alpha)$ and $L(\gamma_A; \alpha)$ are subquasigroups of (G, f) for all $\alpha \in [0, 1]$. Then $A = (\mu_A, \gamma_A)$ is an IFS subquasigroup of (G, f).*

Proof. Let $\alpha \in [0, 1]$. Assume that $U(\mu_A; \alpha) \neq \emptyset$ and $L(\gamma_A; \alpha) \neq \emptyset$ are subquasigroups of an n-ary quasigroup (G, f). We must show that $A = (\mu_A, \gamma_A)$ satisfies the Definition 4.2.

Let $g \in \mathcal{F}$. If the first condition of the Definition 4.2 is false, then there exist $x_1^n \in G$ such that

$$\mu_A(g(x_1^n)) < \min\{\mu_A(x_1), \ldots, \mu_A(x_n)\}.$$

Taking

$$\alpha_0 = \frac{1}{2}\big[\mu_A(g(x_1^n)) + \min\{\mu_A(x_1), \ldots, \mu_A(x_n)\}\big],$$

we have

$$\mu_A(g(x_1^n)) < \alpha_0 < \min\{\mu_A(x_1), \ldots, \mu_A(x_n)\}.$$

It follows that x_1^n are in $U(\mu_A; \alpha_0)$ but $g(x_1^n)$ are not in $U(\mu_A; \alpha_0)$, which is a contradiction.

Assume that the second condition of the Definition 4.2 does not hold. Then

$$\gamma_A(g(x_1^n)) > \max\{\gamma_A(x_1), \ldots, \gamma_A(x_n)\}$$

for some $x_1^n \in G$. Let

$$\beta_0 = \frac{1}{2}\big[\gamma_A(g(x_1^n)) + \max\{\gamma_A(x_1), \ldots, \gamma_A(x_n)\}\big].$$

Then

$$\gamma_A(g(x_1^n)) > \beta_0 > \max\{\gamma_A(x_1), \ldots, \gamma_A(x_n)\}$$

and so $x_1^n \in L(\gamma_A; \beta_0)$ but $g(x_1^n) \notin L(\gamma_A; \beta_0)$. This contradiction completes the proof. □

Proposition 4.5. *If $A = (\mu_A, \gamma_A)$ is an IFS subquasigroup of an n-ary quasigroup (G, f), then for all $i = 1, \ldots, n$, $g \in \mathcal{F}$ we have*

$$\min\{\mu_A(g(x_1^n)), \min\{\bigwedge_{i\neq s} \mu_A(x_i)\}\} = \min\{\mu_A(x_1), ..., \mu_A(x_n)\},$$
$$\max\{\gamma_A(g(x_1^n)), \min\{\bigwedge_{i\neq s} \gamma_A(x_i)\}\} = \max\{\gamma_A(x_1), \ldots, \gamma_A(x_n)\}.$$

Proof. Indeed, for $g = f$ we have

$$\begin{aligned}
&\min\{\mu_A(f(x_1^n)), \min\{\bigwedge_{i\neq s} \mu_A(x_i)\}\} \geqslant \\
&\qquad \min\{\min\{\mu_A(x_1), ..., \mu_A(x_n)\}, \min\{\bigwedge_{i\neq s} \mu_A(x_i)\}\} = \\
&\qquad \min\{\mu_A(x_1), ..., \mu_A(x_n)\} = \\
&\qquad \min\{\mu_A(f^{(s)}(x_1^{s-1}, f(x_1^n), x_{s+1}^n)), \min\{\bigwedge_{i\neq s} \mu_A(x_i)\}\} \geqslant \\
&\qquad \min\{\min\{\mu_A(f(x_1^n)), \min\{\bigwedge_{i\neq s} \mu_A(x_i)\}\}, \min\{\bigwedge_{i\neq s} \mu_A(x_i)\}\} = \\
&\qquad \min\{\mu_A(f(x_1^n)), \min\{\bigwedge_{i\neq s} \mu_A(x_i)\}\},
\end{aligned}$$

which completes the proof in this case. The rest is analogous. □

Theorem 4.6. *If $A = (\mu_A, \gamma_A)$ is an IFS subquasigroup of (G, f), then*

$$\mu_A(x) = \sup\{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\}$$

and

$$\gamma_A(x) = \inf\{\alpha \in [0,1] : x \in L(\gamma_A; \alpha)\}$$

for all $x \in G$.

Proof. Let $\delta = \sup\{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\}$ and let $\varepsilon > 0$ be given. Then $\delta - \varepsilon < \alpha$ for some $\alpha \in [0,1]$ such that $x \in U(\mu_A; \alpha)$. This means that $\delta - \varepsilon < \mu_A(x)$ so that $\delta \leqslant \mu_A(x)$ since ε is arbitrary.

We now show that $\mu_A(x) \leqslant \delta$. If $\mu_A(x) = \beta$, then $x \in U(\mu_A; \beta)$ and so

$$\beta \in \{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\}.$$

Hence

$$\mu_A(x) = \beta \leqslant \sup\{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\} = \delta.$$

Therefore

$$\mu_A(x) = \delta = \sup\{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\}.$$

Now let $\eta = \inf\{\alpha \in [0,1] : x \in L(\gamma_A;\alpha)\}$. Then

$$\inf\{\alpha \in [0,1] : x \in L(\gamma_A;\alpha)\} < \eta + \varepsilon$$

for any $\varepsilon > 0$, and so $\alpha < \eta + \varepsilon$ for some $\alpha \in [0,1]$ with $x \in L(\gamma_A;\alpha)$. Since $\gamma_A(x) \leqslant \alpha$ and ε is arbitrary, it follows that $\gamma_A(x) \leqslant \eta$.

To prove $\gamma_A(x) \geqslant \eta$, let $\gamma_A(x) = \zeta$. Then $x \in L(\gamma_A;\zeta)$ and thus $\zeta \in \{\alpha \in [0,1] : x \in L(\gamma_A;\alpha)\}$. Hence

$$\inf\{\alpha \in [0,1] : x \in L(\gamma_A;\alpha)\} \leqslant \zeta,$$

i.e., $\eta \leqslant \zeta = \gamma_A(x)$. Consequently

$$\gamma_A(x) = \eta = \inf\{\alpha \in [0,1] : x \in L(\gamma_A;\alpha)\},$$

which completes the proof. □

Theorem 4.7. *Let $\{\mathcal{H}_\alpha : \alpha \in \Lambda\}$, where Λ is a nonempty subset of $[0,1]$, be a family of subquasigroups of (G,f) such that*
(a) $G = \bigcup\limits_{\alpha\in\Lambda} H_\alpha$,
(b) $\alpha > \beta \Longleftrightarrow H_\alpha \subset H_\beta$ *for all* $\alpha,\beta \in \Lambda$.
Then an IFS $A = (\mu_A, \gamma_A)$ defined by

$$\mu_A(x) = \sup\{\alpha \in \Lambda : x \in H_\alpha\} \quad \textit{and} \quad \gamma_A(x) = \inf\{\alpha \in \Lambda : x \in H_\alpha\}$$

is an IFS subquasigroup of (G,f).

Proof. According to Theorem 4.4, it is sufficient to show that the nonempty sets $U(\mu_A;\alpha)$ and $L(\gamma_A;\beta)$ are subquasigroups of (G,f).

In order to prove that $U(\mu_A;\alpha) \neq \emptyset$ is a subquasigroup of $\mathcal{G}$, we consider the following two cases:

(i) $\alpha = \sup\{\delta \in \Lambda : \delta < \alpha\}$ and (ii) $\alpha \neq \sup\{\delta \in \Lambda : \delta < \alpha\}$.

Case (i) implies that

$$x \in U(\mu_A;\alpha) \Longleftrightarrow \big(x \in H_\delta \ \forall\, \delta < \alpha\big) \Longleftrightarrow x \in \bigcap_{\delta<\alpha} H_\delta,$$

so that $U(\mu_A;\alpha) = \bigcap\limits_{\delta<\alpha} H_\delta$ which is a subquasigroup of (G,f).

For the case (ii), we claim that

$$U(\mu_A;\alpha) = \bigcup_{\delta\geqslant\alpha} H_\delta.$$

If $x \in \bigcup\limits_{\delta \geqslant \alpha} H_\delta$ then $x \in H_\delta$ for some $\delta \geqslant \alpha$. It follows that $\mu_A(x) \geqslant \delta \geqslant \alpha$, so that $x \in U(\mu_A;\alpha)$. This shows that $\bigcup\limits_{\delta \geqslant \alpha} H_\delta \subseteq U(\mu_A;\alpha)$.

Now assume that $x \notin \bigcup\limits_{\delta \geqslant \alpha} H_\delta$. Then $x \notin H_\delta$ for all $\delta \geqslant \alpha$. Since $\alpha \neq \sup\{\delta \in \Lambda : \delta < \alpha\}$, there exists $\varepsilon > 0$ such that $(\alpha - \varepsilon, \alpha) \cap \Lambda = \emptyset$. Hence $x \notin H_\delta$ for all $\delta > \alpha - \varepsilon$, which means that if $x \in H_\delta$ then $\delta \leqslant \alpha - \varepsilon$. Thus $\mu_A(x) \leqslant \alpha - \varepsilon < \alpha$, and so $x \notin U(\mu_A;\alpha)$. Therefore $U(\mu_A;\alpha) \subseteq \bigcup\limits_{\delta \geqslant \alpha} H_\delta$, and thus $U(\mu_A;\alpha) = \bigcup\limits_{\delta \geqslant \alpha} H_\delta$, which is a subquasigroup of $\mathcal{G}$.

Now we prove that $L(\gamma_A;\beta)$ is a subquasigroup of (G,f). We consider the following two cases:

(iii) $\beta = \inf\{\eta \in \Lambda : \beta < \eta\}$ and (iv) $\beta \neq \inf\{\eta \in \Lambda : \beta < \eta\}$.

For the case (iii) we have

$$x \in L(\gamma_A;\beta) \Longleftrightarrow \big(x \in H_\eta \ \forall\, \eta > \beta\big) \Longleftrightarrow x \in \bigcap_{\eta > \beta} H_\eta$$

and hence $L(\gamma_A;\beta) = \bigcap\limits_{\eta > \beta} H_\eta$ which is a subquasigroup of (G,f).

For the case (iv), there exists $\varepsilon > 0$ such that $(\beta, \beta+\varepsilon) \cap \Lambda = \emptyset$. We will show that $L(\gamma_A;\beta) = \bigcup\limits_{\eta \leqslant \beta} H_\eta$. If $x \in \bigcup\limits_{\eta \leqslant \beta} H_\eta$ then $x \in H_\eta$ for some $\eta \leqslant \beta$. It follows that $\gamma_A(x) \leqslant \eta \leqslant \beta$ so that $x \in L(\gamma_A;\beta)$. Hence $\bigcup\limits_{\eta \leqslant \beta} H_\eta \subseteq L(\gamma_A;\beta)$.

Conversely, if $x \notin \bigcup\limits_{\eta \leqslant \beta} H_\eta$ then $x \notin H_\eta$ for all $\eta \leqslant \beta$, which implies that $x \notin H_\eta$ for all $\eta < \beta + \varepsilon$, i.e., if $x \in H_\eta$ then $\eta \geqslant \beta + \varepsilon$. Thus $\gamma_A(x) \geqslant \beta + \varepsilon > \beta$, i.e., $x \notin L(\gamma_A;\beta)$. Therefore $L(\gamma_A;\beta) \subseteq \bigcup\limits_{\eta \leqslant \beta} H_\eta$ and consequently $L(\gamma_A;\beta) = \bigcup\limits_{\eta \leqslant \beta} H_\eta$ which is a subquasigroup of (G,f). This completes the proof. $\square$

Let $IFS(G,f)$ be the family of all IFS subquasigroups of (G,f) and $\alpha \in [0,1]$ be a fixed real number. For any $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ from $IFS(G,f)$ we define two binary relations $\mathbf{U}^\alpha$ and $\mathbf{L}^\alpha$ on $IFS(G,f)$ as follows:

$$(A,B) \in \mathbf{U}^\alpha \Longleftrightarrow U(\mu_A;\alpha) = U(\mu_B;\alpha)$$

and

$$(A,B) \in \mathbf{L}^\alpha \Longleftrightarrow L(\gamma_A;\alpha) = L(\gamma_B;\alpha)\,.$$

These two relations $\mathbf{U}^{\alpha}$ and $\mathbf{L}^{\alpha}$ are equivalence relations, give rise to partitions of $IFS(G,f)$ into the equivalence classes of $\mathbf{U}^{\alpha}$ and $\mathbf{L}^{\alpha}$, denoted by $[A]_{\mathbf{U}^{\alpha}}$ and $[A]_{\mathbf{L}^{\alpha}}$ for any $A=(\mu_A,\gamma_A)\in IFS(G,f)$, respectively. And we will denote the quotient sets of $IFS(G,f)$ by $\mathbf{U}^{\alpha}$ and $\mathbf{L}^{\alpha}$ as $IFS(G,f)/\mathbf{U}^{\alpha}$ and $IFS(G,f)/\mathbf{L}^{\alpha}$, respectively.

If $\mathcal{S}(G,f)$ is the family of all subquasigroups of (G,f) and $\alpha\in[0,1]$, then we define two maps U_{α} and L_{α} from $IFS(G,f)$ to $\mathcal{S}(G,f)\cup\{\emptyset\}$ as follows:

$$U_{\alpha}(A)=U(\mu_A;\alpha) \quad\text{and}\quad L_{\alpha}(A)=L(\gamma_A;\alpha),$$

respectively, for each $A=(\mu_A,\gamma_A)\in IFS(G,f)$. Then the maps U_{α} and L_{α} are well-defined.

Theorem 4.8. *For any $\alpha\in(0,1)$, the maps U_{α} and L_{α} are surjective from $IFS(G,f)$ onto $\mathcal{S}(G,f)\cup\{\emptyset\}$.*

Proof. Let $\alpha\in(0,1)$. Note that $\mathbf{0}_{\sim}=(\mathbf{0},\mathbf{1})$ is in $IFS(G,f)$, where $\mathbf{0}$ and $\mathbf{1}$ are fuzzy sets in (G,f) defined by $\mathbf{0}(x)=0$ and $\mathbf{1}(x)=1$ for all $x\in G$. Obviously, $U_{\alpha}(\mathbf{0}_{\sim})=L_{\alpha}(\mathbf{0}_{\sim})=\emptyset$. If (H,f) is an n-ary subquasigroup of (G,f), then for the IFS subquasigroup $H=(\chi_H,\overline{\chi_H})$ we have $U_{\alpha}(H)=U(\chi_H;\alpha)=H$ and $L_{\alpha}(H)=L(\overline{\chi_H};\alpha)=H$. Hence U_{α} and L_{α} are surjective. □

Theorem 4.9. *The quotient sets $IFS(G,f)/\mathbf{U}^{\alpha}$ and $IFS(G,f)/\mathbf{L}^{\alpha}$ are equipotent to $\mathcal{S}(G,f)\cup\{\emptyset\}$ for any $\alpha\in(0,1)$.*

Proof. Let $\alpha\in(0,1)$ be fixed and let

$$\overline{U_{\alpha}}: IFS(G,f)/\mathbf{U}^{\alpha}\longrightarrow \mathcal{S}(G,f)\cup\{\emptyset\}$$

and

$$\overline{L_{\alpha}}: IFS(G,f)/\mathbf{L}^{\alpha}\longrightarrow \mathcal{S}(G,f)\cup\{\emptyset\}$$

be the maps defined by

$$\overline{U_{\alpha}}([A]_{\mathbf{U}^{\alpha}})=U_{\alpha}(A) \quad\textit{and}\quad \overline{L_{\alpha}}([A]_{\mathbf{L}^{\alpha}})=L_{\alpha}(A),$$

respectively, for each $A=(\mu_A,\gamma_A)\in IFS(G,f)$.

If $U(\mu_A;\alpha)=U(\mu_B;\alpha)$ and $L(\gamma_A;\alpha)=L(\gamma_B;\alpha)$ for $A=(\mu_A,\gamma_A)$ and $B=(\mu_B,\gamma_B)$ from $IFS(G,f)$, then $(A,B)\in\mathbf{U}^{\alpha}$ and $(A,B)\in\mathbf{L}^{\alpha}$, whence $[A]_{\mathbf{U}^{\alpha}}=[B]_{\mathbf{U}^{\alpha}}$ and $[A]_{\mathbf{L}^{\alpha}}=[B]_{\mathbf{L}^{\alpha}}$. Hence the maps $\overline{U_{\alpha}}$ and $\overline{L_{\alpha}}$ are injective.

To show that the maps $\overline{U_\alpha}$ and $\overline{L_\alpha}$ are surjective, let (H,f) be a subquasigroup of (G,f). Then for $H=(\chi_H,\overline{\chi_H})\in IFS(G,f)$ we have $\overline{U_\alpha}([H]_{\mathbf{U}^\alpha})=U(\chi_H;\alpha)=H$ and $\overline{L_\alpha}([H]_{\mathbf{L}^\alpha})=L(\overline{\chi_H};\alpha)=H$. Also $\mathbf{0}_\sim=(\mathbf{0},\mathbf{1})\in IFS(G,f)$. Moreover $\overline{U_\alpha}([\mathbf{0}_\sim]_{\mathbf{U}^\alpha})=U(\mathbf{0};\alpha)=\emptyset$ and $\overline{L_\alpha}([\mathbf{0}_\sim]_{\mathbf{L}^\alpha})=L(\mathbf{1};\alpha)=\emptyset$. Hence $\overline{U_\alpha}$ and $\overline{L_\alpha}$ are surjective. □

For any $\alpha\in[0,1]$, we define another relation $\mathbf{R}^\alpha$ on $IFS(G,f)$ as following:

$$(A,B)\in\mathbf{R}^\alpha \Longleftrightarrow U(\mu_A;\alpha)\cap L(\gamma_A;\alpha)=U(\mu_B;\alpha)\cap L(\gamma_B;\alpha)$$

for any $A=(\mu_A,\gamma_A)$ and $B=(\mu_B,\gamma_B)$ from $IFS(G,f)$. Then the relation $\mathbf{R}^\alpha$ is also an equivalence relation on $IFS(G,f)$.

Theorem 4.10. *For any $\alpha\in(0,1)$ and any IFS subquasigroup $A=(\mu_A,\gamma_A)$ of (G,f) the map $I_\alpha : IFS(G,f)\longrightarrow\mathcal{S}(G,f)\cup\{\emptyset\}$ defined by*

$$I_\alpha(A)=U_\alpha(A)\cap L_\alpha(A)$$

is suriective.

Proof. Indeed, if $\alpha\in(0,1)$ is fixed, then for $\mathbf{0}_\sim=(\mathbf{0},\mathbf{1})\in IFS(G,f)$ we have

$$I_\alpha(\mathbf{0}_\sim)=U_\alpha(\mathbf{0}_\sim)\cap L_\alpha(\mathbf{0}_\sim)=U(\mathbf{0};\alpha)\cap L(\mathbf{1};\alpha)=\emptyset\,,$$

and for any $H\in\mathcal{S}(G,f)$, there exists $H=(\chi_H,\overline{\chi_H})\in IFS(G,f)$ such that $I_\alpha(H)=U(\chi_H;\alpha)\cap L(\overline{\chi_H};\alpha)=H$. □

Theorem 4.11. *For any $\alpha\in(0,1)$, the quotient set $IFS(G,f)/\mathbf{R}^\alpha$ is equipotent to $\mathcal{S}(G,f)\cup\{\emptyset\}$.*

Proof. Let $\alpha\in(0,1)$ be fixed and let

$$\overline{I_\alpha} : IFS(G,f)/\mathbf{R}^\alpha\longrightarrow\mathcal{S}(G,f)\cup\{\emptyset\}$$

be a map defined by $\overline{I_\alpha}([A]_{\mathbf{R}^\alpha})=I_\alpha(A)$ for each $[A]_{\mathbf{R}^\alpha}\in IFS(G,f)/\mathbf{R}^\alpha$.

If $\overline{I_\alpha}([A]_{\mathbf{R}^\alpha})=\overline{I_\alpha}([B]_{\mathbf{R}^\alpha})$ holds for some $[A]_{\mathbf{R}^\alpha}$ and $[B]_{\mathbf{R}^\alpha}$ from $IFS(G,f)/\mathbf{R}^\alpha$, then

$$U(\mu_A;\alpha)\cap L(\gamma_A;\alpha)=U(\mu_B;\alpha)\cap L(\gamma_B;\alpha),$$

hence $(A,B)\in\mathbf{R}^\alpha$ and $[A]_{\mathbf{R}^\alpha}=[B]_{\mathbf{R}^\alpha}$. It follows that $\overline{I_\alpha}$ is injective.

For $\mathbf{0}_\sim=(\mathbf{0},\mathbf{1})\in IFS(G,f)$ we have $\overline{I_\alpha}(\mathbf{0}_\sim)=I_\alpha(\mathbf{0}_\sim)=\emptyset$. If $H\in\mathcal{S}(G,f)$, then for $H=(\chi_H,\overline{\chi_H})\in IFS(G,f)$, $\overline{I_\alpha}(H)=I_\alpha(H)=H$. Hence $\overline{I_\alpha}$ is a bijective map. □

5. Open problems

The above results show that IFS subsets in n-ary quasigroups can be investigated in a similar way as IFS subsets of universal algebras. The problem is with IFS subgroups of n-ary groups.

As it is well known (cf. [2] or [15]), a nonempty subset S of an n-ary group (G, f) is an n-ary subgroup of (G, f) if it is closed with respect to the operation f and $\overline{x} \in S$ for every $x \in S$, where $\overline{x}$ denotes the solution of the equation $f(x, \ldots, x, \overline{x}) = x$. Since (G, f) is an n-ary group for every x there exists only one $\overline{x}$ satisfying this equation. So, the map $\varphi(x) = \overline{x}$ is well-defined but it is not one-to-one in general. Moreover, there exists n-ary groups in which is one fixed element $a = \overline{x}$ such that $f(y, \ldots, y, a) = y$ is valid for all $y \in G$. An element $\overline{x}$ plays a similar (but not identical) role as an inverse element in classical groups.

Thus, by the analogy to the binary case, an *fuzzy n-ary subgroup* can be defined as an fuzzy subgroupoid μ such that $\mu(\overline{x}) \geqslant \mu(x)$ for all $x \in G$, or as an fuzzy subgroupoid μ such that $\mu(\overline{x}) = \mu(x)$ for all $x \in G$. For $n = 3$ these two concepts are equivalent Because in this case $\overline{\overline{x}} = x$ for every x.

Unfortunately, for $n > 3$ these two concepts are not equivalent. Indeed, as it is not difficult to see, in the unipotent 4-ary group derived from the additive group Z_4 the map μ defined by $\mu(0) = 1$ and $\mu(x) = 0.5$ for all $x \neq 0$ is an example of fuzzy subgroupoid in which $\mu(\overline{x}) \geqslant \mu(x)$ for all $x \in Z_4$. Thus μ is a fuzzy subgroup in the first sense, but not in the second because for $x = 2$ we have $\mu(\overline{x}) > \mu(x)$.

These two concepts are equivalent in n-ary groups in which $\varphi^k(x) = x$ for some fixed $k > 0$ and all x.

Problem 1. *Describe the conditions (for n-ary groups) under which these two concepts are equivalent.*

Problem 2. *Describe the similarities and differences between these two concepts of fuzzy n-ary subgroups (and IFS subgroups).*

E. L. Post proved in [15] that any n-ary group can be embedded into some binary group (called the *covering group*). On the other hand (cf. for example [9]), with any n-ary group (G, f) is connected the family of *binary retracts*, i.e., the family of binary groups $(G, \circ)$ with the operation $x \circ y = f(x, a_2, \ldots, a_{n-1}, y)$, where $a_2, \ldots, a_{n-2} \in G$ are fixed. All such retracts are isomorphic to retracts of the form $x \circ y = f(x, a, \ldots, a, y)$ and induce some properties of the corresponding n-ary group.

Problem 3. *Find the connection between fuzzy subgroups of a given n-ary group and fuzzy subgroups of its binary retracts and its covering group.*

Problem 4. *Describe IFS subgroups of n-ary groups and the connection with IFS subgroups of the corresponding binary groups.*

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