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Affine regular dodecahedron in GS–quasigroups

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Abstract

The concept of the affine regular dodecahedron is defined in any GS–quasigroup by means of twelve ARP relation which are valid for five out of twenty points. A number of statements about the connection of the corresponding vertices of the dodecahedron will be proved. Quaternary relations Par, GST, $DGST$ can be found in these statements. The theorem of the unique determination of the affine regular dodecahedron by means of its four vertices which satisfy certain conditions will be proved. The geometrical interpretation of all mentioned concepts and relations will be given in the GS–quasigroup $C(\frac{1}{2}(1+\sqrt{5}))$.

GS–quasigroups are defined in [2]. In [3], [1] and [4] different geometric structures in GS–quasigroups are defined and investigated. In this paper some new "geometric" concepts in the general GS–quasigroup will be defined.

A quasigroup $(Q,\cdot)$ is said to be GS*–quasigroup* if it is idempotent and if it satisfies the (mutually equivalent) identities

$$a(ab\cdot c)\cdot c=b, \qquad\qquad a\cdot(a\cdot bc)c=b.$$

If C is the set of all points in Euclidean plane and if groupoid $(C,\cdot)$ is defined so that $aa=a$ for any $a\in C$ and for any two different points a, $b\in C$ we define $ab=c$ if the point b divides the pair a, c in the ratio of golden section. In [2] it is proved that $(C,\cdot)$ is a GS–quasigroup. We shall denote that quasigroup by $C(\frac{1}{2}(1+\sqrt{5}))$ because we have $c=\frac{1}{2}(1+\sqrt{5})$ if $a=0$ and $b=1$. The figures in this quasigroup $C(\frac{1}{2}(1+\sqrt{5}))$ can be used for illustration of "geometrical" relations in any GS–quasigroup.

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From now on let $(Q,\cdot)$ be any GS–quasigroup. The elements of the set Q are said to be *points.* The points a, b, c, d are said to be the *vertices of a parallelogram* and we write $\mathrm{Par}(a, b, c, d)$ if the identity $a \cdot b(ca \cdot a) = d$ holds. In [2] numerous properties of the quaternary relation Par on the set Q are proved. Let us mention just the following lemma which we shall use afterwards.

Lemma 1. *From $Par(a, b, c, d)$ and $Par(c, d, e, f)$ follows $Par(a, b, f, e)$.*

In [3] the concept of the GS–trapezoid is defined and explored. The points a, b, c, d successively are said to be the vertices of the *golden section trapezoid* and it is denoted by $\mathrm{GST}(a, b, c, d)$ if the identity $a \cdot ab = d \cdot dc$ holds. In [3] different characterizations of that relation are investigated, we shall mention the following lemmas.

Lemma 2. *$GST(a_1, b_1, b_2, a_2)$,$GST(a_2, b_2, b_3, a_3)$,. . .,$GST(a_{n-1}, b_{n-1}, b_n, a_n)$ $\Rightarrow GST(a_n, b_n, b_1, a_1)$.*

Lemma 3. *$GST(a, b, c, d)$, $GST(a, b, c', d') \Rightarrow GST(d, c, c', d')$.*

Lemma 4. *Any two of the three statements $GST(a, b, c, d)$, $GST(a', b, c, d')$, $Par(a, a', d', d)$ imply the remaining statement.*

Lemma 5. *Any two of the three statements $GST(a, b, c, d)$, $GST(a, b', c', d)$, $Par(b, b', c', c)$ imply the remaining statement.*

In [3] it is proved that any two of the five statements

$$GST(a,b,c,d), GST(b,c,d,e), GST(c,d,e,a), GST(d,e,a,b), GST(e,a,b,c) \quad (1)$$

imply the remaining statements.

In [4] the concept of the affine regular pentagon is defined. The points a, b, c, d, e successively are said to be the vertices of the *affine regular pentagon* and it is denoted by $ARP(a, b, c, d, e)$ if any two (and then all five) of the five statements (1) are valid. In [4] the next properties of the affine regular pentagon are proved.

Lemma 6. *Affine regular pentagon is uniquely determined by any three of its vertices.*

Lemma 7. *If the statement $GST(a, b, c, d)$ is valid then there is one and only one point e such that the statement $ARP(a, b, c, d, e)$ is valid.*

The concept of the DGS–trapezoid is introduced in [1]. Points a, b, c, d are said to be the vertices of the *double golden section trapezoid* or shorter DGS–trapezoid and we write $\mathrm{DGST}(a, b, c, d)$ if the equality $ab = dc$ holds. In [1] it is proved the next connection between GS–trapezoids and DGS–trapezoids in GS–quasigroups.

Lemma 8. *Any two of the three statements $GST(a, e, f, d)$, $GST(e, b, c, f)$ and $DGST(a, b, c, d)$ imply the remaining statement.*

1. Affine regular dodecahedron in GS–quasigroups

Definition 1. We shall say that the points $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5, c_1, c_2, c_3, c_4, c_5, d_1, d_2, d_3, d_4, d_5$ are the vertices of an *affine regular dodecahedron* and we shall write

$$ARD(a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5, c_1, c_2, c_3, c_4, c_5, d_1, d_2, d_3, d_4, d_5)$$

if the following twelve statements are valid (Figure 1)

$$\begin{array}{ll} ARP(a_1, a_2, a_3, a_4, a_5), & ARP(d_1, d_2, d_3, d_4, d_5) \\ ARP(a_3, b_3, c_1, b_4, a_4), & ARP(d_3, c_3, b_1, c_4, d_4), \\ ARP(a_4, b_4, c_2, b_5, a_5), & ARP(d_4, c_4, b_2, c_5, d_5), \\ ARP(a_5, b_5, c_3, b_1, a_1), & ARP(d_5, c_5, b_3, c_1, d_1), \\ ARP(a_1, b_1, c_4, b_2, a_2), & ARP(d_1, c_1, b_4, c_2, d_2), \\ ARP(a_2, b_2, c_5, b_3, a_3), & ARP(d_2, c_2, b_5, c_3, d_3). \end{array}$$

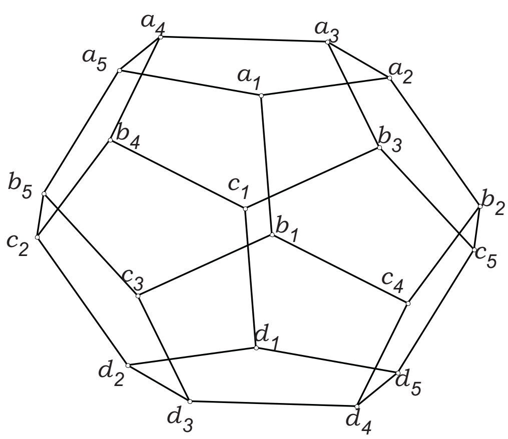

Figure 1.

Let ARD$(a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5, c_1, c_2, c_3, c_4, c_5, d_1, d_2, d_3, d_4, d_5)$ be valid further.

For each $i \in \{1, 2, 3, 4, 5\}$ the vertices a_i and d_i, respectively b_i and c_i are called the *opposite vertices.*

Theorem 1. 30 *statements of the form Par*(a_2, a_5, c_3, c_4) *are valid* (*Figure* 2).

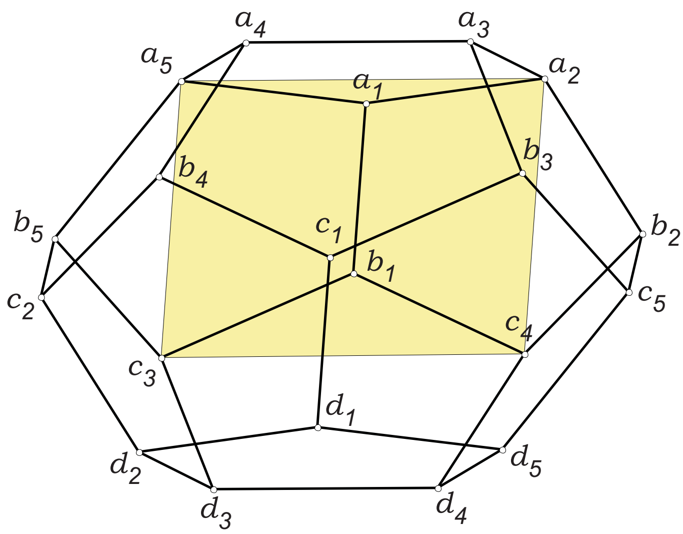

Figure 2.

Proof. The given statement follows from GST(a_2, a_1, b_1, c_4) and GST(a_5, a_1, b_1, c_3) according to Lemma 4. □

For opposite vertices of the ARD the following statement is valid.

Theorem 2. *If x and x', respectively y and y' are opposite vertices of the ARD then Par*(x, y, x', y') *is valid.*

Proof. It is sufficient to prove that, along with the standard symbols, for example the statements Par(a_1, b_1, d_1, c_1) (Figure 3) and Par(a_1, a_3, d_1, d_3) are valid (Figure 4). As GST(a_2, a_1, b_1, c_4) is valid and according to Theorem 1 Par(a_2, b_3, d_5, c_4) too, so by Lemma 4 GST(b_3, a_1, b_1, d_5) follows which together with GST(b_3, c_1, d_1, d_5), based on Lemma 5, implies Par(a_1, c_1, d_1, b_1).

According to Theorem 1 we have Par(a_1, a_3, c_5, c_4) and Par(c_5, c_4, d_3, d_1) from which by Lemma 1 Par(a_1, a_3, d_1, d_3) follows. □

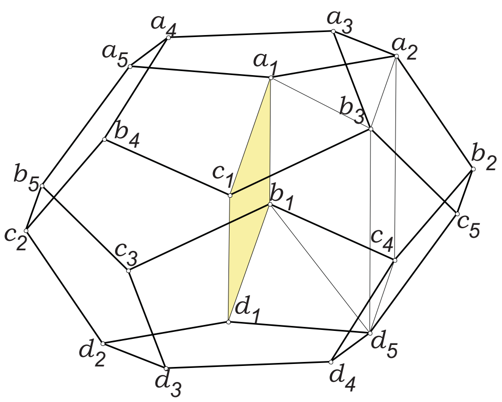

Figure 3.

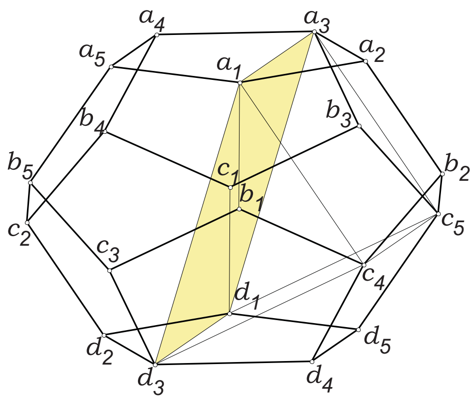

Figure 4.

Theorem 3. 60 *statements of the form* $GST(b_3, a_1, b_1, d_5)$ *are valid* (*see Figure* 5).

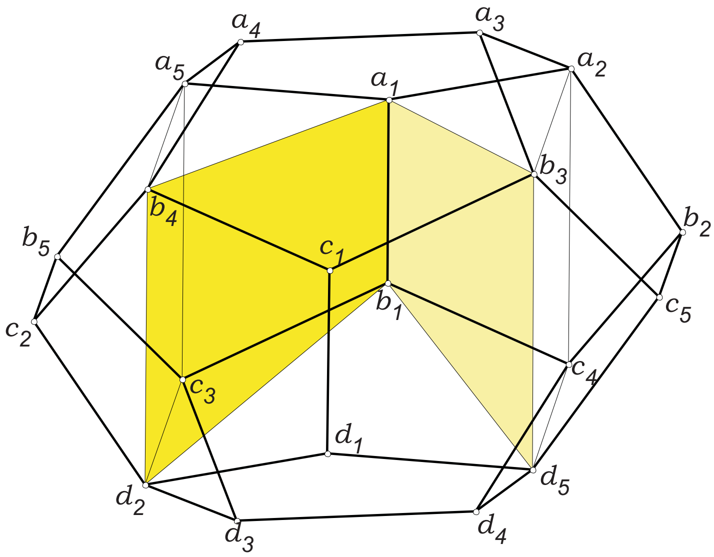

Figure 5.

Proof. The statement is proved in the proof of the previous theorem. □

Theorem 4. 60 *statements of the form* $GST(b_1, a_1, a_3, b_3)$ *are valid* (*see Figure* 6).

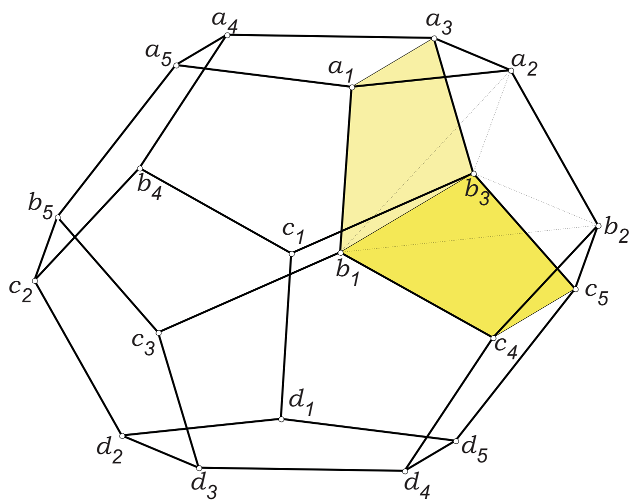

Figure 6.

Proof. The statements follow from $\mathrm{GST}(b_2, a_2, a_1, b_1)$ and $\mathrm{GST}(b_2, a_2, a_3, b_3)$ according to Lemma 3. □

Theorem 5. 60 *statements of the form* $DGST(a_3, a_1, b_1, d_4)$ *are valid* (*see Figure* 7).

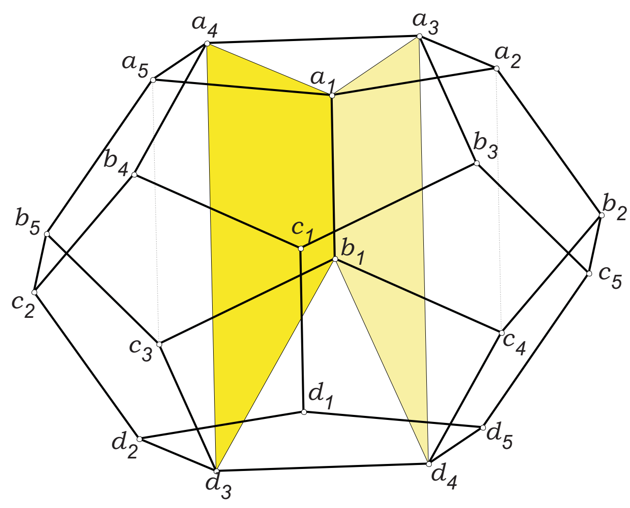

Figure 7.

Proof. According to Theorem 4 GST(a_3, a_2, c_4, d_4) is valid which together with GST$(a_2,\ a_1,\ b_1,\ c_4)$ according to Lemma 8 results in DGST(a_3, a_1, b_1, d_4). □

It is possible to prove that the affine regular dodecahedron is uniquely determined by its four independent vertices i.e. vertices which are not in the relation Par, GST or DGST. We shall prove only the following theorem.

Theorem 6. *For any points a_1, a_2, a_5, b_1 the points $a_3, a_4, b_2, b_3,\ b_4,\ b_5,\ c_1,\ c_2,\ c_3,\ c_4,\ c_5,\ d_1,\ d_2,\ d_3,\ d_4,\ d_5$ are uniquely determined so that $ARD(a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ b_1,\ b_2,\ b_3,\ b_4,\ b_5,\ c_1,\ c_2,\ c_3,\ c_4,\ c_5,\ d_1,\ d_2,\ d_3,\ d_4,\ d_5)$ is valid.*

Proof. Let $a_3, a_4, b_5, c_3, b_2, c_4$ be such points that ARP$(a_1, a_2, a_3, a_4, a_5)$, ARP$(a_5,\ b_5, c_3,\ b_1, a_1)$ and ARP$(a_1, b_1,\ c_4, b_2, a_2)$ and then let the points b_4, c_2, b_3, c_5 be such points that ARP$(a_4, b_4, c_2, b_5, a_5)$ and ARP$(a_2, b_2, c_5, b_3, a_3)$ are valid.

From GST(b_4, a_4, a_5, b_5), GST$(b_5,\ a_5,\ a_1, b_1)$, GST(b_1, a_1, a_2, b_2) and GST(b_2, a_2, a_3, b_3) according to Lemma 2 GST(b_3, a_3, a_4, b_4) follows. According to Lemma 7 there is the point c_1 such that ARP$(a_3, b_3, c_1, b_4, a_4)$ is valid, and then according to Lemma 6 there are such points $d_3,\ d_4$ that ARP$(d_3, c_3, b_1, c_4, d_4)$ is valid. From GST(c_2, b_5, a_5, a_4), GST(a_4, a_5, a_1, a_2), GST(a_2, a_1, b_1, c_4) and GST(c_4, b_1, c_3, d_3) according to Lemma 2 the statement GST(d_3, c_3, b_5, c_2) follows. In the same way from GST(c_5, b_2, a_2, a_3), GST(a_3, a_2, a_1, a_5), GST$(a_5,\ a_1, b_1, c_3)$ and GST$(c_3, b_1, c_4,\ d_4)$ the statement GST(d_4, c_4, b_2, c_5) follows. Therefore according to Lemma 7 there are such points $d_2,\ d_5$ that ARP$(d_2, c_2, b_5, c_3, d_3)$ and ARP$(d_4,\ c_4, b_2, c_5, d_5)$ are valid. From GST(d_5, d_4, c_4, b_2), GST(b_2, c_4, b_1, a_1), GST$(a_1, b_1,\ c_3, b_5)$ and GST(b_5, c_3, d_3, d_2) according to Lemma 2 GST$(d_2,\ d_3, d_4, d_5)$ follows so according to Lemma 7 there is such a point d_1 that ARP$(d_1, d_2, d_3, d_4, d_5)$ is valid.

With the repeated application of Lemma 2 from GST(d_2, c_2, b_5, c_3), GST(c_3, b_5, a_5, a_1), GST(a_1, a_5, a_4, a_3), GST(a_3, a_4, b_4, c_1) follows GST$(c_1,\ b_4, c_2, d_2)$, from GST(d_1, d_2, d_3, d_4), GST(d_4, d_3, c_3, b_1), GST(b_1, c_3, b_5, a_5) and GST(a_5, b_5, c_2, b_4) follows GST(b_4, c_2, d_2, d_1), from GST(d_5, c_5, b_2, c_4), GST(c_4, b_2, a_2, a_1), GST(a_1, a_2, a_3, a_4), GST$(a_4,\ a_3, b_3, c_1)$ follows GST$(c_1,\ b_3, c_5, d_5)$, and from GST(d_1, d_5, d_4, d_3), GST(d_3, d_4, c_4, b_1), GST$(b_1,\ c_4, b_2,\ a_2)$ and GST(a_2, b_2, c_5, b_3) follows GST(b_3, c_5, d_5, d_1) so that we have the final statements ARP$(d_1,\ c_1, b_4, c_2, d_2)$, ARP $(d_5, c_5, b_3, c_1, d_1)$. □

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