Automorphism group of Chein loops

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Abstract

In this paper we describe the automorphism group of Chein loops.

1. Introduction

First, we recall the definition of Chein loops (see [1]). Let $G$ be a group and the element $u$ be an indeterminate. Let $M(G, 2) = G \cup Gu$ be the disjoint union of $G$ and $Gu$ and extend the operation on $G$ to an operation $(.)$ on $M(G, 2)$ by the rules

$$
g.(hu) = (hg)u, \quad (gu).h = (gh^{-1})u, \quad (gu).(hu) = h^{-1}g \quad \forall g, h \in G.
$$

Then $M(G, 2)$ is a Moufang loop, which is a group if and only if $G$ is an abelian group. Moufang loops of this type are called Chein loops.

We mostly use standard notation. If $G$ is a group then we consider the natural action of $\text{Aut}G$ on $G$. This define a semidirect product $\text{Aut}G \times G$ which is called the Holomorph of $G$ and denoted by $\text{Hol}G$. For $g \in G$ and $\varphi \in \text{Aut}G$ we write $g^\varphi$ for the image of $g$ under $\varphi$.

The set

$$
\text{Stab}_{\text{Aut}G}(g) = \{ \varphi \in \text{Aut}G; g^\varphi = g \}
$$

is a subgroup of $\text{Aut}G$, called the stabilizer of $g$ in $\text{Aut}G$. For any $g, h \in G$ we write $[g, h] = g^{-1}h^{-1}gh$.
2. The automorphisms

Consider $\psi \in \text{Aut}(G)$, we extend $\psi$ to $a_\psi : M(G, 2) \rightarrow M(G, 2)$ as follows

$$a_\psi(gu^\lambda) = g\psi u^\lambda, \quad \lambda = 0, 1.$$

Now consider an element $t \in G$ and let

$$d_t(gu) = g(tu) = (tg)u, \quad d_t(g) = g, \quad \forall g \in G.$$

**Lemma 1.** The set $A = \{a_\psi \mid \psi \in \text{Aut}G\}$ is a subgroup of $\text{Aut}M(G, 2)$ isomorphic to $\text{Aut}(G)$ and the set $D = \{d_t \mid t \in G\}$ is a subgroup of $\text{Aut}M(G, 2)$ isomorphic to $G$. Moreover, $[A, D] = D, \quad A \cap D = 1$ and the semidirect splitting extension $AD$ is isomorphic to $\text{Hol}(G)$.

**Proof.** By definition of the operation $(.)$ in $M(G, 2)$ we have

$$a_\psi(g.(hu)) = a_\psi((hg)u) = (hg)^\psi u
\quad a_\psi(g).a_\psi(hu) = g^\psi.(h^\psi u) = (h^\psi g^\psi)u = (hg)^\psi u.$$  \hspace{1cm} (1)

$$d_t(g.(hu)) = d_t((hg)u) = (thg)u,
\quad d_t(g).d_t(hu)) = g.((th)u) = (thg)u.$$ \hspace{1cm} (2)

Analogously, we get

$$a_\psi(gu.h) = a_\psi((gh^{-1})u) = (gh^{-1})^\psi u
\quad a_\psi(gu).a_\psi(h) = g^\psi.u.h^\psi = (g^\psi h^{-\psi})u = (gh^{-1})^\psi u.$$  \hspace{1cm} (3)

$$d_t(gu.h) = d_t((gh^{-1})u) = (tgh^{-1})u
\quad d_t(gu).d_t(h) = (tg)u.h = (thg^{-1})u.$$ \hspace{1cm} (4)

Finally,

$$a_\psi(gu.hu) = a_\psi(h^{-1}g) = (h^{-1})^\psi
\quad a_\psi(gu).a_\psi(hu) = g^\psi.u.h^\psi = (h^{-\psi}g^\psi = (h^{-1})^\psi.$$  \hspace{1cm} (5)

$$d_t(gu.hu) = d_t(h^{-1}g) = h^{-1}g
\quad d_t(gu).d_t(hu) = (tg)u.(th)u = (th)^{-1}tg = h^{-1}g.$$ \hspace{1cm} (6)

Hence $a_\psi$ and $d_t$ are automorphisms. It is easy to see that

$$a_\psi \circ a_\psi = a_{\psi \psi} \quad \text{and} \quad d_t \circ d_h = d_{ht}, \quad a_\psi^{-1} = a_{\psi^{-1}}, \quad d_t^{-1} = d_{t^{-1}}$$
hence \( A = \{a_\psi \mid \psi \in AutG\} \) is a subgroup of \( AutM(G,2) \) isomorphic to \( Aut(G) \) and the set \( D = \{d_t \mid t \in G\} \) is a subgroup of \( AutM(G,2) \) isomorphic to \( G \).

We have \( a_{\psi^{-1}t}d_t a_\psi(h) = h, \ a_{\psi^{-1}t}d_t a_\psi(hu) = t^{\psi^{-1}}h = d_{t^{-1}}(hu). \) Hence \( a_{\psi^{-1}t}d_t a_\psi = d_{t^{-1}}. \) Therefore \( AD \cong Hol(G) \).

Let \( G \) be a generalized dihedral group, i.e. a group such that there exists an abelian subgroup \( G_0 < G \) of index 2 and \( G = G_0 \cup G_0v \), where \( v \not\in G_0, \ v^2 = 1; \ vgv = g^{-1}, \forall g \in G_0. \)

In the Chein loop \( M(G,2) \) we have an abelian subgroup

\[
K = \{1, u, v, w = uv = vu\}
\]

and \( M(G,2) = G_0K. \) For any \( \phi \in AutK = S_3 \) we can define an automorphism of \( M(G,2) \), which we denote by the same letter \( \phi \):

\[
\phi(gx) = gx^\phi \quad \forall x \in K, \ g \in G_0.
\]

We have the following result.

**Theorem 1.** Let \( G \) be a group. If \( G \) is not a dihedral group, then the automorphism group of the corresponding Chein loop \( M(G,2) \) is \( Hol(G) \). If \( G = G_0 \cup G_0v \) is a dihedral group and \( G_0 \) is not a group of period 2, then \( AutM(G,2) = Hol(G)S_3. \)

**Proof.** If \( G \) is not a dihedral group then \( G \) is a characteristic subgroup of \( M(G,2) \). Indeed, if for some \( \phi \in AutM(G,2) \) and \( x \in G \) we have \( y = x^\phi \not\in G \), then \( y^2 = 1 \) and \( yy = g^{-1}, \forall g \in G. \)

Let \( G_0 = \{h \in G \mid h^\phi \in G \} \), then \( G_0 \) is a subgroup of index 2 of \( G \) and \( G^\phi = G_0 \cup G_0y \) is a dihedral group, a contradiction, since \( G \) and \( G^\phi \) are isomorphic.

Let \( \phi \in AutM(G,2) \) and choose \( a_\psi \in A \) such that \( \psi(g) = \phi(g), \forall g \in G. \) Then \( \tau = \phi a_\psi^{-1} \in Stab_{AutM(G,2)}G. \) It is clear that \( Stab_{AutM(G,2)}G = D \) and \( AutM(G,2) = AD = Hol(G) \).

Let \( G = G_0 \cup G_0v \) be a dihedral group and \( N_0 = \{x \in M(G,2) \mid x^2 \neq 1\}, \) \( N = \{x \in M(G,2) \mid [x,N_0] = 1\} \). It is obvious that \( N^\phi = N, \) for any \( \phi \in AutM(M,2), \) and \( N = G_0 \) if \( G \) is not of period 2. As above we have \( AD \subset AutM(G,2). \) If \( \phi \in AutM(G,2), \) then \( u^\phi = ga, \ v^\phi = hb, \) where \( g, h \in G_0, \ a, b \in K. \) Note that \( a \neq b. \) Indeed, if \( a = b, \) then \( (uv)^\phi = gaha = gh^{-1} \in G_0, \) but \( uv \not\in G_0 \) and \( G_0 \) is a characteristic subgroup, a contradiction.

Then there exists \( \psi \in S_3 \) such that \( u^\psi = a, \ v^\psi = b \) and \( \phi \psi^{-1} \in AD. \) This means that \( AutM(G,2) = ADS_3 = Hol(G)S_3. \)
Remark 1. It is easy to see that $\text{Hol}(G) = \mathcal{W}(G_0)$ is a Mikheev group with triality with respect to the action of $S_3$ and the corresponding loop is $G_0$ (see [2]).

References


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