Quasigroups and Related Systems **13** (2005), 109 – 124

Quasigroups with an inverse property and generalized parastrophic identities

Anthony D. Keedwell and Victor A. Shcherbacov

Abstract

We study quasigroups (and loops) which have an inverse property. We show that each such quasigroup satisfies a generalized parastrophic identity and that, when investigating properties related to the nuclei, quasigroups which possess any type of inverse property can all be treated in the same way. By means of our approach using autostrophies, we obtain results concerning isomorphisms between or equality of these nuclei. Also, we find conditions for a groupoid which satisfies a generalized parastrophic identity to be a quasigroup. Some of these results generalize our results on (r, s, t)-inverse quasigroups.

1. Introduction

Almost all the well-known (classical) kinds of quasigroup and loop such as IP-, LIP-, WIP- and CI-loops and quasigroups are included among the classes of quasigroup which have some kind of inverse property. We recall that IP- and LIP-quasigroups and loops were studied, for example, in [7, 17, 18], WIP-loops in [4, 28], CI-loops in [1, 2], WIP-quasigroups in [30], CI-quasigroups in [14, 21], I-, PI-quasigroups and loops in [11, 12].

Most recently, (r, s, t)-inverse quasigroups were defined as a generalization of various kinds of "crossed-inverse" property quasigroup and loop: in particular, they generalize CI-, WIP- and m-inverse loops [20].

In this paper, we show that all the abovementioned kinds of inverse property can be classified into three types which we call λ-inverse, ρ-inverse and (α, β, γ)-inverse. (The last of these three types was introduced in [23, 24].)

2000 Mathematics Subject Classification:Primary: 20N05
Keywords:Quasigroup, parastrophy, autostrophy, inverse quasigroup, nuclei.

A main result is that which of the three nuclei of a loop coincide depends precisely on which of these types the loop belongs to, thus incorporating earlier results of Artzy, Bruck, Belousov, Karklin and Osborn among others.

For a quasigroup, the concept of a nucleus is not well-defined but we consider instead the so-called A-nuclei (autotopy nuclei) and obtain analogous results for these. In the case that the quasigroup is a loop, the latter results reduce to the previous ones.

For example, all three nuclei of any (α, β, γ)-inverse loop in which α or β or γ is the identity map coincide and all three autotopy nuclei of any (α, β, γ)-inverse quasigroup are isomorphic [24].

Our method of proof in [24] and in the present paper using an approach via autostrophisms has its origins in [3, 11] and in the book [29].

Note 1. Various other generalizations of the concept of nucleus applicable to quasigroups have been introduced and studied in the past: for example, in [13], [15], [16] and [25].

2. Definitions

We shall use basic terms and concepts from the books [7], [19] and [29]. However, for completeness, we give the definitions of a groupoid, quasigroup and loop.

Definitions 2.1. A non-empty set Q on which a binary operation $(\circ)$ is defined is called a *groupoid* if, for all $a, b \in Q$, $a \circ b \in Q$.

A groupoid $(Q, \circ)$ is called a *right quasigroup* if, for all $a, b \in Q$, there exists a unique solution $x \in Q$ to the equation $x \circ a = b$, i.e. in this case any right translation of the groupoid $(Q, \circ)$ is a permutation of the set Q.

A groupoid $(Q, \circ)$ is called a *left quasigroup* if, for all $a, b \in Q$, there exists unique solution $y \in Q$ to the equation $a \circ y = b$, i.e. in this case any left translation of the groupoid $(Q, \circ)$ is a permutation of the set Q.

A left and right quasigroup $(Q, \circ)$ is called a *quasigroup*. If also there is an identity element $e \in Q$ such that $e \circ a = a = a \circ e$ for every $a \in Q$, the quasigroup is called a *loop*.

A groupoid $(Q, \circ)$ is isotopic to the groupoid $(Q, \cdot)$ if there exist permutations θ, ϕ, ψ of the set Q such that $(x \circ y)\psi = x\theta \cdot y\phi$ for all $x, y \in Q$.

Any isotope of a groupoid $(Q, \circ)$ is a groupoid [29], any isotope of a left (right) quasigroup $(Q, \circ)$ is a left (right) quasigroup. In fact, let $(Q, \circ)$ be a (θ, ϕ, ψ) isotope of a left quasigroup $(Q, \cdot)$. Then $x \circ y = (x\theta \cdot y\phi)\psi^{-1}$ for all $x, y \in Q$. Hence, $yL_x^{(\circ)} = y\phi L_{x\theta}^{(\cdot)}\psi^{-1}$ and the map $L_x^{(\circ)}$ is a permutation since the map $L_{x\theta}^{(\cdot)}$ is a permutation.

Now we give the definitions of some well-known classes of quasigroups and loops which have an inverse property.

Definitions 2.2.

1) A quasigroup $(Q, \circ)$ has the *left inverse-property* if there exists a permutation $x \to x\lambda$ of the set Q such that

$$x\lambda \circ (x \circ y) = y \tag{2.1}$$

for all $x, y \in Q$ [7];

2) a quasigroup $(Q, \circ)$ has the *right inverse-property* if there exists a permutation $x \to x\rho$ of the set Q such that

$$(x \circ y) \circ y\rho = x \tag{2.2}$$

for all $x, y \in Q$ [7];

3) a quasigroup $(Q, \circ)$ has the *inverse property* (is an *IP-quasigroup* [7, 29]) if both (2.1) and (2.2) hold. If $(Q, \circ)$ is a loop, then $\lambda = \rho$ and we write $x\lambda = x\rho = x^{-1}$. However, in the case when $(Q, \circ)$ is a quasigroup, $x\lambda \neq x\rho$ is possible. See Note 2 below.

4) A quasigroup $(Q, \circ)$ has the *weak-inverse-property* if there exists a permutation $x \to xJ$ of the set Q such that

$$x \circ (y \circ x)J = yJ \tag{2.3}$$

for all $x, y \in Q$ [4, 23, 30];

5) a quasigroup $(Q, \circ)$ has the *crossed-inverse-property* if there exists a permutation $x \to xJ$ of the set Q such that

$$(x \circ y) \circ xJ = y \tag{2.4}$$

for all $x, y \in Q$ [2, 23];

6) a quasigroup $(Q, \circ)$ has the *m-inverse-property* if there exists a permutation $x \to xJ$ of the set Q such that

$$(x \circ y)J^m \circ xJ^{m+1} = yJ^m \tag{2.5}$$

for all $x, y \in Q$ [20, 22];

7) a quasigroup $(Q, \circ)$ has the *(r, s, t)-inverse-property* if there exists a permutation $x \to xJ$ of the set Q such that

$$(x \circ y)J^r \circ xJ^s = yJ^t \tag{2.6}$$

for all $x, y \in Q$ [23, 24].

We can generalize the last definition (which itself generalizes (2.4) and (2.5)) by calling a quasigroup an (α, β, γ)-inverse quasigroup (as was done in [24]) if there exist permutations α, β, γ of the set Q such that

$$(x \circ y)\alpha \circ x\beta = y\gamma \tag{2.7}$$

for all $x, y \in Q$.

In a similar way, we can generalize (2.1) and (2.2) by the following definitions:

8) a quasigroup $(Q, \circ)$ has the λ-*inverse-property* if there exist permutations $\lambda_1, \lambda_2, \lambda_3$ of the set Q such that

$$x\lambda_1 \circ (x \circ y)\lambda_2 = y\lambda_3 \tag{2.8}$$

for all $x, y \in Q$ [11];

9) a quasigroup $(Q, \circ)$ has the ρ-*inverse-property* if there exist permutations ρ_1, ρ_2, ρ_3 of the set Q such that

$$(x \circ y)\rho_1 \circ y\rho_2 = x\rho_3 \tag{2.9}$$

for all $x, y \in Q$ [11].

We shall show that these last three classes of quasigroup (and consequently also the earlier ones) can all be treated in a similar way.

Note 2. Belousov [10] has given an example of a quasigroup which satisfies both (2.1) and (2.2) but with $\lambda \neq \rho$, as follows:

Let$(G, +)$ be a finite abelian group. Define a quasigroup $(Q, \cdot)$ on the ordered pairs of G by $(a_i, b_i) \cdot (a_j, b_j) = (a_i + a_j, b_j - b_i)$. Then $(Q, \cdot)$ is both an LIP-quasigroup with $(a, b)\lambda = (-a, -b)$ and a RIP-quasigroup with $(a, b)\rho = (-a, b)$.

We have

$$\begin{aligned}(a_i, b_i)\lambda \cdot [(a_i, b_i) \cdot (a_j, b_j)] &= (-a_i, -b_i) \cdot (a_i + a_j, b_j - b_i) \\ &= [-a_i + (a_i + a_j), (b_j - b_i) - (-b_i)] = (a_j, b_j)\end{aligned}$$

and

$$\begin{aligned}[(a_j, b_j) \cdot (a_i, b_i)] \cdot (a_i, b_i)\rho &= (a_j + a_i, b_i - b_j) \cdot (-a_i, b_i) \\ &= [(a_j + a_i) - a_i, b_i - (b_i - b_j)] = (a_j, b_j).\end{aligned}$$

Note 3. It might appear that, as well as quasigroups which satisfy (2.7), we should also consider $(\alpha^*, \beta^*, \gamma^*)$-inverse quasigroups which satisfy the identity $x\alpha^* \circ (y \circ x)\beta^* = y\gamma^*$. However, it turns out that every (α, β, γ)-inverse quasigroup is also an $(\alpha^*, \beta^*, \gamma^*)$-inverse quasigroup with $\alpha^* = \beta^{-1}$, $\beta^* = \gamma^{-1}$ and $\gamma^* = \alpha^{-1}$.

Proof. We see that $(x \circ y)\alpha \circ x\beta = y\gamma$ implies that $yL_x\alpha R_z = y\gamma$, where $z = x\beta$. Equivalently, $w\gamma^{-1}L_x\alpha R_z = w$, where $w = y\gamma$. So, $(\gamma^{-1}L_x)(\alpha R_z) = I$, where I denotes the identity mapping and L_x, R_z, α, γ are permutations of Q (and so lie in the symmetric group S_Q).

It follows that $(\alpha R_z)(\gamma^{-1}L_x) = I$, so $y\alpha R_z\gamma^{-1}L_x = y$ or $x \circ (y\alpha \circ x\beta)\gamma^{-1} = y$. Thus, $u\beta^{-1} \circ (v \circ u)\gamma^{-1} = v\alpha^{-1}$, where $u = x\beta$ and $v = y\alpha$. So, $(x \circ y)\alpha \circ x\beta = y\gamma$ for all $x, y \in Q \Longrightarrow x\beta^{-1} \circ (y \circ x)\gamma^{-1} = y\alpha^{-1}$ for all $x, y \in Q$, and conversely by reversing the steps.

As in [24], we define the left, right and middle autotopy nuclei of a quasigroup as the groups of autotopisms of the forms $(\alpha, \varepsilon, \gamma)$, $(\alpha, \beta, \varepsilon)$ and $(\varepsilon, \beta, \gamma)$, where ε is the identity mapping. As before, we shall denote these three groups of mappings by N_l^A, N_m^A and N_r^A respectively and call them A-nuclei.

Also, with any quasigroup $(Q, \otimes)$ it is possible to associate five further quasigroups called *parastrophes* of $(Q, \otimes)$. If we denote the quasigroup operation by the letter A, then with the quasigroup operation A we can associate the following quasigroup operations (see [7, 8, 9, 19, 26, 29]: $A(x_1, x_2) = x_3 \Leftrightarrow A^{(12)}(x_2, x_1) = x_3 \Leftrightarrow A^{(13)}(x_3, x_2) = x_1 \Leftrightarrow A^{(23)}(x_1, x_3) = x_2 \Leftrightarrow A^{(123)}(x_2, x_3) = x_1 \Leftrightarrow A^{(132)}(x_3, x_1) = x_2$.

In other words $A^\sigma(x_{1\sigma}, x_{2\sigma}) = x_{3\sigma} \Leftrightarrow A(x_1, x_2) = x_3$ where $\sigma \in S_3$. For example, $A^{(132)}(x_3, x_1) = x_2 \Leftrightarrow A(x_1, x_2) = x_3$: that is,

$$A^{(132)}(x_{1(132)}, x_{2(132)}) = x_{3(132)} \Leftrightarrow A(x_1, x_2) = x_3.$$

We shall also find it convenient to employ the alternative notation

$$x_1 \otimes x_2 = x_3 \Leftrightarrow x_{1\sigma} \otimes^\sigma x_{2\sigma} = x_{3\sigma},$$

where $\sigma \in S_3$, for parastrophic operations.

A collection of permutations $[\sigma, (\alpha_1, \alpha_2, \alpha_3)] = [\sigma, \alpha]$, where $\sigma \in S_3$ and $\alpha_1, \alpha_2, \alpha_3$ are permutations of the set Q, is called an *autostrophism* of the quasigroup (Q, A) if and only if $A^\sigma(x_1\alpha_1, x_2\alpha_2) = x_3\alpha_3$ [9] for all $(x_1, x_2, x_3) \in A$ or, in our alternative notation, if and only if $x_1 \otimes x_2 = x_3 \Leftrightarrow x_1\alpha_1 \otimes^\sigma x_2\alpha_2 = x_3\alpha_3$.

Lemma 2.1. *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ be isotopisms from (Q, A) to (Q, B). Then $\alpha^\sigma = (\alpha_{1\sigma}, \alpha_{2\sigma}, \alpha_{3\sigma})$ and $\beta^\sigma = (\beta_{1\sigma}, \beta_{2\sigma}, \beta_{3\sigma})$ are isotopisms from (Q, A^σ) to (Q, B^σ) and $(\alpha\beta)^\sigma = \alpha^\sigma\beta^\sigma$, where $\sigma \in S_3$.*

Proof. The first statement is obvious from the definitions. For the second statement, we have

$$(\alpha\beta)^\sigma = (\alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_3)^\sigma = (\alpha_{1\sigma}\beta_{1\sigma}, \alpha_{2\sigma}\beta_{2\sigma}, \alpha_{3\sigma}\beta_{3\sigma}) = \alpha^\sigma\beta^\sigma. \qquad \square$$

Lemma 2.2. *The set of all autostrophisms $Aus(Q, \otimes)$ of the quasigroup $(Q, \otimes)$ form a group with respect to the operation $[\sigma, \alpha] \cdot [\tau, \beta] = [\sigma\tau, \beta\alpha^{\tau}]$. The inverse of the autostrophism $[\sigma, \alpha]$ in this group is the autostrophism $[\sigma^{-1}, (\alpha^{-1})^{\sigma^{-1}}]$.*

Proof. See [24] for the proof. □

Theorem 2.3. *A quasigroup $(Q, \otimes)$ is an (α, β, γ)-inverse quasigroup if and only if it has a $[(1\ 2\ 3), (\beta, \gamma, \alpha)]$ autostrophism. It has the λ-inverse property if and only if it has a $[(2\ 3), (\lambda_1, \lambda_3, \lambda_2)]$ autostrophism. It has the ρ-inverse property if and only if it has a $[(1\ 3), (\rho_3, \rho_2, \rho_1)]$ autostrophism.*

Proof. The first fact follows almost immediately from the definition: for a quasigroup $(Q, \otimes)$ is an (α, β, γ)-quasigroup if and only if there exist permutations α, β, γ of Q such that $(x \otimes y)\alpha \otimes x\beta = y\gamma$ for all $x, y \in Q$. But then $x\beta \otimes^{(1\ 2\ 3)} y\gamma = (x \otimes y)\alpha$ from which the first result follows at once.

$(Q, \otimes)$ has the λ-inverse-property if there exist permutations $\lambda_1, \lambda_2, \lambda_3$ of the set Q such that $x\lambda_1 \otimes (x \otimes y)\lambda_2 = y\lambda_3$ for all $x, y \in Q$. But then $x\lambda_1 \otimes^{(2\ 3))} y\lambda_3 = (x \otimes y)\lambda_2$ from which the second result follows.

$(Q, \otimes)$ has the ρ-inverse-property if there exist permutations ρ_1, ρ_2, ρ_3 of the set Q such that $(x \otimes y)\rho_1 \otimes y\rho_2 = x\rho_3$ for all $x, y \in Q$. Then $x\rho_3 \otimes^{(1\ 3)} y\rho_2 = (x \otimes y)\rho_1$ and so the third result follows. □

Corollary 2.4. *A quasigroup which has the λ-inverse-property has a $(\lambda_1\lambda_1, \lambda_3\lambda_2, \lambda_2\lambda_3)$ autotopism. One which has the ρ-inverse-property has a $(\rho_3\rho_1, \rho_2\rho_2, \rho_1\rho_3)]$ autotopism.*

Proof. By Lemma 2.2,
$[(2\ 3), (\lambda_1, \lambda_3, \lambda_2)].[(2\ 3), (\lambda_1, \lambda_3, \lambda_2)] = [\varepsilon, (\lambda_1, \lambda_3, \lambda_2)(\lambda_1, \lambda_3, \lambda_2)^{(2\ 3)}]$ and so $(\lambda_1\lambda_1, \lambda_3\lambda_2, \lambda_2\lambda_3)$ is an autotopism. Similarly,
$[(1\ 3), (\rho_3, \rho_2, \rho_1)].[(1\ 3), (\rho_3, \rho_2, \rho_1)] = [\varepsilon, (\rho_3, \rho_2, \rho_1)(\rho_3, \rho_2, \rho_1)^{(1\ 3)}]$
$= [\varepsilon, (\rho_3\rho_1, \rho_2\rho_2, \rho_1\rho_3)]$. □

3. Nuclei of (α, β, γ)-inverse quasigroups

Let $(Q, \otimes)$ be an (α, β, γ)-inverse quasigroup. By Theorem 2.3, $[(1\ 2\ 3), (\beta, \gamma, \alpha)]$ is an autostrophism of $(Q, \otimes)$.

Let $H = [(1\ 2\ 3), (\beta, \gamma, \alpha)] = [(1\ 2\ 3), \check{J}]$ say.

Then

$$H^{-1} = [(1\ 3\ 2), (\check{J}^{-1})^{(1\ 3\ 2)}] = [(1\ 3\ 2), (\beta^{-1}, \gamma^{-1}, \alpha^{-1})^{(1\ 3\ 2)}]$$
$$= [(1\ 3\ 2), (\alpha^{-1}, \beta^{-1}, \gamma^{-1})].$$

If $\theta = (\theta_1, \theta_2, \theta_3)$ is an autotopism of $(Q, \otimes)$, this is equivalent to saying that $[\varepsilon, \theta]$ is an autostrophism, so

$$\begin{aligned} H^{-1}[\varepsilon, \theta]H &= [(1\ 3\ 2), (\check{J}^{-1})^{(1\ 3\ 2)}] \cdot [\varepsilon, \theta] \cdot [(1\ 2\ 3), \check{J}] \\ &= [(1\ 3\ 2), \theta(\check{J}^{-1})^{(1\ 3\ 2)}] \cdot [(1\ 2\ 3), \check{J}] \\ &= [(1\ 3\ 2)(1\ 2\ 3), \check{J}[\theta(\check{J}^{-1})^{(1\ 3\ 2)}]^{(1\ 2\ 3)}] \\ &= [\varepsilon, \check{J}\theta^{(1\ 2\ 3)}\check{J}^{-1}] = [\varepsilon, (\beta\theta_2\beta^{-1}, \gamma\theta_3\gamma^{-1}, \alpha\theta_1\alpha^{-1})], \end{aligned}$$

which is another autotopism. Also,

$$\begin{aligned} H[\varepsilon, \theta]H^{-1} &= [(1\ 2\ 3), \check{J}] \cdot [\varepsilon, \theta] \cdot [(1\ 3\ 2), (\check{J}^{-1})^{(1\ 3\ 2)}] \\ &= [(1\ 2\ 3), \theta\check{J}] \cdot [(1\ 3\ 2), (\check{J}^{-1})^{(1\ 3\ 2)}] \\ &= [\varepsilon, (\check{J}^{-1})^{(1\ 3\ 2)}(\theta\check{J})^{(1\ 3\ 2)}] = [\varepsilon, (\check{J}^{-1}\theta\check{J})^{(1\ 3\ 2)}] \\ &= [\varepsilon, (\alpha^{-1}\theta_3\alpha, \beta^{-1}\theta_1\beta, \gamma^{-1}\theta_2\gamma)], \end{aligned}$$

which is again an autotopism.

Let $(Q, \otimes)$ be an (α, β, γ)-inverse quasigroup and let $\nu_l = (\theta_1, \varepsilon, \theta_3)$ be in N_l^A. Then, from above,

$$\begin{aligned} H^{-1}\nu_l H &= (\varepsilon, \gamma\theta_3\gamma^{-1}, \alpha\theta_1\alpha^{-1}) \in N_r^A, \\ H\nu_l H^{-1} &= (\alpha^{-1}\theta_3\alpha, \beta^{-1}\theta_1\beta, \varepsilon) \in N_m^A. \end{aligned}$$

Let $\nu_r = (\varepsilon, \theta_2, \theta_3) \in N_r^A$. Then,

$$\begin{aligned} H^{-1}\nu_r H &= (\beta\theta_2\beta^{-1}, \gamma\theta_3\gamma^{-1}, \varepsilon) \in N_m^A, \\ H\nu_r H^{-1} &= (\alpha^{-1}\theta_3\alpha, \varepsilon, \gamma^{-1}\theta_2\gamma) \in N_l^A. \end{aligned}$$

Let $\nu_m = (\theta_1, \theta_2, \varepsilon) \in N_m^A$. Then

$$\begin{aligned} H^{-1}\nu_m H &= (\beta\theta_2\beta^{-1}, \varepsilon, \alpha\theta_1\alpha^{-1}) \in N_l^A, \\ H\nu_m H^{-1} &= (\varepsilon, \beta^{-1}\theta_1\beta, \gamma^{-1}\theta_2\gamma) \in N_r^A. \end{aligned}$$

Theorem 3.1. *In an (α, β, γ)-inverse quasigroup $(Q, \otimes)$, the left, right and middle A-nuclei are isomorphic in pairs. More exactly, we may say that $N_r^A = H^{-1}N_l^A H$, $N_m^A = H^{-1}N_r^A H$ and $N_l^A = H^{-1}N_m^A H$.*

Proof. From the equalities above, it follows that $H^{-1}N_l^AH \subseteq N_r^A$ and $HN_r^AH^{-1} \subseteq N_l^A$. From the second relation, $N_r^A \subseteq H^{-1}N_l^AH$ since $Aus(Q,\otimes)$ is a group. Hence, $N_r^A = H^{-1}N_l^AH$.

Similarly, $H^{-1}N_r^AH \subseteq N_m^A$ and $HN_m^AH^{-1} \subseteq N_r^A$. Also, $H^{-1}N_m^AH \subseteq N_l^A$ and $HN_l^AH^{-1} \subseteq N_m^A$. Therefore, $N_m^A = H^{-1}N_r^AH$ and $N_l^A = H^{-1}N_m^AH$. □

Notation. From Lemma 5.1 of [24], it follows that the first, second and third components of N_l^A, N_m^A and N_r^A each form groups. For brevity, we shall denote these nine groups by ${}_1N_l^A$, ${}_2N_l^A$, ${}_3N_l^A$, ${}_1N_m^A$, ${}_2N_m^A$, ${}_3N_m^A$, ${}_1N_r^A$, ${}_2N_r^A$ and ${}_3N_r^A$.

Corollary 3.2. *In an (α,β,γ)-inverse quasigroup $(Q,\otimes)$, the following equalities hold:*

$$\gamma({}_3N_l^A)\gamma^{-1} = {}_2N_r^A \tag{3.1}$$

$$\alpha({}_1N_l^A)\alpha^{-1} = {}_3N_r^A \tag{3.2}$$

$$\beta({}_2N_r^A)\beta^{-1} = {}_1N_m^A \tag{3.3}$$

$$\gamma({}_3N_r^A)\gamma^{-1} = {}_2N_m^A \tag{3.4}$$

$$\alpha({}_1N_m^A)\alpha^{-1} = {}_3N_l^A \tag{3.5}$$

$$\beta({}_2N_m^A)\beta^{-1} = {}_1N_l^A \tag{3.6}$$

Proof. We can re-write the equality $H^{-1}N_l^AH = N_r^A$ in the form $[(132),(\alpha^{-1},\beta^{-1},\gamma^{-1})] \cdot [\varepsilon,({}_1N_l^A,\varepsilon,{}_3N_l^A] \cdot [(123),(\beta,\gamma,\alpha)] = [\varepsilon,(\varepsilon,{}_2N_r^A,{}_3N_r^A)]$.

That is, $[\varepsilon,(\varepsilon,\gamma({}_3N_l^A)\gamma^{-1},\alpha({}_1N_l^A)\alpha^{-1})] = [\varepsilon,(\varepsilon,{}_2N_r^A,{}_3N_r^A)]$ and so $\gamma({}_3N_l^A)\gamma^{-1} = {}_2N_r^A$ and $\alpha({}_1N_l^A)\alpha^{-1} = {}_3N_r^A$. The equalities (3.3), (3.4), (3.5) and (3.6) can be proved in a similar way. □

In Theorem 5.2 of [24], we proved that, in a loop $(Q,\otimes)$ with identity element e,

$\lambda_i \in {}_1N_l^A = {}_3N_l^A \Leftrightarrow e\lambda_i \in N_l$,
$\mu_j \in {}_2N_r^A = {}_3N_r^A \Leftrightarrow e\mu_j \in N_r$,
$(\sigma_k,\tau_k,\varepsilon) \in N_m^A \Leftrightarrow e\sigma_k \in N_m$ and $e\sigma_k = e\tau_k^{-1}$.

Equivalently,

${}_1N_l^A = {}_3N_l^A = \langle\lambda_i : i \in I\rangle \Leftrightarrow \langle e\lambda_i : i \in I\rangle = N_l$,
${}_2N_r^A = {}_3N_r^A = \langle\mu_j : j \in J\rangle \Leftrightarrow \langle e\mu_j : j \in J\rangle = N_r$ and
${}_1N_m^A = \langle\sigma_k : k \in K\rangle$,
${}_2N_m^A = \langle\tau_k : k \in K\rangle = \langle\tau_k^{-1} : k \in K\rangle \Leftrightarrow \langle e\sigma_k : k \in K\rangle = N_m$ and

$e\sigma_k = e\tau_k^{-1}$, where $i \in I$ if $(\lambda_i, \varepsilon, \lambda_i) \in N_l^A$, $j \in J$ if $(\varepsilon, \mu_j, \mu_j) \in N_r^A$ and $k \in K$ if $(\sigma_k, \tau_k, \varepsilon) \in N_m^A$.

We shall use this result several times.

Theorem 3.3. *In an* (α, β, γ)*-inverse quasigroup* $(Q, \otimes)$*, the automorphism* $\alpha\beta\gamma$ *commutes with each of the groups* ${}_3N_l^A$ *and* ${}_3N_r^A$*.*

Proof. It follows from equations (3.1), (3.3) and (3.5) that

$$\alpha\beta\gamma({}_3N_l^A)\gamma^{-1}\beta^{-1}\alpha^{-1} = \alpha\beta({}_2N_r^A)\beta^{-1}\alpha^{-1} = \alpha({}_1N_m^A)\alpha^{-1} = {}_3N_l^A$$

and so $\alpha\beta\gamma$ commutes with ${}_3N_l^A$. Also, we have ${}_3N_l^A = {}_1N_l^A$ if $(Q, \otimes)$ is a loop by Theorem 5.2 of [24] (see above).

It follows from equations (3.4), (3.6) and (3.2) that

$$\alpha\beta\gamma({}_3N_r^A)\gamma^{-1}\beta^{-1}\alpha^{-1} = \alpha\beta({}_2N_m^A)\beta^{-1}\alpha^{-1} = \alpha({}_1N_l^A)\alpha^{-1} = {}_3N_r^A$$

and so $\alpha\beta\gamma$ commutes with ${}_3N_r^A$. Also, we have ${}_3N_r^A = {}_2N_r^A$ if $(Q, \otimes)$ is a loop by Theorem 5.2 of [24]. □

Theorem 3.4. *In an* (α, β, γ)*-inverse loop* $(Q, \otimes)$*, the left, right and middle nuclei coincide if*

(i) α *or* β *or* γ *is the identity mapping, or*

(ii) $\alpha\beta$ *or* $\beta\gamma$ *or* $\gamma\alpha$ *is the identity mapping.*

Proof. Proof of (i). If $\alpha = \varepsilon$, it follows from equations (3.2) and (3.5) that ${}_1N_l^A = {}_3N_r^A$ and that ${}_1N_m^A = {}_3N_l^A$ so $\langle \lambda_i : i \in I \rangle = \langle \mu_j : j \in J \rangle$ and $\langle \sigma_k : k \in K \rangle = \langle \lambda_i : i \in I \rangle$ by Theorem 5.2 of [24]. Also, $N_l = \langle e\lambda_i : i \in I \rangle$, $N_r = \langle e\mu_j : j \in J \rangle$ and $N_m = \langle e\sigma_k : k \in K \rangle$ whence $N_l = N_r = N_m$. The proofs for the cases $\beta = \varepsilon$ or $\gamma = \varepsilon$ are similar.

Proof of (ii). From equations (3.3) and (3.5),

$$\alpha\beta({}_2N_r^A)\beta^{-1}\alpha^{-1} = \alpha({}_1N_m^A)\alpha^{-1} = {}_3N_l^A.$$

From Theorem 5.2 of [24] and equations (3.2) and (3.3),

$$\beta\alpha({}_1N_l^A)\alpha^{-1}\beta^{-1} = \beta({}_3N_r^A)\beta^{-1} = \beta({}_2N_r^A)\beta^{-1} = {}_1N_m^A.$$

Since $\alpha\beta = \varepsilon$ implies that $\beta\alpha = \varepsilon$, we have ${}_2N_r^A = {}_3N_l^A$ and ${}_1N_l^A = {}_1N_m^A$. Using Theorem 5.2 of [24] again, $N_r = N_l$ and $N_l = N_m$ when $\alpha\beta = \varepsilon$. The other statements are proved similarly. □

Corollary 3.5. *In a* CI*-loop or a* WIP*-loop,* $N_l = N_r = N_m$.

Proof. Since a CI-loop is a $(\varepsilon, J, \varepsilon)$-inverse loop and a WIP-loop is a $(J^{-1}, \varepsilon, J^{-1})$-inverse loop (where $a \circ aJ = e$ for all a and e is the identity element), the result follows immediately from Theorem 3.4. □

Theorem 3.6. *In a k-inverse loop $(Q, \otimes)$, $N_l = N_m = N_r$.*

Proof. It follows from equations (3.3) and (3.4) that, in a k-inverse loop, $N_m = J^{k+1} N_r J^{-(k+1)}$ and $N_m = J^k N_r J^{-k}$ so $J N_r J^{-1} = N_r$ and hence $N_m = N_r$. Also, from equations (3.5) and (3.6), $N_l = J^{k+1} N_m J^{-(k+1)}$ and $N_l = J^k N_m J^{-k}$ so $J N_m J^{-1} = N_m$ and hence $N_l = N_m$. □

This theorem was first proved in [20].

Also, since every (r, s, t)-inverse loop $(L, \circ)$ in which $a \circ aJ = e$ for all $a \in L$ (where e is the identity element) is an r-inverse loop, Theorem 3.6 shows that the left, right and middle nuclei coincide in every such loop.

Remark. Loops which satisfy the equation (2.6) and for which $a \circ aJ \neq e$ also exist. If $r = 0$, $s = 0$ or $t = 0$ or if $s = -r$, $t = -s$ or $r = -t$ in such a loop then Theorem 3.4 shows that its three nuclei coincide.

4. Nuclei of λ-IP and ρ-IP quasigroups

Let $(Q, \otimes)$ be a quasigroup which has the λ-inverse property. By Theorem 2.3, $[(2\ 3), (\lambda_1, \lambda_3, \lambda_2)]$ is an autostrophism of $(Q, \otimes)$.

Let $K = [(2\ 3), (\lambda_1, \lambda_3, \lambda_2)] = [(2\ 3), J]$ say. Then, obviously, $K^{-1} = [(2\ 3), (J^{-1})^{(2\ 3)}]$.

If $\theta = (\theta_1, \theta_2, \theta_3)$ is an autotopism of a quasigroup $(Q, \otimes)$ which has the λ-inverse-property,

$$\begin{aligned} K^{-1}[\varepsilon, \theta]K &= [(2\ 3), (J^{-1})^{(2\ 3)}] \cdot [\varepsilon, \theta] \cdot [(2\ 3), J] \\ &= [(2\ 3), \theta(J^{-1})^{(2\ 3)}] \cdot [(2\ 3), J] = [(2\ 3)(2\ 3), J\{\theta(J^{-1})^{(2\ 3)}\}^{(2\ 3)}] \\ &= [\varepsilon, J\theta^{(2\ 3)}J^{-1}] = [\varepsilon, (\lambda_1\theta_1\lambda_1^{-1}, \lambda_3\theta_3\lambda_3^{-1}, \lambda_2\theta_2\lambda_2^{-1})], \end{aligned}$$

which is another autotopism. Also,

$$\begin{aligned} K[\varepsilon, \theta]K^{-1} &= [(2\ 3), J] \cdot [\varepsilon, \theta] \cdot [(2\ 3), (J^{-1})^{(2\ 3)}] \\ &= [(2\ 3), \theta J] \cdot [(2\ 3), (J^{-1})^{(2\ 3)}] = [(2\ 3)(2\ 3), (J^{-1})^{(2\ 3)}(\theta J)^{(2\ 3)}] \\ &= [\varepsilon, (J^{-1})^{(2\ 3)}\theta^{(2\ 3)}J^{(2\ 3)}] = [\varepsilon, (\lambda_1^{-1}\theta_1\lambda_1, \lambda_2^{-1}\theta_3\lambda_2, \lambda_3^{-1}\theta_2\lambda_3)], \end{aligned}$$

which is again an autotopism.

Let $(Q,\otimes)$ be a quasigrooup with the λ-inverse property and let $\nu_l = (\theta_1,\varepsilon,\theta_3) \in N_l^A$. Then, from above,

$$K^{-1}\nu_l K = (\lambda_1\theta_1\lambda_1^{-1}, \lambda_3\theta_3\lambda_3^{-1}, \varepsilon) \in N_m^A, \tag{4.1}$$

$$K\nu_l K^{-1} = (\lambda_1^{-1}\theta_1\lambda_1, \lambda_2^{-1}\theta_3\lambda_2, \varepsilon) \in N_m^A. \tag{4.2}$$

Let $\nu_r = (\varepsilon,\theta_2,\theta_3) \in N_r^A$. Then,

$$K^{-1}\nu_r K = (\varepsilon, \lambda_3\theta_3\lambda_3^{-1}, \lambda_2\theta_2\lambda_2^{-1}) \in N_r^A, \tag{4.3}$$

$$K\nu_r K^{-1} = (\varepsilon, \lambda_2^{-1}\theta_3\lambda_2, \lambda_3^{-1}\theta_2\lambda 3) \in N_r^A. \tag{4.4}$$

Let $\nu_m = (\theta_1,\theta_2,\varepsilon) \in N_m^A$. Then

$$K^{-1}\nu_m K = (\lambda_1\theta_1\lambda_1^{-1}, \varepsilon, \lambda_2\theta_2\lambda_2^{-1}) \in N_l^A, \tag{4.5}$$

$$K\nu_m K^{-1} = (\lambda_1^{-1}\theta_1\lambda_1, \varepsilon, \lambda_3^{-1}\theta_2\lambda_3) \in N_l^A. \tag{4.6}$$

Theorem 4.1. *In a quasigroup with the λ-inverse property, the left and middle A-nuclei are isomorphic. More exactly, we may say that $N_m^A = K^{-1}N_l^A K = KN_l^A K^{-1}$.*

Proof. From the equalities (4.1) and (4.6) above, it follows that $K^{-1}N_l^A K \subseteq N_m^A$ and $KN_m^A K^{-1} \subseteq N_l^A$. From the second relation, $N_m^A \subseteq K^{-1}N_l^A K$ since $Aus(Q,\otimes)$ is a group. Hence, $N_m^A = K^{-1}N_l^A K$. Also, from the equalities (4.2) and (4.5), it follows that $N_m^A = KN_l^A K^{-1}$. We conclude that $K^{-1}N_l^A K = KN_l^A K^{-1}$ or $N_l^A K^2 = K^2 N_l^A$ so the autotopism $K^2 = (\lambda_1\lambda_1, \lambda_3\lambda_2, \lambda_2\lambda_3)$ commutes with N_l^A and similarly it also commutes with N_m^A. □

Corollary 4.2. *In a quasigroup with the λ-inverse property, the following equalities hold:*

$$\lambda_1({}_1N_l^A)\lambda_1^{-1} = {}_1N_m^A = \lambda_1^{-1}({}_1N_l^A)\lambda_1 \tag{4.7}$$

$$\lambda_3({}_3N_l^A)\lambda_3^{-1} = {}_2N_m^A = \lambda_2^{-1}({}_3N_l^A)\lambda_2 \tag{4.8}$$

Proof. We can re-write the equality $K^{-1}N_l^A K = N_m^A$ in the form

$$[(2\ 3), (\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1})] \cdot [\varepsilon, ({}_1N_l^A,\ \varepsilon,\ {}_3N_l^A)] \cdot [(2\ 3), (\lambda_1, \lambda_3, \lambda_2)] = [\varepsilon, ({}_1N_m^A,\ {}_2N_m^A,\ \varepsilon)].$$

That is,

$$[\varepsilon, (\lambda_1({}_1N_l^A)\lambda_1^{-1}, \lambda_3({}_3N_l^A)\lambda_3^{-1}, \varepsilon)] = [\varepsilon, ({}_1N_m^A,\ {}_2N_m^A,\ \varepsilon)]$$

and so $\lambda_1({}_1N_l^A)\lambda_1^{-1} = {}_1N_m^A$ and $\lambda_3({}_3N_l^A)\lambda_3^{-1} = {}_2N_m^A$.

We can re-write the equality $KN_l^AK^{-1} = N_m^A$ in the form

$$[(2\ 3), (\lambda_1, \lambda_3, \lambda_2)] \cdot [\varepsilon, ({}_1N_l^A,\ \varepsilon,\ {}_3N_l^A)] \cdot [(2\ 3), (\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1})] = [\varepsilon, ({}_1N_m^A,\ {}_2N_m^A,\ \varepsilon)].$$

That is,

$$[\varepsilon, (\lambda_1^{-1}({}_1N_l^A)\lambda_1, \lambda_2^{-1}({}_3N_l^A)\lambda_2, \varepsilon)] = [\varepsilon, ({}_1N_m^A,\ {}_2N_m^A,\ \varepsilon)]$$

and so $\lambda_1^{-1}({}_1N_l^A)\lambda_1 = {}_1N_m^A$ and $\lambda_2^{-1}({}_3N_l^A)\lambda_2 = {}_2N_m^A$. □

Theorem 4.3. *In a λ-inverse-property loop, the left and middle nuclei coincide if any one of λ_1, λ_2 or $\lambda_3 = I$.*

Proof. This follows immediately from Corollary 4.2 and Theorem 5.2 of [24] (which we stated in the previous Section). □

Corollary 4.4. *The left and middle nuclei coincide in an LIP-loop.*

Proof. In every such loop, both λ_2 and $\lambda_3 = I$. (See Equation 2.1.) □

Remark. Corollary 4.4 is well known, see, for example, [3, 27].

By means of an exactly similar analysis of ρ-inverse-property quasigroups to that just made for λ-inverse-property quasigroups, it is easy to prove that

Theorem 4.5.

(*i*) *In a quasigroup with the ρ-inverse property, the middle and right A-nuclei are isomorphic. More exactly, we may say that $N_m^A = K^{-1}N_r^AK = KN_r^AK^{-1}$, where $K = [(1\ 3), (\rho_3, \rho_2, \rho_1)$.*

(*ii*) *In a ρ-inverse property loop, the middle and right nuclei coincide if any one of ρ_1, ρ_2 or $\rho_3 = I$.*

(*iii*) *The middle and right nuclei coincide in an RIP-loop.*

From Corollary 4.4 and Theorem 4.5(iii), we may deduce among other things the well-known fact that $N_l = N_r = N_m$ in every IP-loop.

5. Generalized balanced parastrophic identities of length two

If we examine the equations (2.1)-(2.9) carefully, we notice that they are all of the form

$$A^\sigma(x\nu_1, y\nu_2) = [A(x,y)]\nu_3, \tag{5.1}$$

where A^σ is some parastrophe of the operation A and ν_1, ν_2, ν_3 are permutations of the set Q, i.e. $[(\sigma, (\nu_1, \nu_2, \nu_3)]$ is an autostrophy of the quasigroup (Q, A).

For example, equation(2.3) is $x \otimes (y \otimes x)J = yJ$ and is equivalent to $x \otimes^{(2\,3)} yJ = (y \otimes x)J$ or to $x \otimes^{(2\,3)} yJ = (x \otimes^{(1\,2)} y)J$. If we write $\otimes^{(1\,2)} = \oplus$, we get $\otimes^{(2\,3)} = \oplus^{(1\,2)(2\,3)} = \oplus^{(1\,3\,2)}$ and so $x \oplus^{(1\,3\,2)} yJ = (x \oplus y)J$ which is of the above form.

This fact suggests that we should make the following generalization:

Definition 5.1. An identity of the form

$$[A^\sigma(x\nu_1, y\nu_2)]\nu_3 = [A^\tau(x\nu_4, y\nu_5)]\nu_6 \tag{5.2}$$

on a groupoid (Q, A) where A^σ, A^τ are some parastrophes of the operation A, $x, y \in Q$ and ν_i for $i = 1, 2, ..., 6$ are some permutations of the set Q, will be called a *generalized balanced parastrophic identity of length two* on the groupoid (Q, A).

Note 4. In calling identity (5.2) *balanced* and *of length two* we follow [6].

Theorem 5.1. *Any generalized balanced parastrophic identity of length two on a quasigroup* (Q, A) *is equivalent to an identity of type* (5.1).

Proof. It will be convenient to use (x_1, x_2, x_3) in place of (x, y, z) and to write $A(x_1, x_2, x_3)$ to denote that $A(x_1, x_2) = x_3$ in the quasigroup (Q, A).

From the identity (5.2), we have $[A^\sigma(x_1\nu_1, x_2\nu_2)]\nu_3\nu_6^{-1} = A^\tau(x_1\nu_4, x_2\nu_5)$. Put $y_1 = x_1\nu_4$, $y_2 = x_2\nu_5$. Then $[A^\sigma(y_1\nu_4^{-1}\nu_1, y_2\nu_5^{-1}\nu_2)]\nu_3\nu_6^{-1} = A^\tau(y_1, y_2)$.

That is, $[A^\sigma(y_1\theta_1, y_2\theta_2)]\theta_3^{-1} = A^\tau(y_1, y_2)$, where $\theta_1 = \nu_4^{-1}\nu_1$, $\theta_2 = \nu_5^{-1}\nu_2$ and $\theta_3 = \nu_3\nu_6^{-1}$. So there exists an isotopism $(\theta_1, \theta_2, \theta_3)$ from (Q, A^σ) to (Q, A^τ) and we have

$$A^\sigma(y_1\theta_1, y_2\theta_2, y_3\theta_3) \Leftrightarrow A^\tau(y_1, y_2, y_3) \Leftrightarrow A(y_{1\tau^{-1}}, y_{2\tau^{-1}}, y_{3\tau^{-1}}).$$

Therefore,

$$A^\sigma(z_{1\tau}\theta_1, z_{2\tau}\theta_2, z_{3\tau}\theta_3) \Leftrightarrow A(z_1, z_2, z_3), \text{ where } z_i = y_{i\tau^{-1}}.$$

So, $A^{\sigma\tau^{-1}}(z_1\theta_{1\tau^{-1}}, z_2\theta_{2\tau^{-1}}, z_3\theta_{3\tau^{-1}}) \Leftrightarrow A(z_1, z_2, z_3)$.

That is, $A^{\sigma\tau^{-1}}(z_1\theta_{1\tau^{-1}}, z_2\theta_{2\tau^{-1}})\theta^{-1}_{3\tau^{-1}} = A(z_1, z_2)$
or $A^{\sigma\tau^{-1}}(z_1\theta_{1\tau^{-1}}, z_2\theta_{2\tau^{-1}}) = [A(z_1, z_2)]\theta_{3\tau^{-1}}$ which is of the form (5.1). □

Theorem 5.2. *If a groupoid (Q, A) has a* (1 3)*-autostrophy, it is a right quasigroup. If it has a* (2 3)*-autostrophy, it is a left quasigroup. If it has a* (1 2 3)*-autostrophy, it also has a* (1 3 2)*-autostrophy and is a quasigroup.*

Proof. Suppose that the groupoid (Q, A) has the autostrophy $[(1\ 3), (\alpha_1, \alpha_2, \alpha_3)]$. This is equivalent to saying that (Q, A) is isotopic to $(Q, A^{(1\ 3)})$ and so the latter is a groupoid. That is, for all $a, b \in Q$, $A^{(1\ 3)}(a, b) = x$ is uniquely soluble for x. Equivalently, $A(x, b) = a$ is uniquely soluble for x and so (Q, A) is a right quasigroup.

The proof of the second statement is similar.

For the third statement, we note that, by Lemma 2.2, the product of a (1 2 3)-autostrophy with itself is a (1 3 2)-autostrophy. The remainder of the proof is similar to the foregoing. □

Corollary 5.3. *A right quasigroup which has a* (2 3)*-autostrophy is a quasigroup. Likewise, a left quasigroup which has a* (1 3)*-autostrophy is a quasigroup.*

Note 5. The concepts of isotopy and parastrophy have well-known geometrical interpretations in the language of 3-nets(3-webs). The definition and some properties of nets, including their inter-relationships with quasigroups, are given in [5, 7, 8, 19, 29]. In particular, both [8] and Section 10 of the extensive paper [5] include mention of the geometrical interpretation of isostrophy in terms of collineations.

Acknowledgement. The authors thank the referee for his helpful comments. Also, the authors wish to thank the London Mathematical Society for the award of a Scheme 5 grant which provided the setting for joint research by enabling the second author to visit the UK.

References

[1] **R. Artzy**: *On loops with a special property*, Proc. Amer. Math. Soc., **6** (1955), 448 – 453.

[2] **R. Artzy**: *Crossed-inverse and related loops*, Trans. Amer. Math. Soc., **91** (1959), 480 – 492.

[3] **R. Artzy**: *Relation between loop identities*, Proc. Amer. Math. Soc., **11** (1960), 847 – 851.

[4] **R. Baer**: *Nets and groups, II*, Trans. Amer. Math. Soc., **47** (1940), 435−439.

[5] **A. Barlotti, K. Strambach**: *The geometry of binary systems.* Adv. in Math. **49** (1983), 1 − 105.

[6] **V. D. Belousov**: *Balanced identities on quasigroups*, Mat. sbornik, **70** (112):1, (1966), 55 − 97, (in Russian).

[7] **V. D. Belousov**: *Foundations of the theory of quasigroups and loops*, Nauka, Moscow, 1967 (in Russian).

[8] **V. D. Belousov**: *Algebraic nets and quasigroups*, Shtiintsa, Kishinev, 1971 (in Russian).

[9] **V. D. Belousov**: *n-Ary quasigroups*, Shtiintsa, Kishinev, 1972 (in Russian).

[10] **V. D. Belousov**: *Elements of quasigroup theory: a special course*, Shtiintsa, Kishinev, 1981 (in Russian).

[11] **V. D. Belousov**: *Inverse quasigroups*, Quasigroups, Mat. Issled., Kishinev, Shtiinta, **95** (1987), 3 − 22, (in Russian).

[12] **V. D. Belousov**: *Autotopies in inverse lops*, Mat. Issled., Kishinev, Shtiinta, **120** (1991), 30 − 44, (in Russian).

[13] **V. D. Belousov, A. A. Gvaramiya**: *Partial identities and nuclei of quasigroups*, Bulletin of the Academy of Sciences of the Georgian SSR, **65**, no. 2, (1972), 277 − 279, (in Russian).

[14] **V. D. Belousov, B. V. Tsurkan**: *Crossed-inverse quasigroups* (*CI-quasigroups*), Izvestiya Vysshih Uchebnyh Zavedenii, Matematika, no. 3, **82**, (1969), 21 − 27 (in Russian).

[15] **G. B. Belyavskaya**: *Nuclei and centre of a quasigroup*, Mat. Issled., Kishinev, Shtiinta, **102** (1988), 37 − 52, (in Russian).

[16] **G. B. Belyavskaya**: *Quasigroup theory: nuclei, centre, commutants*, Bul. A.Ş. a R.M., matematica, 1996, No. 2(21), 47 − 71 (in Russian).

[17] **R. H. Bruck**: *Some results in the theory of quasigroups*, Trans. Amer. Math. Soc., **55** (1944), 19 − 52.

[18] **R. H. Bruck**: *A survey of binary systems, Third Printing, Corrected*, Springer, New York, 1971.

[19] **J. Dénes, A. D. Keedwell**: *Latin Squares and their Applications*, Acadêmiai Kiadó, Budapest, 1974.

[20] **B. B. Karklin's, V. B. Karklin'**: *Inverse loops*, Mat. Issled., Kishinev, Shtiinta, **39** (1976), 87 − 101 (in Russian).

[21] **A. D. Keedwell**: *Crossed-inverse quasigroups with long inverse cycles and applications to cryptography.* Australas. J. Comb. **20** (1999), 241 − 250.

[22] **A. D. Keedwell, V. A. Shcherbacov**: *On m-inverse loops and quasigroups with a long inverse cycle*, Australas. J. Combin. **26** (2002), 99 – 119.

[23] **A. D. Keedwell, V. A. Shcherbacov**: *Construction and properties of (r, s, t)-inverse quasigroups I*, Discrete Math. **266** (2003), 275 – 291.

[24] **A. D. Keedwell, V. A. Shcherbacov**: *Construction and properties of (r, s, t)-inverse quasigroups II*, Discrete Math. **288** (2004), 61 – 71.

[25] **M. D. Kitoroagè**: *Nuclei in quasigroups*, Mat. Issled., Kishinev, Shtiinta, **7** (1972), 60 – 71 (in Russian).

[26] **I. V. Leakh**: *On transformations of orthogonal systems of operations and algebraic nets*, Ph.D. Dissertation, Kishinev, Institute of Mathematics, (1986), 108 pages, (in Russian).

[27] **G. Nagy**: *Collineation groups of the smallest Bol 3-nets*, Note Mat. **14** (1994), 115 – 129.

[28] **J. M. Osborn**: *Loops with the weak inverse property*, Pacific J. Math. **10** (1960), 295 – 304.

[29] **H. O. Pflugfelder**: *Quasigroups and loops: Introduction*, Berlin, Heldermann Verlag, 1990.

[30] **M. Steinberger**: *On loops with a general weak inverse property*, Mitt. Math. Ges. Hamburg, **10** (1979), 573 – 586.

Received February 8, 2005

A.D. Keedwell
Department of Mathematics
and Statistics
University of Surrey
Guildford Surrey, GU2 7XH
UK
e-mail: A.Keedwell@surrey.ac.uk

V.A. Shcherbacov
Institute of Mathematics
and Computer Science
Academy of Sciences of Moldova
Academiei 5, MD-2028
Moldova
scerb@math.md