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Subassociative groupoids

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Dedicated to the memory of Eva Ruth Silberger, 1962 – 2006

Abstract

When $\langle G;\diamond\rangle$ is a groupoid with binary operation $\diamond : G^2 \to G$, and when $k \in \mathbf{N} := \{1,2,3,\ldots\}$, then $F^\sigma(k)$ denotes the set of all formal products $\mathbf{u}$ on k independent variables. It is well known that $|F^\sigma(k)| = C(k)$, where $C(k)$ is the kth Catalan number.

Each word $\mathbf{u} \in F^\sigma(k)$ induces a function $\mathbf{u} : G^k \to G$ given by $\mathbf{u} : \vec{g} \mapsto \mathbf{u}(\diamond, \vec{g})$, where $\mathbf{u}(\diamond,\vec{g})$ is the interpretation in $\langle G;\diamond\rangle$ of $\mathbf{u}$ as a $\diamond$–product of the sequence $\vec{g} := \langle g_0, g_1, \ldots, g_{k-1}\rangle \in G^k$.

Write $\mathbf{u} =_\diamond \mathbf{v}$ for $\{\mathbf{u},\mathbf{v}\} \subseteq F^\sigma(k)$ iff $\mathbf{u}(\diamond,\vec{g}) = \mathbf{v}(\diamond,\vec{g})$ whenever $\vec{g} \in G^k$. This $=_\diamond$ is an equivalence relation on the set $F^\sigma := \bigcup\{F^\sigma(k) : k \in \mathbf{N}\}$. The sequence $\mathbf{SaT}(\langle G;\diamond\rangle) := \langle |F^\sigma(k)/=_\diamond|\rangle_{k=2}^{\infty}$ presents the subassociativity types of $\langle G;\diamond\rangle$.

We calculate $\mathbf{SaT}(G)$ for a few evocative groupoids $G := \langle G;\diamond\rangle$, and we initiate a study of the partitions $F^\sigma(k)/=_\diamond$. Each equivalence class of the completely free groupoid F^σ is a singleton, and so F^σ realizes the theoretical minimum k–associativity for each $k \in \mathbf{N}$. We propose for each k a minimally k–associative class of finite groupoids.

Introduction

Given a set G and a *binary operation* $\diamond : G \times G \to G$ on G, it is customary to write $\diamond(x,y)$ in the form $x \diamond y$ when $\langle x,y\rangle \in G^2 := G \times G$. The pair $\langle G;\diamond\rangle$ is said to be a *groupoid.*

We say that a triple $\langle g_0,g_1,g_2\rangle \in G^3$ of elements in G *associates* under the binary operation $\diamond$ iff $(g_0 \diamond g_1) \diamond g_2 = g_0 \diamond (g_1 \diamond g_2)$. If every triple of

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elements in G associates under $\diamond$, the binary operation $\diamond$ itself is said to be *associative*, and the groupoid $\langle G; \diamond\rangle$ is called a *semigroup*.

For $\langle G; \diamond\rangle$ a semigroup, each finite sequence $g_0, g_1, \ldots, g_{k-1}$ of elements in G determines under $\diamond$ a unique element in G as its product. We can write this product in the simplified form $g_0 \diamond g_1 \diamond \cdots \diamond g_{k-1}$ because parentheses are not needed to avoid ambiguity.

Of course, the great majority of groupoids are not semigroups. Each such nonsemigroup has at least one triple of elements which fails to associate. This failure of some triples to associate induces diversity among products of the longer strings as well.

Our paper's principal focus is upon this diversity of products.

For instance, if $\langle G; \diamond\rangle$ is not a semigroup then we expect that some quadruples $\langle g_0, g_1, g_2, g_3\rangle \in G^4$ may in a general sense also fail to associate. However, whereas there are at most two potentially distinct products for a triple of elements in a groupoid, there are five potentially distinct products of a quadruple of such elements, fourteen potentially distinct 5–tuple products, and in general there are $C(k)$ potentially distinct k–products, where $C(k)$ is the $k\underline{\text{th}}$ Catalan number.

That is, when the binary operation $\diamond$ of a groupoid lacks 3–associativity, then $\diamond$ may lack k–associativity for sundry integers $k \geqslant 4$ as well.

In §1 we introduce Reverse Polish Notation, which provides a convenient tool for specifying the potentially different k–products under $\diamond$ of a length k sequence of elements in G. This leads to our presentation in §2 of the notion of a formal k–product, and of a completely free groupoid in which every formal k–product of a length k sequence in G produces a *de facto* distinct element in the groupoid, and enables our development in §3 of a measure of the subassociativity of an arbitrary groupoid; this measure is given as an infinite sequence of positive integers which we call the subassociativity type of the groupoid. In §3 we calculate the subassociativity type of each of several important nonassociative groupoids, including that of the groupoid of integers under subtraction. Related to the subassociativity type of a groupoid is its size sequence, which appears interestingly complicated even when the subassociativity type of the groupoid is regular and simple in form.

§4 concentrates upon those groupoids in which k–associativity is minimal for every integer $k \geqslant 3$.

Our paper presents a variety of natural problems.

1. Reverse Polish Notation

For a nonassociative binary operation $\diamond$, if $\vec{g} := \langle g_0, g_1, \dots, g_{k-1}\rangle$ is a finite sequence of elements in G then it may happen that $\mathbf{w}(\diamond, \vec{g}) \neq \mathbf{v}(\diamond, \vec{g})$, where $\mathbf{w}$ and $\mathbf{v}$ are some two "appropriate parenthesizations" of the augmented sequence $\diamond(\vec{g}) := \langle g_0, \diamond, g_1, \diamond, \cdots, \diamond, g_{k-1}\rangle$.

We call a parenthesization of $\diamond(\vec{g})$ *appropriate* if it enables the $k-1$ occurrences of the symbol $\diamond$ unambiguously to serve as a binary operation in $\diamond(\vec{g})$. For instance, the two parenthesizations in (1), below, are appropriate; and, if $\diamond$ is associative, then we can believe that

$$((g_0 \diamond (g_1 \diamond g_2)) \diamond (g_3 \diamond g_4)) = ((g_0 \diamond g_1) \diamond ((g_2 \diamond g_3) \diamond g_4)). \tag{1}$$

We have enclosed each of the two expressions, balanced by the $=$ sign in (1), with an external, conventionally unnecessary, parenthesis pair, whose purpose is to assure that each $\diamond$–multiplication is consistent in its form; namely, $(a \diamond b)$, instead of $a \diamond b$ as sometimes abbreviated. Our reason for this ostensible redundancy of parentheses should become clear after our discussion, in the next few paragraphs, of Reverse Polish Notation (RPN).

RPN is sometimes more convenient than parenthesized expressions of the sort in (1). For people who are uneasy with RPN we provide a gradual approach to this, parenthesis-free, notation. In two steps we will convert the usual-form equality (1) into its equivalent RPN version, (3).

First, we remove left parentheses from (1). A routine proof shows that there is exactly one way to restore left parentheses to the resulting left-parenthesis deprived expression, (2), so as to regain an appropriately parenthesized augmented sequence. Here then is (2):

$$g_0 \diamond g_1 \diamond g_2)) \diamond g_3 \diamond g_4)) = g_0 \diamond g_1) \diamond g_2 \diamond g_3) \diamond g_4)) \tag{2}$$

Each of the $=$ expressions in (1) and (2) has 5 terms, g_i, which are elements in G. It is no accident that each of those expressions has also exactly 4 occurrences of $\diamond$ and exactly 4 right parentheses. In order to create from the expressions in (2) their equivalent RPN expressions we merely eliminate the 4 occurrences in (2) of $\diamond$, and then in the $\diamond$–free resulting expression we replace each right parenthesis with a new occurrence of $\diamond$. Thus, finally, we obtain the RPN equation which is equivalent to (1):

$$g_0 g_1 g_2 \diamond \diamond g_3 g_4 \diamond \diamond = g_0 g_1 \diamond g_2 g_3 \diamond g_4 \diamond \diamond. \tag{3}$$

Comparing (1) and (3), we see that (3) is shorter than (1). RPN is an efficient way of representing lengthy $\diamond$–products. One can safely remove all parentheses from (1) and maintain a *bona fide* equality if and only if $\diamond$ is associative. But with the RPN expression, (3), there are no parentheses to remove, and when $\diamond$ is associative then every RPN product constructed from the sequence $\langle g_0, g_1, g_2, g_3, g_4\rangle \in G^5$ is equal to that given by

$$g_0g_1g_2g_3g_4 \diamond^4 .$$

This too is shorter than the usual product expression

$$g_0 \diamond g_1 \diamond g_2 \diamond g_3 \diamond g_4.$$

RPN confers a more important advantage: It facilitates our classification of the "subassociativity" of groupoids.

2. Formal Products

We will define a groupoid, F^σ, inspired by the idea of the "completely free" groupoid $F := F(x, \bullet)$ generated by a two-letter alphabet, $\{x, \bullet\}$.

$F \subset \{x, \bullet\}^*$, where $\{x, \bullet\}^*$ is the semigroup under concatenation of all finite words with letters in $\{x, \bullet\}$.

We write $\mathbf{a} = \mathbf{b}$ to say that the word $\mathbf{a}$ is spelled the same as the word $\mathbf{b}$, for $\{\mathbf{a}, \mathbf{b}\} \subseteq F$.

$\#(\mathbf{u}, z)$ denotes the number of occurrences of a letter z in the word $\mathbf{u}$. A nonempty word $\mathbf{w} \in \{\bullet, x\}^*$ is an element in F if and only if

(i) $\#(\mathbf{w}, x) - \#(\mathbf{w}, \bullet) = 1$.

(ii) If $\mathbf{p}$ is a nonempty prefix of $\mathbf{w}$ then $\#(\mathbf{p}, x) > \#(\mathbf{p}, \bullet)$.

It is easy to see that $x \in F$, and that $\{\mathbf{u}, \mathbf{v}\} \subseteq F \Rightarrow \mathbf{uv}\bullet \in F$. Thus $\bullet$ serves in F as an operator symbol, providing a binary operation $\bullet : \langle \mathbf{u}, \mathbf{v}\rangle \mapsto \mathbf{uv}\bullet$ for F in RPN format.

The relevant property of the groupoid $\langle F, \bullet\rangle$ is that if $\langle \mathbf{p}, \mathbf{s}\rangle \neq \langle \mathbf{p}', \mathbf{s}'\rangle$ with $\{\mathbf{p}, \mathbf{p}', \mathbf{s}, \mathbf{s}'\} \subset F$ then $\mathbf{ps} \neq \mathbf{p}'\mathbf{s}'$. An easy related fact is that the binary operation $\bullet$ is antiassociative; *i.e.*, that no triples in F associate:

Theorem 2.1. *Let* $\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle \in F^3$. *Then* $\mathbf{abc} \bullet \bullet \neq \mathbf{ab} \bullet \mathbf{c} \bullet$.

Proof. Clearly the "free" semigroup $\{x, \bullet\}^*$ has the cancellation property. Thus, if $\mathbf{abc} \bullet \bullet = \mathbf{ab} \bullet \mathbf{c} \bullet$ then $\mathbf{c} \bullet \bullet = \bullet \mathbf{c} \bullet$. But $\mathbf{c} \bullet \bullet \neq \bullet \mathbf{c} \bullet$, since x is a prefix of $\mathbf{c}$ and since $x \neq \bullet$. □

Notice that, if $\mathbf{w} \in F$, then either $\mathbf{w} = x$ or there exists exactly one pair $\langle \mathbf{p}, \mathbf{s} \rangle \in F \times F$ such that $\mathbf{w} = \mathbf{ps}\bullet$.

Henceforth $\vec{x} := \langle x_0, x_1, x_2, \ldots \rangle$ is a sequence of distinct variables, and $\bullet$ is an operator symbol. Let k be a positive integer. We now modify F:

Definition 2.2. By a *formal k–product* we mean any word $\mathbf{w}$ of length $2k-1$ in the alphabet $\{x_0, x_1, \ldots, x_{k-1}, \bullet\}$, satisfying three conditions:

(i) $x_0 x_1 \ldots x_{k-1}$ is a subword of $\mathbf{w}$.

(ii) $\mathbf{w}$ has exactly $k-1$ occurrences of the operator symbol $\bullet$.

(iii) If $\mathbf{p}$ is a nonempty prefix of $\mathbf{w}$ then $\mathbf{p}$ has fewer occurrences of the operator symbol $\bullet$ than it has of variable symbols x_i.

As usual $\mathbf{N} := \{1, 2, 3, \ldots\}$. For $k \in \mathbf{N}$ the expression $F^\sigma(k)$ denotes the set of all formal k–products. Finally, we define the infinite set F^σ by

$$F^\sigma := \bigcup \{F^\sigma(k) : k \in \mathbf{N}\}.$$

It is well-known, *viz* [1, 2, 3, 4], that for each $k \in \mathbf{N}$ the number $|F^\sigma(k)|$ is the k<u>th</u> term of the Catalan sequence, which is to say that

$$|F^\sigma(k)| = C(k) := \frac{1}{2k-1}\binom{2k-1}{k}.$$

Henceforth $\omega := \mathbf{N} \cup \{0\}$, and $k := \{0, 1, \ldots, k-1\}$ when $k \in \mathbf{N}$.

For $\langle k, j \rangle \in \mathbf{N} \times \omega$ and $\mathbf{w} \in F^\sigma(k)$, the expression $\mathbf{w}_j$ denotes the word obtained by replacing the letter x_i in $\mathbf{w}$ with the letter x_{j+i} for each $i \in k$. We write $F^\sigma_j(k) := \{\mathbf{u}_j : \mathbf{u} \in F^\sigma(k)\}$.

<u>Illustrative Example 1</u>: When $\mathbf{w} := x_0 x_1 \bullet x_2 x_3 \bullet x_4 \bullet \bullet$ then $\mathbf{w} \in F^\sigma(5)$, and $\mathbf{w}_{13} = x_{13} x_{14} \bullet x_{15} x_{16} \bullet x_{17} \bullet \bullet \in F^\sigma_{13}(5)$.

Observe that $\langle F^\sigma; \odot \rangle$ is a groupoid where the binary operation $\odot$ is defined thus: When $\langle \mathbf{u}, \mathbf{v} \rangle \in F^\sigma(k) \times F^\sigma(j)$, then $\mathbf{uv}\odot := \mathbf{uv}_k \bullet$. It is easy to see that then $\mathbf{uv}\odot \in F^\sigma(k+j)$.

Indeed, $F^\sigma(1) = \{x_0\}$, while for $2 \leqslant k \in \mathbf{N}$ one could show that

$$F^\sigma(k) = \bigcup \{F^\sigma(i)F^\sigma(k-i)\odot :\ 1 \leqslant i \leqslant k-1\},$$

where $F^\sigma(i)F^\sigma(k-i)\odot := \{\mathbf{uv}\odot :\ \mathbf{u} \in F^\sigma(i) \wedge \mathbf{v} \in F^\sigma(k-i)\}$.

Illustrative Example 2: When $\mathbf{u} := x_0x_1 \bullet x_2x_3 \bullet\bullet \in F^\sigma(4)$, and when $\mathbf{v} := x_0x_1x_2x_3 \bullet x_4 \bullet\bullet\bullet \in F^\sigma(5)$, we have that

$$\begin{aligned}\mathbf{uv}\odot &= x_0x_1 \bullet x_2x_3 \bullet\bullet \quad x_0x_1x_2x_3 \bullet x_4 \bullet\bullet\bullet \odot \\ &= x_0x_1 \bullet x_2x_3 \bullet\bullet x_{4+0}x_{4+1}x_{4+2}x_{4+3} \bullet x_{4+4} \bullet\bullet\bullet\bullet \\ &= x_0x_1 \bullet x_2x_3 \bullet\bullet x_4x_5x_6x_7 \bullet x_8 \bullet\bullet\bullet\bullet \in F^\sigma(4+5).\end{aligned}$$

Theorem 2.3. *If* $\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle \in (F^\sigma)^3$ *then* $\mathbf{ab} \odot \mathbf{c} \odot \neq \mathbf{abc} \odot \odot$.

Proof. $\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle \in F^\sigma(i) \times F^\sigma(j) \times F^\sigma(t)$ for some $\langle i, j, t\rangle \in \mathbf{N}^3$. Thus $\mathbf{ab} \odot \mathbf{c} \odot = \mathbf{ab}_i \bullet \mathbf{c} \odot = \mathbf{ab}_i \bullet \mathbf{c}_{i+j} \bullet$ and $\mathbf{abc} \odot \odot = \mathbf{abc}_j \bullet \odot = \mathbf{ab}_i\mathbf{c}_{i+j} \bullet\bullet$. So if $\mathbf{ab} \odot \mathbf{c} \odot = \mathbf{abc} \odot \odot$ then $\bullet \mathbf{c}_{i+j} \bullet = \mathbf{c}_{i+j} \bullet\bullet$, an impossibility. $\square$

In our view, the groupoids $\langle F; \bullet\rangle$ and $\langle F^\sigma; \odot\rangle$ lie at an opposite extreme from the class of semigroups. For, no triple of elements in either of these two groupoids associates. However, every triple in a semigroup associates.

For $\langle G; \diamond\rangle$ a semigroup, each sequence $\vec{g} := \langle g_0, g_1, \ldots, g_{k-1}\rangle \in G^k$ unambiguously determines the element in G obtained by the conventionally presented but unparenthesized product $g_0 \diamond g_1 \diamond \cdots \diamond g_{k-1}$.

If we endowed F^σ with the relation, $x_0x_0 \odot x_0 \odot \approx x_0x_0x_0 \odot \odot$, then $|F^\sigma(k)/\approx| = 1$ for each $k \in \mathbf{N}$, where $F^\sigma(k)/\approx$ is the family of $\approx$–equivalence classes.

Plainly every groupoid falls between the extremes represented by $\langle F^\sigma; \odot\rangle$ on one end, and by the class of semigroups on the other. We believe that every finite nonassociative groupoid lies strictly between these extremes.

We next propose a scheme for using F^σ in order to pin down this idea.

3. The Subassociativity Type of a Groupoid

Let $\langle G; \diamond\rangle$ be an arbitrary groupoid, let $\mathbf{w} \in F^\sigma(k)$ for a given $k \in \mathbf{N}$, and let $\vec{g} := \langle g_0, g_1, g_2, \ldots\rangle \in G^\infty$. Then $\mathbf{w}(\diamond, \vec{g})$ denotes the element in G obtained by replacing in $\mathbf{w}$ the operator $\bullet$ with the operation $\diamond$, and the symbol x_i with the element g_i for each $i \in k$, and then executing the $k-1$ binary operations $\diamond$ as indicated in the modified version of $\mathbf{w}$.

Illustrative Example 3: Consider the groupoid $\langle \mathbf{Z};\, -\,\rangle$ of integers under subtraction, and the formal 5–product $\mathbf{w} := x_0x_1 \bullet x_2x_3x_4 \bullet\bullet\bullet \in F^\sigma(5)$. Let $\vec{g} := \langle 2, 7, 0, 1, -5, g_5, g_6, \ldots\rangle$. Then $\mathbf{w}(\,-\,, \vec{g}) = 2\,7\, -\, 0\,1\,(-5)\, -\, -\,-$, where we append parentheses to eliminate ambiguity. In conventional form $\mathbf{w}(\,-\,, \vec{g}) = (2-7) - (0 - (1 - (-5)))$, and hence $\mathbf{w}(\,-\,, \vec{g}) = 1$.

Definition 3.1. Let $k \geqslant 3$, and let $\{\mathbf{u}, \mathbf{v}\} \subseteq F^\sigma(k)$. Let $\langle G; \diamond\rangle$ be a groupoid. We say that $\mathbf{u}$ is $\diamond$*–equivalent* to $\mathbf{v}$, in which event we write $\mathbf{u} \approx_\diamond \mathbf{v}$, iff $\mathbf{u}(\diamond, \vec{g}) = \mathbf{v}(\diamond, \vec{g})$ for all $\vec{g} \in G^\infty$. The expressions $F^\sigma(k)/\diamond$ and $F^\sigma(k)/\approx_\diamond$ denote the family of $\approx_\diamond$–equivalence classes $[\mathbf{w}]_\diamond$ of $F^\sigma(k)$.

Definition 3.2. We call a groupoid $\langle G; \diamond\rangle$ *completely free* iff $[\mathbf{w}]_\diamond = \{\mathbf{w}\}$ for every $\mathbf{w} \in F^\sigma$.

Illustrative Example 4: Returning to the groupoid $\langle \mathbf{Z};\, -\,\rangle$ of Example 3, we easily see for $k \in \{1, 2, 3\}$ that $F^\sigma(k)/\approx_- = \{\{\mathbf{w}\} : \mathbf{w} \in F^\sigma(k)\}$. But for $k = 4$ the situation complicates slightly. As we will proceed to show, $\mathbf{u} \approx_- \mathbf{v}$ where $\mathbf{u} := x_0x_1x_2x_3 \bullet\bullet\bullet$ and where $\mathbf{v} := x_0x_1x_2 \bullet\bullet x_3 \bullet$:

Switching back and forth between RPN and ordinary terminology as convenience dictates, we note for an arbitrary $\vec{g} \in \mathbf{Z}^\infty$ that $\mathbf{u}(\,-\,, \vec{g}) = g_0g_1g_2g_3 - - - = g_0 - (g_1 - (g_2 - g_3)) = g_0 - g_1 + g_2 - g_3 = (g_0 - (g_1 - g_2)) - g_3 = g_0g_1g_2 - - g_3 - = \mathbf{v}(\,-\,, \vec{g})$. Similar calculations establish that $F^\sigma(4)/\approx_- = \{\{\mathbf{u}, \mathbf{v}\}, \{\mathbf{a}\}, \{\mathbf{b}\}, \{\mathbf{c}\}\}$, where $\mathbf{a} := x_0x_1x_2 \bullet x_3 \bullet\bullet$ and $\mathbf{b} := x_0x_1 \bullet x_2x_3 \bullet\bullet$ and $\mathbf{c} := x_0x_1 \bullet x_2 \bullet x_3 \bullet$. Thus $|F^\sigma(4)/\approx_-| = 4 < 5 = |F^\sigma(4)|$. So the groupoid $\langle \mathbf{Z};\, -\,\rangle$ is neither a semigroup, nor is it completely free.

Theorem 3.3. *Let $\langle G; \diamond\rangle$ be a groupoid, and let $\{\mathbf{s}, \mathbf{t}, \mathbf{s}', \mathbf{t}'\} \subset F^\sigma$. Let $\mathbf{s} \approx_\diamond \mathbf{s}'$, and let $\mathbf{t} \approx_\diamond \mathbf{t}'$. Then $\mathbf{st}\odot \approx_\diamond \mathbf{s}'\mathbf{t}'\odot$.*

Proof. There exist $k \in \mathbf{N}$ such that $\{\mathbf{s}, \mathbf{s}'\} \subseteq F^\sigma(k)$, and $j \in \mathbf{N}$ such that $\{\mathbf{t}, \mathbf{t}'\} \subseteq F^\sigma(j)$. Pick $\vec{g} := \langle g_0, g_1, \ldots, g_{k-1}, g_k, \ldots, g_{k+j-1}, \ldots\rangle \in G^\infty$. Let $\vec{b} := \langle g_k, g_{k+1}, \ldots, g_{k+j-1}, \ldots\rangle \in G^\infty$.

By hypothesis $\mathbf{s}(\diamond, \vec{g}) = \mathbf{s}'(\diamond, \vec{g})$ and $\mathbf{t}(\diamond, \vec{b}) = \mathbf{t}'(\diamond, \vec{b})$. Therefore

$$(\mathbf{st}\odot)(\diamond, \vec{g}) = (\mathbf{st_k}\bullet)(\diamond, \vec{g}) = [\mathbf{s}(\diamond, \vec{g})][\mathbf{t}_k(\diamond, \vec{g})]\diamond = [\mathbf{s}(\diamond, \vec{g})][\mathbf{t}(\diamond, \vec{b})]\diamond =$$

$$[\mathbf{s}'(\diamond, \vec{g})][\mathbf{t}'(\diamond, \vec{b})]\diamond = [\mathbf{s}'(\diamond, \vec{g})][\mathbf{t}'_k(\diamond, \vec{g})]\diamond = (\mathbf{s}'\mathbf{t}'_k\bullet)(\diamond, \vec{g}) = (\mathbf{s}'\mathbf{t}'\odot)(\diamond, \vec{g}),$$

with parentheses and brackets appended only to aid the reader. $\square$

Recall that $C(k)$ denotes the kth Catalan number. The following is an easy consequence of Theorem 3.3 and Illustrative Example 4.

Corollary 3.4. *If* $k \in \{1,2,3\}$ *then* $|F^\sigma(k)/\approx_- | = C(k)$. *However,* $|F^\sigma(j)/\approx_- | < C(j)$ *for every integer* $j \geqslant 4$.

Definition 3.5. For $\langle G;\diamond\rangle$ a groupoid, we define the *subassociativity type* of this groupoid to be the infinite sequence in $\mathbf{N}$, written

$$\mathbf{SaT}(\langle G;\diamond\rangle) := \langle\, |F^\sigma(k)/\approx_\diamond|\,\rangle_{k=2}^\infty .$$

For $\langle S;\cdot\,\rangle$ a semigroup, obviously $\mathbf{SaT}(\langle S;\cdot\,\rangle) = \langle 1,1,1,\ldots\rangle$.
As we remarked, $\mathbf{SaT}(\langle F^\sigma;\odot\rangle) = \langle C(n)\rangle_{n=2}^\infty$.

Theorem 3.6. $\mathbf{SaT}(\langle \mathbf{Z};\,-\,\rangle) = \langle 2^{k-2}\rangle_{k=2}^\infty$.

Proof. For each integer $k \geqslant 2$, for each $\mathbf{w} \in F^\sigma(k)$, and for each $\vec{g} \in \mathbf{Z}^k$, we observe that $\mathbf{w}(\,-\,,\vec{g}) = g_0 - g_1 \pm_1 g_2 \pm_2 g_3 \pm_3 \cdots \pm_{k-3} g_{k-2} \pm_{k-2} g_{k-1}$ for some "sign" sequence $\langle \pm_1, \pm_2, \ldots, \pm_{k-2}\rangle \in \{-,+\}^{k-2}$, where we present the expression to the right of the symbol $=$ in ordinary terminology. Indeed, since there are only 2^{k-2} distinct sign sequences of length $k-2$, we see that $|F^\sigma(k)/\approx_- | \le 2^{k-2}$. So it suffices to show that $|F^\sigma(k)/\approx_- | \not< 2^{k-2}$.

<u>Claim</u>: For every sign sequence $\langle \pm_1, \pm_2, \ldots, \pm_{k-2}\rangle \in \{-,+\}^{k-2}$, there exists $\mathbf{r} \in F^\sigma(k)$ such that $\mathbf{r}(\,-\,,\vec{g}) = g_0 - g_1 \pm_1 g_2 \pm_2 \cdots g_{k-2} \pm_{k-2} g_{k-1}$, thus "realizing" the sign sequence $\langle \pm_i\rangle_{i=1}^{k-2}$.

We argue by induction on $k \geqslant 2$. The claim is trivial for $k = 2$.

For $2 \leqslant n \in \mathbf{N}$, suppose the claim holds when $k = n$. Pick a sign sequence $\langle \pm_i\rangle_{i=1}^{n-2} \in \{-1,1\}^{n-2}$ and a sequence $\vec{h} := \langle h_0, h_1, \ldots, h_n\rangle \in \mathbf{Z}^{n+1}$. We are required only to supply $\{\mathbf{w}_-\,, \mathbf{w}_+\} \subset F^\sigma(n+1)$ such that

$$\mathbf{w}_-(\,-\,,\vec{h}) = h_0 - h_1 \pm_1 h_2 \pm_2 \cdots \pm_{n-3} h_{n-2} \pm_{n-2} h_{n-1} - h_n$$

and such that

$$\mathbf{w}_+(\,-\,,\vec{h}) = h_0 - h_1 \pm_1 h_2 \pm_2 \cdots \pm_{n-3} h_{n-2} \pm_{n-2} h_{n-1} + h_n.$$

For each positive integer $i \leqslant n-2$ let $\mp_i := -\pm_i$. And now define $\vec{p} := \langle h_0, h_1, h_2, \ldots, h_{n-2}, h_{n-1}, \ldots\rangle$ and $\vec{s} := \langle h_1, \pm_1 h_2, h_3, \ldots, h_{n-1}, h_n, \ldots\rangle$. Both $\vec{p}$ and $\vec{s}$ are sequences in $\mathbf{Z}^\infty$.

By the inductive hypothesis there exists $\mathbf{u} \in F^\sigma(n)$ such that

$$\mathbf{u}(\,-\,,\vec{p}) = h_0 - h_1 \pm_1 h_2 \pm_2 \cdots h_{n-2} \pm_{n-2} h_{n-1}.$$

Let $\mathbf{w}_- := \mathbf{u}x_0\odot$. Then, with ordinary terminology when convenient,

$$\mathbf{w}_-(\,-\,,\vec{h}) = (\mathbf{u}x_n\bullet)(\,-\,,\vec{h}) = (\mathbf{u}(\,-\,,\vec{p}))-h_n = h_0-h_1\pm_1\cdots\pm_{n-2}h_{n-1}-h_n\,.$$

Again by the inductive hypothesis there exists $\mathbf{v} \in F^\sigma(n)$ such that

$$\mathbf{v}(\,-\,,\vec{s}) = h_1 - (\pm_1 h_2) \mp_2 h_3 \mp_3 \cdots \mp_{n-2} h_{n-1} - h_n.$$

Let $\mathbf{w}_+ := x_0\mathbf{v}\odot$. Then

$$\mathbf{w}_+(\,-\,,\vec{h}) = (x_0\mathbf{v}_1\bullet)(\,-\,,\vec{h}) = h_0 - (\mathbf{v}(\,-\,,\vec{s})) =$$

$$h_0 - (h_1 - (\pm_1 h_2) \mp_2 h_3 \mp_3 \cdots \mp_{n-2} h_{n-1} - h_n) =$$

$$h_0 - h_1 \pm_1 h_2 \pm_2 h_3 \pm_3 \cdots \pm_{n-2} h_{n-1} + h_n.$$

The theorem follows. □

Corollary 3.4 applies to each groupoid $\langle G; \diamond\rangle$. If $|F^\sigma(k)/\approx_\diamond| < C(k)$ then $|F^\sigma(j)/\approx_\diamond| < C(j)$ for all $j > k$.

The subtraction of integers is an issue for the very young. Surely one ought to be able to settle every relevant question about the groupoid $\langle \mathbf{Z};\,-\,\rangle$.

Theorem 3.6 suggests further scrutiny. Since the average size

$$\frac{C(k)}{2^{k-2}}$$

of the equivalence classes $[\mathbf{w}]_- \in F^\sigma(k)/\approx_-$ increases without bound as k increases, it is reasonable to wonder how the sizes of those equivalence classes are distributed

Definition 3.7. For each $k \geqslant 2$ we say that the sequence $\langle\,\langle \nu_k(i), i\rangle\,\rangle_{i=1}^\infty$ in $\omega\times\mathbf{N}$ is the *size sequence* for k *of* $\langle\mathbf{Z};\,-\rangle$ when $F^\sigma(k)/\approx_-$ contains exactly $\nu_k(i)$ member sets of size i, for each $i \in \mathbf{N}$.

Of course in any size sequence, $\nu_k(i) > 0$ for only finitely many i. For a size sequence we list only those terms with positive first coordinates.

Here are the size sequences and other relevant numerical data about $F^\sigma(k)/\approx_-$ for the cases $k \in \{4, 5, 6\}$:

$|F^\sigma(4)| = 5$ and $|F^\sigma(4)/\approx_-| = 4$. Size sequence: $\langle 3, 1\rangle$, $\langle 1, 2\rangle$.

$|F^\sigma(5)| = 14$ and $|F^\sigma(5)/\approx_-| = 8$. Size sequence: $\langle 4, 1\rangle$, $\langle 3, 2\rangle$, $\langle 1, 4\rangle$.

$|F^\sigma(6)| = 42$ and $|F^\sigma(6)/\approx_-| = 16$. Size sequence: $\langle 5, 1\rangle$, $\langle 3, 2\rangle$, $\langle 4, 3\rangle$, $\langle 2, 4\rangle$, $\langle 1, 5\rangle$, $\langle 1, 6\rangle$.

Problem 3.8. *Specify the size sequences of* $\langle \mathbf{Z};\, - \rangle$ *for each* $k \geqslant 2$.

Observe that a groupoid $\langle G; \diamond\rangle$ is completely free if and only if $|F^\sigma(k)| = |F^\sigma(k)/\approx_\diamond|$ for all $k \in \mathbf{N}$.

Suggestive Example 5: Let the binary operation $\triangleleft$ on the set $2 := \{0,1\}$ be given by $0t\triangleleft := 1$ and $1t\triangleleft := 0$ for each $t \in 2$.

It is easily checked that the groupoid, $\langle 2;\triangleleft\rangle$, is antiassociative. Hence, $F^\sigma(3)/\approx_\triangleleft\, = \{\{x_0x_1x_2\bullet\bullet\}, \{x_0x_1\bullet x_2\bullet\}\}$. $F^\sigma(4)/\approx_\triangleleft\, := \{A, B\}$ where $|A| = 3$ and $|B| = 2$. In fact

$$A = \{x_0x_1\bullet x_2\bullet x_3\bullet,\ x_0x_1x_2\bullet x_3\bullet\bullet,\ x_0x_1x_2x_3\bullet\bullet\bullet\},$$
$$B = \{x_0x_1x_2\bullet\bullet x_3\bullet,\ x_0x_1\bullet x_2x_3\bullet\bullet\}.$$

$F^\sigma(5)/\approx_\triangleleft\, = \{C, D\}$, where C contains 8 of the elements in $F^\sigma(5)$ while D contains the other 6 formal 5–products.

Theorem 3.9. $|F^\sigma(k)/\approx_\triangleleft| = 2$ *for all* $k \geqslant 3$.

Proof. Choose $k \geqslant 3$. Our "test sequence" is $\vec{0} := \langle 0, 0, \ldots\rangle$. For $\mathbf{w} \in F^\sigma(k)$ we write $\mathbf{w} \in A_k$ to mean that $\mathbf{w}(\triangleleft, \vec{0}) = 0$, and we define $B_k := F^\sigma(k)\setminus A_k$.

For every positive integer pair $\langle i, j\rangle$ such that $i + j = k$, we have that $A_iF^\sigma(j)\odot \subset B_k$. Indeed

$$\bigcup_{i=1}^{k-1} A_iF^\sigma(k-i)\odot = B_k$$

and similarly

$$\bigcup_{i=1}^{k-1} B_iF^\sigma(k-i)\odot = A_k.$$

So $\{A_k, B_k\}$ is a partition of $F^\sigma(k)$, since

$$F^\sigma(k) = \bigcup_{i=1}^{k-1} F^\sigma(i)F^\sigma(k-i)\odot = \bigcup_{i=1}^{k-1}(A_i\dot{\cup}B_i)F^\sigma(k-i)\odot = B_k\dot{\cup}A_k.$$

It remains only to show that we were wise in our choice of $\vec{0}$ as a test sequence in 2^∞. For an arbitrary pair $\mathbf{u}$ and $\mathbf{v}$ of formal k–products we must prove that $\mathbf{u}(\triangleleft, \vec{0}) = \mathbf{v}(\triangleleft, \vec{0}) \Rightarrow \mathbf{u} \approx_\triangleleft \mathbf{v}$. So, pick any $\vec{g} \in 2^\infty$.

Recall that $\mathbf{u} = \mathbf{p}_0\mathbf{s}_0\odot$ for a unique pair $\langle \mathbf{p}_0, \mathbf{s}_0\rangle \in F^\sigma \times F^\sigma$. Likewise there is a unique $\langle \mathbf{p}_1, \mathbf{s}_1\rangle \in F^\sigma \times F^\sigma$ with $\mathbf{p}_0 = \mathbf{p}_1\mathbf{s}_1\odot$. Proceeding, we obtain a unique descending sequence of formal product prefixes of $\mathbf{u}$. There exists $\lambda := \lambda(\mathbf{u}) \in \mathbf{N}$ for which $\mathbf{p}_\lambda$ is the final and shortest term $\neq x_0$ of the sequence. Notice that $\mathbf{p}_\lambda(\triangleleft, \vec{g}) = \overline{g_0}$ where $\overline{0} := 1$ and $\overline{1} = 0$. Furthermore,

$$\mathbf{u} = \mathbf{p}_\lambda \mathbf{s}_\lambda \odot \mathbf{s}_{\lambda-1} \odot \cdots \odot \mathbf{s}_1 \odot \mathbf{s}_0 \odot .$$

Thus we see that $\mathbf{u}(\triangleleft, \vec{g}) = g_0$ if the integer $\lambda(\mathbf{u})$ is even, but that $\mathbf{u}(\triangleleft, \vec{g}) = \overline{g_0}$ if $\lambda(\mathbf{u})$ is odd. A parallel analysis holds for $\mathbf{v}$. Thus $\mathbf{u}(\triangleleft, \vec{g}) = \mathbf{v}(\triangleleft, \vec{g})$ if and only if $\mathbf{u}(\triangleleft, \vec{0}) = \mathbf{v}(\triangleleft, \vec{0})$.

So now, for $\{\mathbf{u}, \mathbf{v}\} \subseteq F^\sigma(k)$, we see that

$$\mathbf{u} \approx_\triangleleft \mathbf{v} \;\Leftrightarrow\; \mathbf{u}(\triangleleft, \vec{0}) = \mathbf{v}(\triangleleft, \vec{0}) \;\Leftrightarrow\; \lambda(\mathbf{u}) + \lambda(\mathbf{v}) \quad \text{is even} \;\Leftrightarrow$$

$$(\{\mathbf{u}, \mathbf{v}\} \subseteq A_k \,\vee\, \{\mathbf{u}, \mathbf{v}\} \subseteq B_k)\,.$$

The theorem follows. □

Theorem 3.9 generalizes to an infinite class of antiassociative groupoids $\langle n; \hat{\delta}\rangle$ for $2 \leqslant n \in \mathbf{N}$. Indeed, the groupoid in Theorem 3.9 is the smallest example of the sort we will call "vertically deranged".

The expression $\mathrm{Sym}(G)$ denotes the collection of all permutations on the set G. And $\mathrm{Drn}(G)$ denotes the set of all *derangements* of G, which are those $f \in \mathrm{Sym}(G)$ such that $x \neq xf$ for every $x \in G$, where xf denotes the image — often written $f(x)$ — of x under the function f.

We call a groupoid $\langle G; \hat{\delta}\rangle$ *vertically deranged* if there is a derangement $\delta \in \mathrm{Drn}(n)$ such that $xy\hat{\delta} := x\delta$ for every $\langle x, y\rangle \in n \times n$, and we say that δ induces $\hat{\delta}$. Remarks analogous to those below apply to the horizontally deranged groupoid $\langle G; \check{\delta}\rangle$ where $xy\check{\delta} := y\delta$.

Theorem 3.10. *Every vertically deranged groupoid is antiassociative.*

Proof. Let $\langle G; \hat{\delta}\rangle$ be vertically deranged *via* some $\delta \in \mathrm{Drn}(G)$. Let $\vec{g} \in G^3$. Then $g_0 g_1 g_2 \hat{\delta}\hat{\delta} = g_0 g_1 \delta\hat{\delta} = g_0\delta \neq g_0\delta\delta = g_0 g_1 \hat{\delta}\delta = g_0 g_1 \hat{\delta} g_2 \hat{\delta}$. □

In the interest of maximizing the size of the families $F^\sigma(k)/\approx_\diamond$ induced by the groupoids $\langle n; \diamond\rangle$ for a fixed $n \in \mathbf{N}$, it seems prudent first to consider those $\langle n; \diamond\rangle$ which are antiassociative. The vertically deranged $\langle n; \diamond\rangle$ constitute a convenient class of finite antiassociative groupoids.

If $2 \leqslant n \in \mathbf{N}$ and if δ is a cyclic permutation of n, then for $i \in \omega$ it is evident that $\delta^i \in \mathrm{Drn}(n)$ if and only if i is not a multiple of n.

Theorem 3.11. *Let $2 \leqslant n \in \mathbf{N}$. Let $\delta \in Sym(n)$ be cyclic. Then*

$$|F^\sigma(k)/\approx_{\hat{\delta}}| = \min\{k-1, n\}.$$

Proof. By Theorem 3.9 we can take it that $n \geqslant 3$.

Let $k \in \{2, \ldots, n\}$. Pick $\mathbf{w} \in F^\sigma(k)$ and $\vec{g} := \langle g_0, g_1, g_2, \ldots\rangle \in n^\infty$.

Claim One: $\mathbf{w}(\hat{\delta}, \vec{g}) = g_0\delta^i$ for some positive integer $i < k$.

First, for $k = 2$ observe that $(x_0x_0\odot)(\hat{\delta}, \vec{g}) = g_0g_1\hat{\delta} := g_0\delta =: g_0\delta^1$.

Choose $k \geqslant 3$. Suppose that whenever $2 \leqslant j < k$,

$$\mathbf{u} \in F^\sigma(j) \;\Rightarrow\; \mathbf{u}(\hat{\delta}, \vec{g}) = g_0\delta^t$$

for some $t \in \{1, 2, \ldots, j-1\}$. Factor $\mathbf{w}$ in $\langle F^\sigma, \odot\rangle$: $\mathbf{w} = \mathbf{ps}\odot$. Since $\mathbf{p} \in F^\sigma(j)$ for some $j < k$, by hypothesis there exists t with $1 \leqslant t \leqslant j-1$ such that $\mathbf{p}(\hat{\delta}, \vec{g}) = g_0\delta^t$. From earlier calculations, $(\mathbf{ps}\odot)(\hat{\delta}, \vec{g}) = \mathbf{p}(\hat{\delta}, \vec{g})\,\mathbf{s}(\hat{\delta}, \vec{b})\hat{\delta}$, where $\vec{b} := \langle g_j, g_{j+1}, \ldots\rangle$. So $\mathbf{w}(\hat{\delta}, \vec{g}) = \mathbf{p}(\hat{\delta}, \vec{g})\,\mathbf{s}(\hat{\delta}, \vec{g})\hat{\delta} = \mathbf{p}(\hat{\delta}, \vec{g})\delta = g_0\delta^t\delta = g_0\delta^{t+1}$. Furthermore, $t+1 \leqslant j \leqslant k-1$. Claim One is established.

Claim Two: Since $k \leqslant n$, for every $i < k$ there exists $\mathbf{v} \in F^\sigma(k)$ such that $\mathbf{v}(\hat{\delta}, \vec{g}) = g_0^i$.

Pick an appropriate i. Let $\mathbf{v} := x_0x_1 \bullet x_2 \bullet x_3 \bullet \cdots \bullet x_{i-1} \bullet \mathbf{r} \odot$ where $\mathbf{r} \in F^\sigma(k-i)$. Then $\mathbf{v}(\hat{\delta}, \vec{g}) =$

$$g_0g_1\hat{\delta}g_2\hat{\delta}\cdots\hat{\delta}g_{i-2}\hat{\delta}g_{i-1}\hat{\delta}\delta = g_0\delta g_2\hat{\delta}\cdots g_{i-2}\hat{\delta}g_{i-1}\hat{\delta}\delta =$$
$$g_0\delta^2g_3\hat{\delta}\cdots\hat{\delta}g_{i-1}\hat{\delta}\delta = \cdots = g_0\delta^{i-1}\delta = g_0\delta^i.$$

Claim Two is established.

If $k > n+1$ then $\delta^{k-1} = \delta^j$ for some $j \in \{1, 2, \ldots, n\}$. In the light of Claims One and Two each of the n distinct elements $g_0\delta^j \in n$ determines a distinct equivalence class $[\mathbf{w}]_{\hat{\delta}} \in F^\sigma(k)/\approx_{\hat{\delta}}$, and all of the elements in $F^\sigma(k)/\approx_{\hat{\delta}}$ are thus determined if $k > n$. □

Corollary 3.12. *If $C(k) > n$ then there exist distinct formal k–products $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u} \approx_{\hat{\delta}} \mathbf{v}$.*

Proof. $|F^\sigma(k)| = C(k)$. Therefore if $C(k) > n$ then the pigeonhole principle applies, since $|F^\sigma(k)/\approx_{\hat{\delta}}| \leqslant n$ by Theorem 3.11. □

Conjecture 3.13. *No finite groupoid is completely free.*

Under the assumption that our conjecture is correct, it becomes relevant to raise the following question:

Problem 3.14. *Given $2 \leqslant n \in \mathbf{N}$, what is the smallest integer $\tau(n)$ such that, for every integer $k \geqslant \tau(n)$ and for every groupoid $\langle n; \diamond\rangle$, there exist elements $\mathbf{a} \neq \mathbf{b}$ in $F^\sigma(k)$ for which $\mathbf{a} \approx_\diamond \mathbf{b}$?*

4. k–Anti-Associativity

In RPN an ordered triple $\langle x, y, z\rangle$ of elements in G associates under μ iff $xy\mu z\mu = xyz\mu\mu$. RPN confers other conveniences besides relieving us of parenthesis jungles. We use it to express the complex products involving the nonassociative binary operations of concern in this section.

If the groupoid $\langle G; \mu\rangle$ happens to be a semigroup then, for every pair $\{\mathbf{p}, \mathbf{s}\} \subseteq F^\sigma(k)$ of formal k–products, we get that $\mathbf{p}(\mu, \vec{g}) = \mathbf{s}(\mu, \vec{g})$ whenever $\vec{g} \in G^\infty$. That is, in a semigroup, all formal k–products are μ–equivalent. So we focus on non-semigroups. We seek groupoids which are, indeed, "as anti-associative as possible". The following remarks elaborate.

The concept of k–anti-associativity, as it pertains to a groupoid $\langle G; \mu\rangle$, is trivial for $1 \leqslant k \leqslant 2$. Henceforth we take it that $k \geqslant 3$.

$\langle G; \mu\rangle$ is 3–*anti-associative* iff $xyz\mu\mu \neq xy\mu z\mu$ for every ordered triple $\langle x, y, z\rangle \in G^3$. Theorem 3.10, and a comment preceding it on horizontal derangements, provides $2 \cdot |\mathrm{Drn}(n)|$ distinct 3–anti-associative groupoids on the set n, when $2 \leqslant n \in \mathbf{N}$.

$\langle G; \mu\rangle$ is 4–*anti-associative* iff, for every $\vec{g} := \langle g_0, g_1, g_2, g_3, \ldots\rangle \in G^\infty$, the subset,

$$\{g_0g_1g_2g_3\mu\mu\mu,\ g_0g_1g_2\mu g_3\mu\mu,\ g_0g_1\mu g_2g_3\mu\mu,\ g_0g_1g_2\mu\mu g_3\mu,\ g_0g_1\mu g_2\mu g_3\mu\},$$

of G is 5–membered. If $\langle G; \mu\rangle$ is 4–anti-associative, clearly $|G| \geqslant 5$.

Do there exist 4–anti-associative groupoids?

Definition 4.1. $\langle G; \mu\rangle$ is *k–anti-associative* iff $\mathbf{u}(\mu, \vec{g}) \neq \mathbf{v}(\mu, \vec{g})$ for all $\langle \mathbf{u}, \mathbf{v}, \vec{g}\rangle \in F^\sigma(k) \times F^\sigma(k) \times G^\infty$ with $\mathbf{u} \neq \mathbf{v}$.

Theorem 4.2. *The groupoid $\langle F^\sigma; \odot\rangle$ is k–anti-associative if $k \geqslant 3$.*

Proof. Fix $k \geqslant 3$. We normally use $\langle F^\sigma; \odot\rangle$ as a tool for evaluating the subassociativity of other groupoids. Since our argument here requires $\langle F^\sigma; \odot\rangle$ itself to be evaluated, we relabel this groupoid *qua* instrument in order to distinguish it from the same groupoid *qua* entity scrutinized.

$\langle \overline{F}^\sigma; \overline{\odot}\rangle$ is the tool version; its elements are words in the alphabet $\{\overline{\bullet}, \overline{x}_0, \overline{x}_1, \overline{x}_2, \ldots\}$. Let $\vec{\mathbf{g}} := \langle \mathbf{g}0, \mathbf{g}1, \mathbf{g}2, \ldots\rangle \in (F^\sigma)^\infty$ be any infinite sequence of finite formal products. We must prove that $\overline{\mathbf{u}}(\odot, \vec{\mathbf{g}}) \neq \overline{\mathbf{v}}(\odot, \vec{\mathbf{g}})$ for every $\{\overline{\mathbf{u}}, \overline{\mathbf{v}}\} \subseteq \overline{F}^\sigma(k)$ with $\overline{\mathbf{u}} \neq \overline{\mathbf{v}}$.

For $\{\mathbf{r}, \mathbf{s}\} \subseteq F^\sigma$, recall that $\mathbf{r} = \mathbf{s}$ iff $\mathbf{r}$ and $\mathbf{s}$ are spelled alike as finite words in the infinite alphabet $\{\bullet, x_0, x_1, x_2, \ldots\}$, in which event there exists $j \in \mathbf{N}$ such that $\{\mathbf{r}, \mathbf{s}\} \subseteq F^\sigma(j)$.

Now choose any $\langle \overline{\mathbf{u}}, \overline{\mathbf{v}} \rangle \in \overline{F}^{\sigma}(k) \times \overline{F}^{\sigma}(k)$ such that $\overline{\mathbf{u}} \neq \overline{\mathbf{v}}$. Obviously $|\overline{\mathbf{u}}(\odot, \vec{\mathbf{g}})| = |\overline{\mathbf{v}}(\odot, \vec{\mathbf{g}})|$. So, in order to prove that $\overline{\mathbf{u}}(\odot, \vec{\mathbf{g}}) \neq \overline{\mathbf{v}}(\odot, \vec{\mathbf{g}})$, we must show that the words $\overline{\mathbf{u}}(\odot, \vec{\mathbf{g}})$ and $\overline{\mathbf{v}}(\odot, \vec{\mathbf{g}})$ are spelled differently.

The following remarks should be viewed in the light of Definition 2.2 and the material between that definition and the statement of Theorem 2.3.

Among the possibly many occurrences of the letter $\bullet$ in the word $\overline{\mathbf{u}}(\odot, \vec{g}) \in F^{\sigma} \subseteq \{\bullet, x_0, x_1, x_2, \ldots\}^*$ are exactly $k-1$ of them which derive from transformations of $\odot$ into $\bullet$. The same is true of the word $\overline{\mathbf{v}}(\odot, \vec{\mathbf{g}}) \in F^{\sigma}$. We tag those crucial occurrences of $\bullet$ in order to keep track of them: We write them as $\bullet'$.

If we removed all of the $k-1$ occurrences of $\bullet'$ from $\overline{\mathbf{u}}(\odot, \vec{\mathbf{g}})$, and all of the $k-1$ occurrences of $\bullet'$ from $\overline{\mathbf{v}}(\odot, \vec{\mathbf{g}})$, then the two resulting shortened words would be identical. It is the differing placements of those $k-1$ vagrant tagged $\bullet'$ in $\overline{\mathbf{u}}(\odot, \vec{\mathbf{g}})$ and in $\overline{\mathbf{v}}(\odot, \vec{\mathbf{g}})$ that make the two words differ in their spellings. We now establish this orthographic distinction.

Since by hypothesis $\overline{\mathbf{u}} \neq \overline{\mathbf{v}}$, there is a smallest integer m for which $m = |\overline{\mathbf{p}}_{\overline{\mathbf{u}}}| = |\overline{\mathbf{p}}_{\overline{\mathbf{v}}}|$, but for which $\overline{\mathbf{p}}_{\overline{\mathbf{u}}} \neq \overline{\mathbf{p}}_{\overline{\mathbf{v}}}$, where $\overline{\mathbf{p}}_{\overline{\mathbf{u}}}$ and $\overline{\mathbf{p}}_{\overline{\mathbf{v}}}$ are prefixes respectively of $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$. Let $\mathbf{p_u}$ be the prefix that is generated in $\mathbf{u}(\odot, \vec{g})$ from $\overline{\mathbf{p}}_{\overline{\mathbf{u}}}$ under the mapping $\overline{\mathbf{u}} \mapsto \overline{\mathbf{u}}(\odot, \vec{\mathbf{g}})$. Let $\mathbf{p_v}$ be similarly obtained from $\overline{\mathbf{p}}_{\overline{\mathbf{v}}}$.

Since surely both $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$ have $\overline{x}_0$ as a prefix, we have that $m \geqslant 2$. Furthermore, by our choice of m, if $\overline{\mathbf{q}}$ is a prefix of $\overline{\mathbf{p}}_{\overline{\mathbf{u}}}$ with $|\overline{\mathbf{q}}| = m-1$, then $\overline{\mathbf{q}}$ is a prefix also of $\overline{\mathbf{p}}_{\overline{\mathbf{v}}}$. Therefore the length-one suffix of $\overline{\mathbf{p}}_{\overline{\mathbf{u}}}$ differs from the length-one suffix of $\overline{\mathbf{p}}_{\overline{\mathbf{v}}}$.

Without loss of generality we suppose that $\overline{\mathbf{p}}_{\overline{\mathbf{u}}}$ has $\overline{\bullet}$ as its length-one suffix. Then $\mathbf{p_u}$ has $\bullet'$ as its length-one suffix. Moreover, $\overline{\mathbf{p}}_{\overline{\mathbf{v}}}$ has some $\overline{x}_c$ as its length-one suffix. There are two cases.

Case: $\mathbf{g}c = x_0$. Then $|\mathbf{p_u}| = |\mathbf{p_v}|$ and $\mathbf{p_v}$ has a length-one suffix of the sort $x_d \notin \{\bullet', \bullet\}$. So the length–$|\mathbf{p_u}|$ prefix of $\overline{\mathbf{u}}(\odot, \vec{\mathbf{g}})$ differs from the length–$|\mathbf{p_u}|$ prefix of $\overline{\mathbf{v}}(\odot, \vec{\mathbf{g}})$, whence $\overline{\mathbf{u}}(\odot, \vec{\mathbf{g}}) \neq \overline{\mathbf{v}}(\odot, \vec{\mathbf{g}})$.

Case: $|\mathbf{g}c| \geqslant 3$, and so $|\mathbf{p_u}| < |\mathbf{p_v}|$. Then the length–$|\mathbf{p_u}|$ prefix of $\overline{\mathbf{v}}(\odot, \vec{\mathbf{g}})$ has some $x_d \notin \{\bullet', \bullet\}$ as its length-one suffix. So the words $\overline{\mathbf{u}}(\odot, \vec{\mathbf{g}})$ and $\overline{\mathbf{v}}(\odot, \vec{\mathbf{g}})$ have distinct length–$|\mathbf{p_u}|$ prefixes, and are therefore themselves distinct.

These cases are exhaustive, and in both cases $\overline{\mathbf{u}}(\odot, \vec{\mathbf{g}}) \neq \overline{\mathbf{v}}(\odot, \vec{\mathbf{g}})$. □

Illustrative Example 6: Let $\mathbf{g}0 := x_0 x_1 \bullet$ and $\mathbf{g}1 := \mathbf{g}2 := x_0$ and $\mathbf{g}3 := x_0 x_1 \bullet x_2 \bullet$ be the first four terms in a sequence $\vec{\mathbf{g}} \in (F^{\sigma})^{\infty}$ of formal products. Consider the actions on $\vec{\mathbf{g}}$ of two elements $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$ in the set $\overline{F}^{\sigma}(4)$; to wit, the words $\overline{\mathbf{u}} := \overline{x}_0\overline{x}_1\overline{\bullet}\,\overline{x}_2\overline{x}_3\overline{\bullet}\,\overline{\bullet}$ and $\overline{\mathbf{v}} := \overline{x}_0\overline{x}_1\overline{x}_2\overline{x}_3\overline{\bullet}\,\overline{\bullet}\,\overline{\bullet}$.

Now $\overline{\mathbf{u}}(\odot, \vec{\mathbf{g}}) = \mathbf{g}0\,\mathbf{g}1 \odot \mathbf{g}2\,\mathbf{g}3 \odot \odot = x_0x_1 \bullet x_0 \odot x_0x_0x_1 \bullet x_2 \bullet \odot\odot = x_0x_1 \bullet x_2 \bullet'\, x_0x_1x_2 \bullet x_3 \bullet \bullet' \odot = x_0x_1 \bullet x_2 \bullet'\, x_3x_4x_5 \bullet x_6 \bullet \bullet'\bullet'$, with the tags $'$ appended to those instances in the word $\overline{\mathbf{u}}(\odot, \vec{\mathbf{g}})$ of the letter $\bullet$ which came from transformed operator symbols $\odot$, which in their turn replaced the occurrences of the letter $\overline{\bullet}$ in the word $\overline{\mathbf{u}}$. In summary,

$$\overline{\mathbf{u}}(\odot, \vec{\mathbf{g}}) = x_0x_1 \bullet x_2 \bullet'\, x_3x_4x_5 \bullet x_6 \bullet \bullet'\, \bullet' .$$

Likewise, $\overline{\mathbf{v}}(\odot, \vec{\mathbf{g}}) = \mathbf{g}0\,\mathbf{g}1\,\mathbf{g}2\,\mathbf{g}3 \odot \odot \odot$, and so eventually

$$\overline{\mathbf{v}}(\odot, \vec{\mathbf{g}}) = x_0x_1 \bullet x_2x_3x_4x_5 \bullet x_6 \bullet \bullet'\, \bullet'\, \bullet' .$$

Notice: $\overline{\mathbf{u}}(\odot, \vec{\mathbf{g}}) \neq \overline{\mathbf{v}}(\odot, \vec{\mathbf{g}})$ because the $\bullet'$ occur differently in each word.

Since $\langle F^{\sigma}; \odot\rangle$ achieves the theoretical extreme of anti-associativity, and since in a semigroup everything is k–associative for every k, we imagine a hierarchy of groupoids between these extremes. Of course, the set F^{σ} is infinite, rendering anti-associativity fairly straightforward to produce.

Recall that $|F^{\sigma}(k)| = C(k)$. Thus

Theorem 4.3. *If $C(k) > n$, no groupoid $\langle n; \diamond\rangle$ is k–anti-associative. So, no finite groupoid is k–anti-associative for every $k \in \mathbf{N}$.*

Problem 4.4. *For each $k \geqslant 3$ is there some $n := n(k) \in \mathbf{N}$ and some $\beta : n \times n \to n$ such that the groupoid $\langle n; \beta\rangle$ is k–anti-associative?*

Our inquiry refines and extends in natural ways. Here is one:

For integers $n \geqslant 2$ and $k \geqslant 3$ and a binary operation $\diamond : n^2 \to n$ let

$$\Psi(n, k, \diamond) := |\{\vec{g}{\restriction}k : \vec{g} \in n^{\infty} \wedge \forall\{\mathbf{u}, \mathbf{v}\} \subseteq F^{\sigma}(k)\,(\,\mathbf{u}(\diamond, \vec{g}) = \mathbf{v}(\diamond, \vec{g})\,)\}|.$$

Given an arbitrary rational number $q \in [0, 1]$ does there exist a relevant triple $\langle n, k, \diamond\rangle$ such that

$$q = \frac{\Psi(n, k, \diamond)}{n^k}\,?$$

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References

[1] **J. H. Folkman and R. L. Graham**: *On highly non-associative groupoids*, Colloq. Mathematicum **24** (1972), $1-10$.

[2] **H. W. Gould**: *Research Bibliography of Two Special Sequences*, Combinatorial Research Institute, West Virginia University, Morgantown, 1977.

[3] **I. M. Niven**: *Mathematics of Choice or How to Count without Counting*, final chapter. New Mathematical Library, Vol. **15**, 1965. Random House, New York.

[4] **D. M. Silberger**: *Occurrences of the integer* $(2n-2)!/n!(n-1)!$, Prace Mat. **13** (1969), $91-96$.

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