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Skew endomorphisms on n-ary groups

Nikolay A. Shchuchkin

Abstract

Let $\overline{x}^{(k)}$ denote this element of an n-ary group G which is skew to $\overline{x}^{(k-1)}$, where $k \geqslant 1$ and $\overline{x}^{(0)} = x$. We find the identities defining the variety of all n-ary groups for which the operation $^{-(k)} : x \mapsto \overline{x}^{(k)}$ is an endomorphism.

1. Introduction

According to the general convention used in the theory of n-ary systems the sequence of elements $x_i, x_{i+1}, \ldots, x_j$ will be denoted by x_i^j. In the case $j < i$ it will be the empty symbol. If $x_{i+1} = x_{i+2} = \ldots = x_{i+t} = x$, then instead of x_{i+1}^{i+t} we shall write $\overset{(t)}{x}$. In this convention $f(x_1, \ldots, x_n) = f(x_1^n)$ and

$$f(x_1, \ldots, x_i, \underbrace{x, \ldots, x}_{t}, x_{i+t+1}, \ldots, x_n) = f(x_1^i, \overset{(t)}{x}, x_{i+t+1}^n).$$

If $m = k(n-1)+1$, then the m-ary operation g of the form

$$g(x_1^{k(n-1)+1}) := \underbrace{f(f(\ldots,f(f}_{k}(x_1^n), x_{n+1}^{2n-1}), \ldots), x_{(k-1)(n-1)+2}^{k(n-1)+1})$$

will be denoted by $f_{(k)}$. In certain situations, when the arity of g does not play a crucial role or when it will differ depending on additional assumptions, we will write $f_{(.)}$ to mean $f_{(k)}$ for some $k = 1, 2, \ldots$.

For $n \geqslant 3$, there are several equivalent definitions of an n-ary group (see for example, [2], [6], [8], [10]). The definition given in [1] generalizes the definition of a binary group as follows:

The algebra $\langle G, f \rangle$ with the n-ary operation f is called an *n-ary group* if for every $i = 1, 2, \ldots, n$ the following two conditions are satisfied:

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1. the operation f satisfies the general associative law:

$$f(f(x_1^n), x_{n+1}^{2n-1}) = f(x_1^i, f(x_{i+1}^{i+n}), x_{i+n+1}^{2n-1}), \tag{1}$$

2. the equation $f(a_1^{i-1}, x, a_{i+1}^n) = b$ has a unique solution $x \in G$ for all $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, b \in G^n$.

An algebra $\langle G, f\rangle$ satisfying (1) for all $i = 1, 2, \ldots, n$ is called an *n-ary semigroup*.

In an n-ary group $\langle G, f\rangle$ the solution z of the equation

$$f(\overset{(n-1)}{x}, z) = x,$$

is denoted by $\overline{x}$ and is called the *skew element* of x.

One can prove (see for example [1]) that

$$f(\overset{(i-1)}{x}, \overline{x}, \overset{(n-i)}{x}) = x, \quad 1 \leqslant i \leqslant n,$$

$$f(y\overset{(n-j-1)}{x}, \overline{x}, \overset{(j-1)}{x}) = y, \quad 1 \leqslant j \leqslant n-1 \tag{2}$$

$$f(\overset{(i-1)}{x}, \overline{x}, \overset{(n-i-1)}{x}, y) = y, \quad 1 \leqslant i \leqslant n-1 \tag{3}$$

for all $x, y \in G$.

Identities (1), (2) and (3) can be used as identities defining the variety of all n-ary groups (see [2], [6], [8], [10]).

For example, in [6] the following theorem is proved.

Theorem 1.1. *An n-ary $(n > 2)$ semigroup $\langle G, f\rangle$ with the unary operation $^{-} : x \to \overline{x}$ is an n-ary group if and only if the identities* (2) *and* (3) *hold in G for some $1 \leqslant i, j \leqslant n-1$.*

Following Post [11], we say that two sequences a_1^{n-1} and $b_1^{k(n-1)}$ of elements of G are *equivalent* in an n-ary group $\langle G, f\rangle$ if the equation

$$f(x, a_1^{n-1}) = f_{(k)}(x, b_1^{k(n-1)}) \tag{4}$$

is valid for some $x \in G$.

Lemma 1.2. *If in an n-ary group $\langle G, f\rangle$ the sequences a_1^{n-1} and $b_1^{k(n-1)}$ are equivalent, then the equation* (4) *is valid for all $x \in G$.*

Proof. Indeed, if this equality holds for some $x, a_1^{n-1}, b_1^{k(n-1)} \in G$, then

$$f(y, \overset{(n-3)}{x}, \overline{x}, f(x, a_1^{n-1})) = f(y, \overset{(n-3)}{x}, \overline{x}, f_{(k)}(x, b_1^{k(n-1)}))$$

is valid for all $y \in G$. Whence, according to the associativity of f, we obtain

$$f(f(y, \overset{(n-3)}{x}, \overline{x}, x), a_1^{n-1}) = f_{(k)}(f(y, \overset{(n-3)}{x}, \overline{x}, x), b_1^{k(n-1)}).$$

This, by (2), implies

$$f(y, a_1^{n-1}) = f_{(k)}(y, b_1^{k(n-1)}),$$

which completes the proof. □

2. Skew endomorphisms

W. A. Dudek posed in ([5]) several problems on the operation $^{-} : x \to \overline{x}$ on n-ary groups. He asks (see also [4]) when this operation is an endomorphism, i.e., in which n-ary groups the identity

$$\overline{f(x_1^n)} = f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n) \tag{5}$$

is satisfied.

The partial answer was given in [5]. Other answer is given in [13]. Namely, in [13] the following theorem is proved.

Theorem 2.1. *The operation* $^{-} : x \to \overline{x}$ *is an endomorphism of an n-ary group* $\langle G, f\rangle$ *if and only if*

$$f(f(x, \overset{(n-1)}{u}, y), \ldots, f(x, \overset{(n-1)}{u}, y), \overset{(2)}{u}) = f(\overset{(n-1)}{y}, f(u, f(x, \overset{(n)}{u}), \ldots, f(x, \overset{(n)}{u}), x, u), u)$$

and

$$f(\overset{n}{u}, f(\overset{n-1}{x}, u, u)) = f(f(\overset{n-1}{x}, u, u), \overset{n}{u})$$

hold for all $x, y, u \in G$.

It is clear that $^{-} : x \to \overline{x}$ is an endomorphism in all commutative n-ary groups. Obviously, it is an endomorphism in all idempotent (also non-commutative) n-ary groups. Głazek and Gleichgewicht proved in [9] that it is an endomorphism in all *medial* n-ary groups, i.e., in n-ary groups satisfying the identity

$$f(\{f(x_{i1}^{in})\}_{i=1}^{i=n}) = f(\{f(x_{1i}^{ni})\}_{i=1}^{i=n}). \tag{6}$$

One can prove (see [2]) that an n-ary group $\langle G, f\rangle$ is medial if there exists an element $a \in G$ such that

$$f(x, \overset{(n-2)}{a}, y) = f(y, \overset{(n-2)}{a}, x) \tag{7}$$

holds for all $x, y \in G$.

Using (7) and the associativity of the operation f it is not difficult to verify that the following theorem is true.

Theorem 2.2. *Each medial n-ary group satisfies the identity*

$$\begin{aligned} f_{(n-1)}(x_1, \overset{(n-2)}{x_2}, \overset{(n-2)}{x_3}, \ldots, \overset{(n-2)}{x_{n+1}}, x_{n+2}) = \\ f(x_1, \underbrace{f(x_{n+1}, x_n, ..., x_2), \ldots, f(x_{n+1}, x_n, ..., x_2)}_{n-2 \quad times}, x_{n+2}). \end{aligned} \tag{8}$$

The identity (8) describes the class of n-ary groups for which $\bar{}: x \to \overline{x}$ is an endomorphism.

Theorem 2.3. *The operation* $\bar{}: x \to \overline{x}$ *is an endomorphism of an n-ary group* $\langle G, f\rangle$ *if and only if* $\langle G, f\rangle$ *satisfies* (8).

Proof. Let $\bar{}: x \to \overline{x}$ be an endomorphism of an n-ary group $\langle G, f\rangle$, i.e., let (5) be satisfied. Then, according to (2) and (3), for any $x_2^{n+2} \in G$ we have

$$f_{(n-1)}(f(\overline{x}_{n+1}, \overline{x}_n, \ldots, \overline{x}_2), \overset{(n-2)}{x_2}, \overset{(n-2)}{x_3}, \ldots, \overset{(n-2)}{x_{n+1}}, x_{n+2}) = x_{n+2}$$

and

$$f(\overline{f(x_{n+1}, x_n, ..., x_2)}, \underbrace{f(x_{n+1}, x_n, ..., x_2), \ldots, f(x_{n+1}, x_n, ..., x_2)}_{n-2 \quad times}, x_{n+2}) = x_{n+2}$$

for all elements $x_2^{n+1} \in G$, which, by (5), means that the sequences $\overset{(n-2)}{x_2}, \overset{(n-2)}{x_3}, ..., \overset{(n-2)}{x_{n+1}}, x_{n+2}$ and $\underbrace{f(x_{n+1}, x_n, ..., x_2), ..., f(x_{n+1}, x_n, ..., x_2)}_{n-2 \quad times}, x_{n+2}$ are equivalent. So, in view of Lemma 1.1, the equality (8) is valid for all $x_1^{n+1} \in G$.

Conversely, let (8) be satisfied in an n-ary group $\langle G, f\rangle$. Then putting $x_1 = f(\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_n)$, $x_{n+2} = \overline{f(y_1^n)}$ and $x_k = y_{n+2-k}$ for $2 \leqslant k \leqslant n+1$ we see that the left hand side of (8) has the form

$$f_{(n-1)}(f(\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_n), \overset{(n-2)}{y_n}, \overset{(n-2)}{y_{n-1}}, \ldots, \overset{(n-2)}{y_1}, \overline{f(y_1^n)}) = \overline{f(y_1^n)}.$$

On the right side of (8) we obtain

$$f(f(\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_n), \underbrace{f(y_1^n), \ldots, f(y_1^n)}_{n-2 \quad times}, \overline{f(y_1^n)}) = f(\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_n).$$

So, $\overline{f(y_1^n)} = f(\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_n)$ for all $y_1^n \in G$. This completes the proof. $\square$

This theorem proves that the converse of Theorem 2.2 is not true. Indeed, in any idempotent n-ary group the operation $^- : x \to \overline{x}$ is the identity endomorphism but not any idempotent n-ary group is medial [11].

Let $\overline{x}^{(k)}$ be the skew element to $\overline{x}^{(k-1)}$, where $k \geqslant 1$ and $\overline{x}^{(0)} = x$, i.e., let $\overline{x}^{(1)} = \overline{x}$, $\overline{x}^{(2)} = \overline{\overline{x}}$, and so on. If $^- : x \to \overline{x}$ is an endomorphism of an n-ary group $\langle G, f\rangle$, then obviously $^{-(k)} : x \to \overline{x}^{(k)}$ is an endomorphism too. In some cases it is an automorphism (see [4] and [5]). However, the converse is not true. For example, in all ternary groups $\overline{\overline{x}} = x$, i.e., the operation $^{-(2)} : x \to \overline{\overline{x}}$ is the identity endomorphism, but in a ternary group $\langle S_3, f\rangle$ defined on the symmetric group S_3, where f is the composition on three permutations, we have

$$\overline{f((12),(13),(123))} \neq (132) = f(\overline{(12)}, \overline{(13)}, \overline{(132)}).$$

Hence $^- : x \to \overline{x}$ is not an endomorphism of this group.

Since in ternary groups $\overline{\overline{x}} = x$ for all x, we have $\overline{x}^{(k)} = x$ if k is even, and $\overline{x}^{(k)} = \overline{x}$ if k is odd. Therefore, the operation $^{-(k)} : x \to \overline{x}^{(k)}$ is the identity endomorphism or coincides with the operation $^- : x \to \overline{x}$. From the last theorem it follows that $^- : x \to \overline{x}$ is an endomorphism of a ternary group if and only if this group is medial. In this case $^- : x \to \overline{x}$ is an automorphism.

Other important properties of operations $^{-(k)} : x \to \overline{x}^{(k)}$ in n-ary groups satisfying some additional properties are described in [3] and [4].

Following Post [11] an *n-ary power* of an element x in an n-ary group $\langle G, f\rangle$ is defined as $x^{<0>} = x$ and $x^{<k+1>} = f(\overset{(n-1)}{x}, x^{<k>})$ for all $k > 0$.

In this convention $x^{<-k>}$ means $z \in G$ such that $f(x^{<k-1>}, \overset{(n-2)}{x}, z) = x^{<0>} = x$.

It is not difficult to verify that the following exponential laws hold

$$f(x^{<s_1>}, x^{<s_2>}, \ldots, x^{<s_n>}) = x^{<s_1+s_2+\ldots+s_n+1>},$$
$$(x^{<r>})^{<s>} = x^{<rs(n-1)+s+r>} = (x^{<s>})^{<r>}.$$

Using the above laws we can see that $\overline{x} = x^{<-1>}$ and, consequently

$$\overline{x}^{(2)} = (x^{<-1>})^{<-1>} = x^{<n-3>},$$
$$\overline{x}^{(3)} = ((x^{<-1>})^{<-1>})^{<-1>},$$

and so on. Generally: $\overline{x}^{(k)} = (\overline{x}^{(k-1)})^{<-1>}$ for all $k \geqslant 1$. This implies (see [3] or [4]) that $\overline{x}^{(k)} = x^{<S_k>}$ for

$$S_k = -\sum_{i=0}^{k-1}(2-n)^i = \frac{(2-n)^k - 1}{n-1}\,.$$

For even k we have $S_k = \frac{(n-2)^k - 1}{n-1}$. Hence

$$\overline{x}^{(k)} = f_{(\cdot)}(\overset{((n-2)^k)}{x}) \tag{9}$$

for even k. In particular $\overline{\overline{x}} = x^{<n-3>} = f_{(n-3)}(\overset{((n-2)^2)}{x})$. Thus the operation ${}^{-(k)} : x \to \overline{x}^{(k)}$ coincides with the operation ${}^{<S_k>} : x \to x^{<S_k>}$. So, the operation ${}^{-(k)} : x \to \overline{x}^{(k)}$ is an endomorphism if and only if

$$f(x_1^n)^{<S_k>} = f(x_1^{<S_k>}, x_2^{<S_k>}, \ldots, x_n^{<S_k>})$$

is valid for all $x_1^n \in G$. This implies

Theorem 2.4. *For even k the operation ${}^{-(k)} : x \to \overline{x}^{(k)}$ is an endomorphism of an n-ary group $\langle G, f\rangle$ if and only if the identity*

$$f_{(\cdot)}(\underbrace{f(x_1^n), \ldots, f(x_1^n)}_{(n-2)^k}) = f_{(\cdot)}(\overset{((n-2)^k)}{x_1}, \overset{((n-2)^k)}{x_2}, \ldots, \overset{((n-2)^k)}{x_n})$$

is satisfied.

Theorem 2.5. *For odd k the operation $\overline{}^{(k)} : x \to \overline{x}^{(k)}$ is an endomorphism of an n-ary group $\langle G, f\rangle$ if and only if the identity*

$$f_{(\cdot)}(x_1, \overset{((n-2)^k)}{x_2}, \overset{((n-2)^k)}{x_3}, \ldots, \overset{((n-2)^k)}{x_{n+1}}, x_{n+2}) = \\ f_{(\cdot)}(x_1, \underbrace{f(x_{n+1}, x_n, \ldots, x_2), \ldots, f(x_{n+1}, x_n, \ldots, x_2)}_{(n-2)^k}, x_{n+2}), \qquad (10)$$

is satisfied.

Proof. Let k be odd and let $\overline{}^{(k)} : x \to \overline{x}^{(k)}$ be an endomorphism of an n-ary group $\langle G, f\rangle$. From (2), (3) we get

$$f_{(\cdot)}(y, \overset{((n-2)^k)}{x}, \overset{((n-2)^{k-1})}{\overline{x}}) = f_{(\cdot)}(y, \overset{(n-2)}{x}, \overline{x}, \ldots, \overset{(n-2)}{x}, \overline{x}) = y, \qquad (11)$$

$$f_{(\cdot)}(\overset{((n-2)^{k-1})}{\overline{x}}, \overset{((n-2)^k)}{x}, y) = f_{(\cdot)}(\overline{x}, \overset{(n-2)}{x}, \ldots, \overline{x}, \overset{(n-2)}{x}, y) = y. \qquad (12)$$

Consequently

$$f_{(\cdot)}(f_{(\cdot)}(\overset{((n-2)^{k-1})}{\overline{x}_{n+1}}, \overset{((n-2)^{k-1})}{\overline{x}_n}, \ldots, \overset{((n-2)^{k-1})}{\overline{x}_2}), \overset{((n-2)^k)}{x_2}, \ldots, \overset{((n-2)^k)}{x_{n+1}}, x_{n+2}) = x_{n+2}$$

and

$$f_{(\cdot)}(\underbrace{\overline{f(x_{n+1}, x_n, \ldots, x_2)}, \ldots, \overline{f(x_{n+1}, x_n, \ldots, x_2)}}_{(n-2)^{k-1}}, \\ \underbrace{f(x_{n+1}, x_n, \ldots, x_2), \ldots, f(x_{n+1}, x_n, \ldots, x_2)}_{(n-2)^k}, x_{n+2}) = x_{n+2}.$$

Since $k-1$ is even, by (9) we have $\overline{x}^{(k)} = \overline{(\overline{x})}^{(k-1)} = f_{(\cdot)}(\overset{((n-2)^{k-1})}{\overline{x}})$ for all $x \in G$. Thus

$$\overline{f(x_{n+1}, x_n, \ldots, x_2)}^{(k)} = f_{(\cdot)}(\underbrace{\overline{f(x_{n+1}, x_n, \ldots, x_2)}, \ldots, \overline{f(x_{n+1}, x_n, \ldots, x_2)}}_{(n-2)^{k-1}})$$

and

$$f_{(\cdot)}(\overset{((n-2)^{k-1})}{\overline{x}_{n+1}}, \overset{((n-2)^{k-1})}{\overline{x}_n}, \ldots, \overset{((n-2)^{k-1})}{\overline{x}_2}) = f_{(\cdot)}(\overline{x}_{n+1}^{(k)}, \overline{x}_n^{(k)}, \ldots, \overline{x}_2^{(k)}),$$

whence

$$f_{(\cdot)}(\overset{((n-2)^{k-1})}{\overline{x}_{n+1}}, \overset{((n-2)^{k-1})}{\overline{x}_n}, \ldots, \overset{((n-2)^{k-1})}{\overline{x}_2}) =$$

$$f_{(\cdot)}(\underbrace{\overline{f(x_{n+1}, x_n, \ldots, x_2)}, \ldots, \overline{f(x_{n+1}, x_n, \ldots, x_2)}}_{(n-2)^{k-1}}).$$

This, together with the above two identities containing x_{n+2}, means that the sequences:

$$\overset{((n-2)^k)}{x_2}, \overset{((n-2)^k)}{x_3}, \ldots, \overset{((n-2)^k)}{x_{n+1}}, x_{n+2}$$

and

$$\underbrace{f(x_{n+1}, x_n, \ldots, x_2), \ldots, f(x_{n+1}, x_n, \ldots, x_2)}_{(n-2)^k}, x_{n+2}$$

are equivalent. Hence, by Lemma 1.2, the equality (10) is valid for all $x_1^{n+2} \in G$.

On the other hand, if (10) is valid for all $x_1^{n+2} \in G$, then for

$$x_1 = f_{(\cdot)}(\overset{((n-2)^{k-1})}{\overline{y}_1}, \overset{((n-2)^{k-1})}{\overline{y}_3}, \ldots, \overset{((n-2)^{k-1})}{\overline{y}_n}),$$

$$x_k = y_{n+2-k}, \quad \text{for} \quad k = 2, 3, \ldots, n+1,$$

$$x_{n+2} = f_{(\cdot)}(\underbrace{\overline{f(y_1^n)}, \overline{f(y_1^n)}, \ldots, \overline{f(y_1^n)}}_{(n-2)^{k-1}}) = f_{(\cdot)}(\overset{((n-2)^{k-1})}{\overline{f(y_1^n)}}),$$

it has the form

$$f_{(\cdot)}(f_{(\cdot)}(\overset{((n-2)^{k-1})}{\overline{y}_1}, \ldots, \overset{((n-2)^{k-1})}{\overline{y}_n}), \overset{((n-2)^k)}{y_n}, \ldots, \overset{((n-2)^k)}{y_2}, \overset{((n-2)^k)}{y_1}, f_{(\cdot)}(\overset{((n-2)^{k-1})}{\overline{f(y_1^n)}}) =$$

$$f_{(\cdot)}(f_{(\cdot)}(\overset{((n-2)^{k-1})}{\overline{y}_1}, \ldots, \overset{((n-2)^{k-1})}{\overline{y}_n}), \overset{((n-2)^k)}{f(y_1^n)}, f_{(\cdot)}(\overset{((n-2)^{k-1})}{\overline{f(y_1^n)}})).$$

Whence, applying (11) and (12), we obtain

$$f_{(\cdot)}(\overset{((n-2)^{k-1})}{\overline{f(y_1^n)}}) = f_{(\cdot)}(\overset{((n-2)^{k-1})}{\overline{y}_1}, \ldots, \overset{((n-2)^{k-1})}{\overline{y}_n}).$$

But, by (9), for all $y \in G$ we have $f_{(\cdot)}(\overset{((n-2)^{k-1})}{\overline{y}}) = \overline{(\overline{y})}^{(k-1)} = \overline{y}^{(k)}$. Thus, the last identity implies

$$\overline{f(y_1^n)}^{(k)} = f(\overline{y}_1^{(k)}, \overline{y}_2^{(k)}, \ldots, \overline{y}_n^{(k)}).$$

Therefore, ${}^{-(k)} : x \to \overline{x}^{(k)}$ is an endomorphism. □

Note that for any finite n-ary group there exists a natural number m such that $\overline{x}^{(m)} = x$ holds for all $x \in G$. The same holds also in some infinite n-ary groups (see for example [3]). In these groups endomorphisms ${}^{-(k)} : x \to \overline{x}^{(k)}$ are automorphisms.

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Volgograd State Pedagogical University
Lenina prosp., 27
400131 Volgograd
Russia
e-mail: shchuchkin@fizmat.vspu.ru