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Loops in Relativistic Dynamics

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Abstract

The Einstein velocity addition loop and the symmetric velocity addition loop are used to develop relativistic dynamic equations. Since these loops are highly non-commutative, the question arises whether one should use the left or the right translations of these loops. We show that while the left translations are well-suited to relativistic dynamics, the right translations are problematic. We hypothesize that using the left translations is equivalent to a generalized form of the Equivalence Principle.

1. Introduction

This paper is about two loops which play a central role in Special Relativity. The first is the Einstein velocity addition loop $(D_v, \oplus_E)$, where

$$D_v = \{\mathbf{v} \in \mathbb{R}^3 : |\mathbf{v}| < c\}$$

($c =$ the speed of light) is the ball of relativistically admissible velocities and $\mathbf{v} \oplus_E \mathbf{u}$ is the relativistic sum of the two velocities $\mathbf{v}$ and $\mathbf{u}$. This is a left Bruck loop. The loop operation $\oplus_E$ is constructed from the Lorentz transformations between two inertial systems. This construction will be carried out in Section 2. Einstein velocity addition is, in general, not commutative. In fact, $\mathbf{v} \oplus_E \mathbf{u} = \mathbf{u} \oplus_E \mathbf{v}$ if and only if $\mathbf{v}$ and $\mathbf{u}$ are parallel.

The second loop under investigation involves a new dynamic variable, called *symmetric velocity*, defined as follows. If the relative velocity between two inertial systems is $\mathbf{v}$, then the symmetric velocity between the systems is the unique velocity $\mathbf{w}$ such that $\mathbf{w} \oplus_E \mathbf{w} = \mathbf{v}$. Thus the symmetric velocity is the *relativistic half* of the given velocity. Let $D_s = \{\mathbf{v} \in \mathbb{R}^3 : |\mathbf{v}| < 1\}$

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denote the set of relativistically admissible *symmetric* velocities (normalized to $c = 1$). The set D_s admits a binary operation $\oplus_s$, the addition of symmetric velocities, which makes $(D_s, \oplus_s)$ a loop.

The two loops $(D_v, \oplus_E)$ and $(D_s, \oplus_s)$ are isotopic as *topological loops*. Indeed, the function $\Psi : D_v \to D_s$ which maps a given velocity $\mathbf{v}$ to its corresponding symmetric velocity $\mathbf{w}$ is a homeomorphism which also respects the loop operations:

$$\Psi(\mathbf{v} \oplus_E \mathbf{u}) = \Psi(\mathbf{v}) \oplus_s \Psi(\mathbf{u}). \tag{1}$$

See Section 5 for explicit definitions of Ψ and Ψ^{-1}.

Despite the above isotopy, these loops *are* different. $(D_v, \oplus_E)$ is a left Bruck loop, whereas $(D_s, \oplus_s)$ is not. Moreover, these two loops behave differently *geometrically*. Friedman and Semon [2] have already exploited this difference. They used symmetric velocity and obtained an *analytic* solution for the motion of an electric charge in a uniform, constant electromagnetic field $\mathbf{E}, \mathbf{B}$ in which $\mathbf{E}$ and $\mathbf{B}$ are perpendicular. The first *explicit* solution to this problem was found in 2002 by Takeuchi [5].

The left translations of $(D_v, \oplus_E)$ (respectively, $(D_s, \oplus_s)$) generate a group of automorphisms of D_v (respectively, D_s). In turns out that in the case of $(D_v, \oplus_E)$, the automorphisms are *projective* (also called *affine*). This means that line segments are mapped to line segments. In this way, D_v can be seen as a subset of *projective space* $\mathbb{P}_3$. In contrast, the automorphisms induced by *symmetric* velocity are *conformal*. Thus while the two automorphism groups are isomorphic as groups, they are quite different *geometrically*.

The use of these two loops in developing relativistic dynamics is new and brings with it an interesting *dilemma*. Relativistic dynamics is concerned with describing the motion of an object whose velocity is changing with time due to a force. Since the velocities are bounded by c, they must be added relativistically. Over an infinitesimal time period dt, the force adds a change $d\mathbf{v}$ to the velocity $\mathbf{v}$. The new velocity will be $\mathbf{v} \oplus_E d\mathbf{v}$. Thus, velocity addition lies at the heart of relativistic dynamics, and it is natural to use the loop $(D_v, \oplus_E)$ to develop relativistic dynamics.

Now comes the dilemma. Is the new velocity really $\mathbf{v} \oplus_E d\mathbf{v}$? Or is it $d\mathbf{v} \oplus_E \mathbf{v}$? Since Einstein velocity addition is, in general, not commutative, we must choose between having the force act on the left or on the right. At first glance, this choice seems arbitrary. There is no *a priori* preference. Why should we prefer one over the other? And how does the *force* know which side to act on? Furthermore, does it matter? Does the dynamics based on left translations coincide with the dynamics based on right translations? The answer to this last question will be interesting either way. Agreement of "left" and "right" dynamics would be fascinating given the highly non-commutative nature of the velocity addition. On the other hand, if the two dynamics are at odds, we will then be faced with two additional questions: *Which* dynamics does nature use? *Why* does nature use this one?

Unfortunately, we cannot yet compare "left" and "right" dynamics because no one to date has succeeded in using the right translations to develop relativistic dynamics. Indeed, Friedman [1] uses the *left* translations of the Einstein velocity addition loop $(D_v, \oplus_E)$ to derive the relative dynamics equation

$$m_0 \frac{d\mathbf{v}(\tau)}{d\tau} = q(\mathbf{E} + \mathbf{v}(\tau) \times \mathbf{B} - c^{-2}\langle \mathbf{v}(\tau)|\mathbf{E}\rangle \mathbf{v}(\tau)) \tag{2}$$

for a particle of charge q and rest-mass m_0 in an electromagnetic field $\mathbf{E}, \mathbf{B}$. Here, τ is the proper time of the particle. Friedman's development is straightforward. The *right* translations, on the other hand, possess some inherent pathologies. We will attempt to explain this asymmetry in terms of the physical interpretation of the loop operations.

Note that the traditional approach to relativistic dynamics does not encounter the above dilemma. In fact, relativistic dynamics is usually developed without reference to Einstein velocity addition at all. In [3], for example, one starts with the assumption that the force on, say, a charged particle is equal to the rate of change of the particle's relativistic momentum. Since a particle with charge q and velocity $\mathbf{v}$ in an electromagnetic field $\mathbf{E}, \mathbf{B}$ experiences a force $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, the relativistic dynamics equation in this case is

$$m_0 \frac{d(\gamma \mathbf{v})}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{3}$$

In [1], it is shown that (2) and (3) are equivalent. Note that although the traditional approach avoids our dilemma, the equivalence of (2) and (3) means that the traditional approach implicitly assumes that the force acts on the *left*. See also [4].

This paper is organized as follows. In the next section, we construct the Einstein velocity addition loop from the Lorentz transformations. In Section 3, the left translations of this loop are used to derive the relativistic dynamics equation (2). Section 4 describes the difficulties inherent in using the right translations to develop relativistic dynamics. Section 5 is devoted to the symmetric velocity addition loop. Here, also, we will see that the left translations are preferred over the right. In Section 6, we discuss possible reasons why the left and right translations should behave so differently. The final section offers suggestions for further research. One direction is to develop relativistic dynamics using the triple product to overcome the difficulties of the right translations. Another approach is to show that using the left translations is actually equivalent to the Equivalence Principle. The latter idea will be taken up in a forthcoming paper.

2. Construction of the Einstein velocity addition loop

In this section, we will construct the Einstein velocity addition loop from the Lorentz spacetime transformation between two inertial systems K and K'. We assume that the spatial axes of K are parallel to those of K' and that at time $t = 0$, the origins of the two systems coincided. The spacetime coordinates of an event in K will be denoted by $\begin{pmatrix} t \\ \mathbf{r} \end{pmatrix}$, where $t \in \mathbb{R}$ is the time of the event and $\mathbf{r} \in \mathbb{R}^3$ represents the location of the event. The coordinates of the same event in K' will be denoted by $\begin{pmatrix} t' \\ \mathbf{r}' \end{pmatrix}$.

Suppose that the velocity of K' with respect to K is $\mathbf{v}$. Then the Lorentz transformation from K' to K is

$$\begin{pmatrix} t \\ \mathbf{r} \end{pmatrix} = \gamma \begin{pmatrix} 1 & c^{-2}\mathbf{v}^T \\ \mathbf{v} & P_{\mathbf{v}} + \alpha(I - P_{\mathbf{v}}) \end{pmatrix} \begin{pmatrix} t' \\ \mathbf{r}' \end{pmatrix}, \tag{4}$$

where $\gamma = \gamma(\mathbf{v}) = \frac{1}{\sqrt{1-\frac{|\mathbf{v}|^2}{c^2}}}$, $\alpha = \alpha(\mathbf{v}) = \frac{1}{\gamma(\mathbf{v})}$, $\mathbf{v}^T$ denotes the transpose of $\mathbf{v}$, and $P_{\mathbf{v}}$ denotes the projection operator onto $\mathbf{v}$.

The physical definition of the Einstein velocity addition is as follows. We are given that the velocity of K' with respect to K is $\mathbf{v}$. Suppose that an observer at rest in system K' measures an object's velocity as $\mathbf{u}$. Then the velocity of this object as measured by an observer at rest in system K is called the *relativistic sum* of $\mathbf{v}$ and $\mathbf{u}$ and is denoted by $\mathbf{v} \oplus_E \mathbf{u}$.

Consider motion with uniform velocity $\mathbf{u}$ in system K'. The world-line of this motion is $\begin{pmatrix} t' \\ \mathbf{u}t' \end{pmatrix}$. From (4), this world-line in system K is

$$\gamma \begin{pmatrix} t' + \frac{\mathbf{v}^T\mathbf{u}t'}{c^2} \\ \mathbf{v}t' + t'P_{\mathbf{v}}\mathbf{u} + \alpha t'(I - P_{\mathbf{v}})\mathbf{u} \end{pmatrix} \tag{5}$$

or

$$\gamma t' \begin{pmatrix} 1 + \frac{\langle \mathbf{v}|\mathbf{u}\rangle}{c^2} \\ \mathbf{v} + \mathbf{u}_{\|} + \alpha \mathbf{u}_{\perp} \end{pmatrix}, \tag{6}$$

where $\mathbf{u}_{\|} = P_{\mathbf{v}}\mathbf{u}$ denotes the component of $\mathbf{u}$ parallel to $\mathbf{v}$ and $\mathbf{u}_{\perp} = (I - P_{\mathbf{v}})\mathbf{u}$ denotes the component of $\mathbf{u}$ perpendicular to $\mathbf{v}$. This defines a uniform motion in system K with velocity

$$\mathbf{v} \oplus_E \mathbf{u} = \frac{\mathbf{v} + \mathbf{u}_{\|} + \alpha \mathbf{u}_{\perp}}{1 + \frac{\langle \mathbf{v}|\mathbf{u}\rangle}{c^2}}, \tag{7}$$

with $\alpha = \alpha(\mathbf{v}) = \sqrt{1 - \frac{|\mathbf{v}|^2}{c^2}}$. This defines the binary operation $\oplus_E$ on D_v. The pair $(D_v, \oplus_E)$ is a left Bruck loop.

In case $\mathbf{v}$ and $\mathbf{u}$ are parallel, the Einstein velocity addition reduces to

$$\mathbf{v} \oplus_E \mathbf{u} = \frac{\mathbf{v} + \mathbf{u}}{1 + \frac{vu}{c^2}}, \tag{8}$$

where $v = |\mathbf{v}|$ and $u = |\mathbf{u}|$. In case $\mathbf{v}$ and $\mathbf{u}$ are perpendicular, the formula becomes

$$\mathbf{v} \oplus_E \mathbf{u} = \mathbf{v} + \alpha(\mathbf{v})\mathbf{u}. \tag{9}$$

Note that the velocity addition is commutative only for parallel velocities.

3. Left translations

In [1], the left translations of the loop $(D_v, \oplus_E)$ are used to obtain the relativistic dynamics equation

$$m_0 \frac{d\mathbf{v}(\tau)}{d\tau} = q(\mathbf{E} + \mathbf{v}(\tau) \times \mathbf{B} - c^{-2}\langle \mathbf{v}(\tau)|\mathbf{E}\rangle \mathbf{v}(\tau)), \tag{10}$$

where τ is the proper time of the particle. It is then shown that (10) is equivalent to (3). Here we give an outline of that development. For details, see [1].

For each $\mathbf{v}$ in the velocity ball D_v, we define the left translation $L_{\mathbf{v}} : D_v \to D_v$ by

$$L_{\mathbf{v}}(\mathbf{u}) = \mathbf{v} \oplus_E \mathbf{u}. \tag{11}$$

See Figure 1.

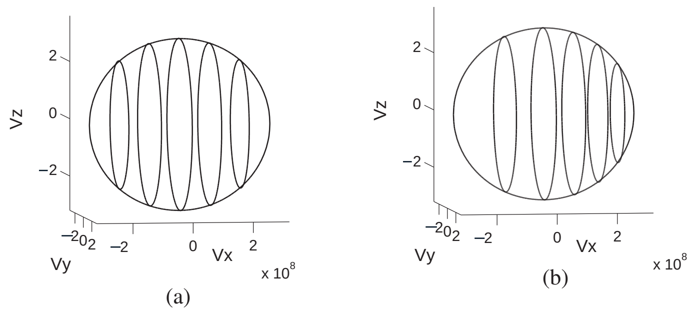

Figure 1. *Action of the velocity addition on* D_v.
(a) A set of 5 uniformly spaced discs Δ_j obtained by intersecting the three-dimensional velocity ball D_v of radius $c = 3\cdot 10^8\mathrm{m/s}$ with y-z planes at $x = 0, \pm 10^8, \pm 2\cdot 10^8\mathrm{m/s}$. (b) The images of these Δ_j under the left translation $L_{\mathbf{v}}(\mathbf{u}) = \mathbf{v} \oplus_E \mathbf{u}$, with $\mathbf{v} = (10^8, 0, 0)\mathrm{m/s}$. Note that $L_{\mathbf{v}}(\Delta_j)$ is also a disc in D_v, perpendicular to $\mathbf{v}$ and moved in the direction of $\mathbf{v}$. On each disc Δ_j, the map $L_{\mathbf{v}}$ acts as multiplication by a constant in the component of $\mathbf{u}$ perpendicular to $\mathbf{v}$.

The left translations have some nice properties. First, each left translation $L_{\mathbf{v}}$ is a *projective* automorphism of D_v. To appreciate the projective geometry at work here, envision the action of $L_{\mathbf{v}}$ on D_v as follows. Fix a velocity $\mathbf{u} \in D_v$. Identify $\mathbf{u}$ with the intersection of the world-line $L = \begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix}$ in the inertial system K and the plane $\Pi = \{(1, \mathbf{r}) : \mathbf{r} \in \mathbb{R}\}$. Let K' be an inertial system moving with relative velocity $\mathbf{v}$ with respect to K. Applying the Lorentz transformation from K to K' to the line L yields a line L' in K' whose intersection with Π is $\mathbf{v} \oplus_E \mathbf{u}$.

The second nice property is closure under inverses. In fact $L_{\mathbf{v}}^{-1} = L_{-\mathbf{v}}$. The above two properties combine to make the following useful characterization of the group $Aut_p(D_v)$ of all projective automorphisms of D_v. Let ψ be any projective automorphism of D_v. Set $\mathbf{v} = \psi(0)$ and $U = L_{\mathbf{v}}^{-1}\psi$. Then U is a projective map that maps $0 \to 0$ and is thus a linear map which can be represented by a 3×3 matrix. Since U maps D_v onto itself, it is an isometry and is represented by an orthogonal matrix. Since $\psi = L_{\mathbf{v}}U$, the group $Aut_p(D_v)$ is defined by

$$Aut_p(D_v) = \{L_{\mathbf{v}}U : \mathbf{v} \in D_v,\ U \in O(3)\}. \tag{12}$$

We write $L_{\mathbf{v},U}$ instead of $L_{\mathbf{v}}U$.

The group $Aut_p(D_v)$ is a representation of the Lorentz group by affine maps. It is a real Lie group of dimension 6, since any element of the group is determined by an element $\mathbf{v}$ of the three-dimensional open ball of radius c in $\mathbb{R}^3$ and an element U of the three-dimensional orthogonal group $O(3)$.

The dynamics equation (10) will be constructed from the elements of the Lie algebra $aut_p(D_v)$ of $Aut_p(D_v)$. The elements of a Lie algebra are, by definition, the tangent space of the identity of the corresponding Lie group. To obtain the elements of $aut_p(D_v)$, let $g(s)$ be a differentiable curve from a neighborhood I_0 of 0 into $Aut_p(D_v)$, with $g(0) = L_{\mathbf{0},I}$, the identity of $Aut_p(D_v)$. Then $g(s)$ has the form

$$g(s) = L_{\mathbf{v}(s),U(s)}, \tag{13}$$

where $\mathbf{v} : I_0 \to D_v$ is a differentiable function satisfying $\mathbf{v}(0) = \mathbf{0}$ and $U(s) : I_0 \to O(3)$ is differentiable and satisfies $U(0) = I$.

For an example of such a curve $g(s)$, fix $\mathbf{v} \in D_v$. Let $\mathbf{j} = \mathbf{v}/|\mathbf{v}|$ and define $k = \tanh^{-1}(|\mathbf{v}|/c)$. For $s \in \mathbb{R}$, define $\mathbf{b}(s) = \tanh(sk)c\mathbf{j}$. The resulting curve $g(s) := L_{\mathbf{b}(s)}$ is called the *one-parameter subgroup generated by* $L_{\mathbf{v}}$. See Figure 2.

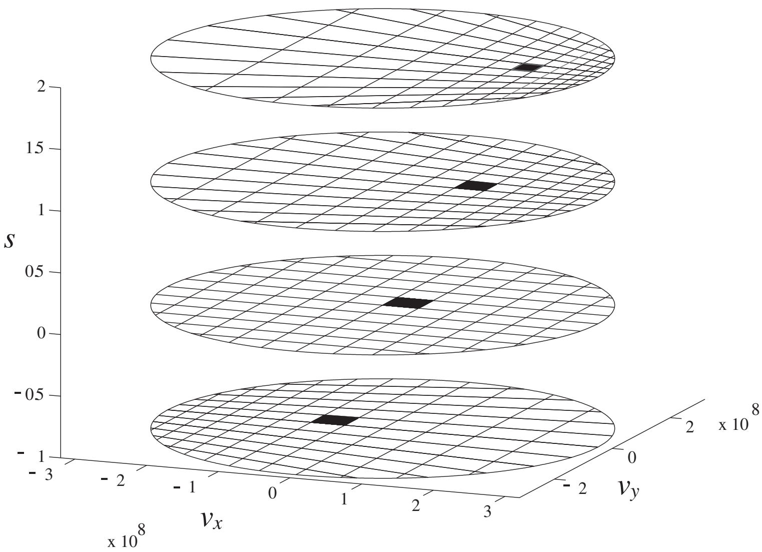

Figure 2. Action of a one-parameter subgroup on D_v.

The effect on a two-dimensional section of D_v by the one-parameter subgroup $g(s)$ generated by the map $L_{\mathbf{v}}$, for $s = -1, 0, 1, 2$. One cell of the grid has been darkened along with its images to help visualize the effect of the transformation. Note that $g(-1) = L_{\mathbf{v}}^{-1} = L_{-\mathbf{v}}, g(0) = I$-the identity, $g(1) = L_{\mathbf{v}}$ and $g(2) = L_{\mathbf{v}}^2 = L_{\mathbf{v}\oplus_E\mathbf{v}}$.

We denote by δ the element of $aut_p(D_v)$ generated by $g(s)$. For any fixed $\mathbf{u} \in D_v$, $g(s)(\mathbf{u})$ is a smooth curve in D_v, with $g(0)(\mathbf{u}) = \mathbf{u}$, and $\delta(\mathbf{u})$ is a tangent vector to this line. Thus, the elements of $aut_p(D_v)$ are vector fields $\delta(\mathbf{u})$ on D_v defined by

$$\delta(\mathbf{u}) = \frac{d}{ds} g(s)(\mathbf{u})\Big|_{s=0}. \tag{14}$$

Note that (14) is equivalent to using $d\mathbf{v} \oplus_E \mathbf{v}$ and *not* $\mathbf{v} \oplus_E d\mathbf{v}$ for the velocity at time $t + dt$.

The explicit form of $\delta(\mathbf{u})$ is calculated in [1]. There it is shown that the Lie algebra

$$aut_p(D_v) = \{\delta_{\mathbf{E},\mathbf{B}} : \mathbf{E}, \mathbf{B} \in \mathbb{R}^3\}, \tag{15}$$

where $\delta_{\mathbf{E},\mathbf{B}} : D_v \to \mathbb{R}^3$ is the vector field defined by

$$\delta_{\mathbf{E},\mathbf{B}}(\mathbf{u}) = \mathbf{E} + \mathbf{u} \times \mathbf{B} - c^{-2}\langle \mathbf{u} \,|\mathbf{E}\rangle \mathbf{u}. \tag{16}$$

Note that each generator $\delta_{\mathbf{E},\mathbf{B}}(\mathbf{u})$ is a second-degree polynomial in $\mathbf{u}$. The quadratic term can be used to derive the *triple product* associated with D_v as a *Bounded Symmetric Domain.* Moreover, these generators give the correct formulas for the transformation of an electromagnetic field between two inertial systems. Two examples of these vector fields are shown in Figures 3 and 4.

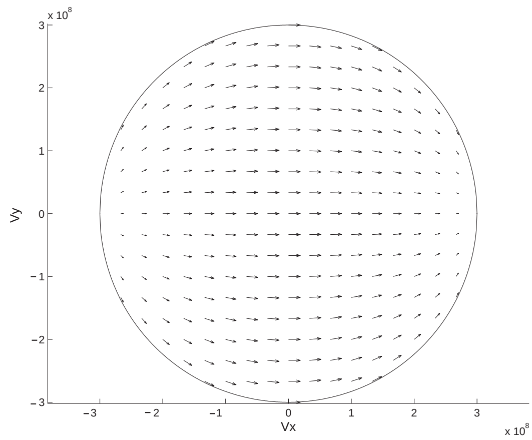

Figure 3. The vector field generated by an electric field $\mathbf{E}$*.*
The vector field $q/m \cdot \delta_{\mathbf{E},\mathbf{B}}$ on a two-dimensional section of D_v, with $q/m = 10^7 C/kg$, $\mathbf{E} = (2, 0, 0)V/m$ and $\mathbf{B} = 0$. Since $\mathbf{E}$ is in the positive direction of the v_x-axis, the

field tends to move particles in this direction. However, near the edge of D_v, the vectors either shrink to zero magnitude or become nearly tangent to D_v, reflecting the fact that the flow generated by this field cannot leave D_v.

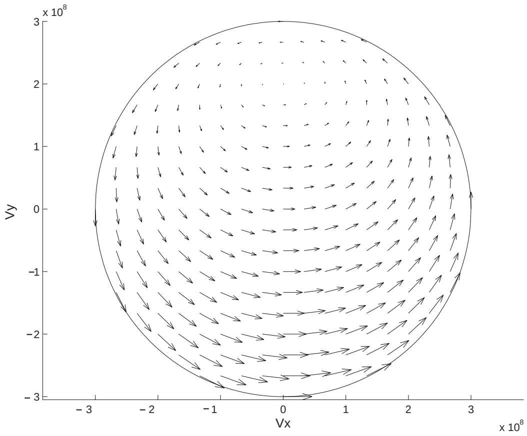

Figure 4. *The vector field generated by an electromagnetic field* $\mathbf{E}, \mathbf{B}$. The vector field $q/m \cdot \delta_{\mathbf{E},\mathbf{B}}$ on a two-dimensional section of D_v, with $q/m = 10^7 C/kg$, $\mathbf{E} = (2,0,0)V/m$ and $c\mathbf{B} = (0,0,3)V/m$. Here, the addition of a magnetic field $\mathbf{B}$ causes a rotation.

Using the generator $\delta_{\mathbf{E},\mathbf{B}} \in aut_p(D_v)$ defined by (16) to represent the force on a particle with charge q and rest-mass m_0, we obtain the relativistic dynamics equation

$$\frac{d\mathbf{v}(\tau)}{d\tau} = \frac{q}{m_0}\delta_{\mathbf{E},\mathbf{B}}(\mathbf{v}(\tau))$$

or

$$m_0\frac{d\mathbf{v}(\tau)}{d\tau} = q(\mathbf{E} + \mathbf{v}(\tau) \times \mathbf{B} - c^{-2}\langle\mathbf{v}(\tau)|\mathbf{E}\rangle\mathbf{v}(\tau)), \tag{17}$$

where τ is the proper time of the particle. It is shown in [1] that (17) is equivalent to (3).

4. Right translations

When we try to mimic the development of the previous section using the right translations, we run into problems.

For each $\mathbf{v} \in D_v$, we define the right translation $R_{\mathbf{v}} : D_v \to D_v$ by

$$R_{\mathbf{v}}(\mathbf{u}) = \mathbf{u} \oplus_E \mathbf{v}. \tag{18}$$

Unfortunately, the right translations do not possess any of the nice properties of the left translations. The map $R_{\mathbf{v}}$ is not projective. It's not even analytic. Moreover,

$$R_{\mathbf{v}}^{-1} \neq R_{-\mathbf{v}}. \tag{19}$$

In fact, $R_{\mathbf{v}}^{-1}$ is not a right translation at all. We will express $R_{\mathbf{v}}^{-1}(\mathbf{u})$ using Ungar's gyration operator [6]. For $\mathbf{x}, \mathbf{y} \in D_v$, define $\mathrm{gyr}[\mathbf{x}, \mathbf{y}] : D_v \to D_v$ by

$$\mathrm{gyr}[\mathbf{x}, \mathbf{y}](\mathbf{z}) = -(\mathbf{x} \oplus_E \mathbf{y}) \oplus_E (\mathbf{x} \oplus_E (\mathbf{y} \oplus_E \mathbf{z})). \tag{20}$$

Then

$$R_{\mathbf{v}}^{-1}(\mathbf{u}) = \mathbf{u} \oplus_E -\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{v}. \tag{21}$$

This last equation shows that $R_{\mathbf{v}}^{-1}$ is not a right translation. It is not clear at all how to proceed from this point in developing relativistic dynamics. We think the difficulties might be overcome by using the triple product associated with D_v as a Bounded Symmetric Domain, but this approach is still in the beginning stages. See Section 7.

5. Symmetric velocity addition

In this section, we construct the loop $(D_s, \oplus_s)$ of symmetric velocities. We derive the formula for the addition of symmetric velocities from the physical definition of this addition. The left translations of $(D_s, \oplus_s)$, which belong to the group $Aut_c(D_s)$ of all *conformal* automorphisms of D_s, are then used to derive the relativistic dynamics equation for symmetric velocities. The elements of the Lie algebra $aut_c(D_s)$ will be expressed in terms of a *triple product*. We also obtain a very useful two-dimensional version of $(D_s, \oplus_s)$. This version is usually simpler to work with and yet captures all of the properties of the full three-dimensional version. Here too, in the case of symmetric velocity, we will see that while the *left* translations yield a nice development of relativistic dynamics, the *right* translations are rather problematic, even in the simpler, two-dimensional case.

The definition of symmetric velocity is as follows. If the relative velocity between two inertial systems is $\mathbf{v}$, then the symmetric velocity between the systems is the unique velocity $\mathbf{w}_1$ such that

$$\mathbf{v} = \mathbf{w}_1 \oplus_E \mathbf{w}_1 = \frac{\mathbf{w}_1 + \mathbf{w}_1}{1 + \frac{|\mathbf{w}_1|}{c}\frac{|\mathbf{w}_1|}{c}} = \frac{2\mathbf{w}_1}{1 + |\mathbf{w}_1|^2/c^2}.$$

Instead of $\mathbf{w}_1$, we use a dimensionless vector $\mathbf{w} = \mathbf{w}_1/c$ and call it *s-velocity*. Thus, the relationship between an s-velocity $\mathbf{w}$ and its corresponding velocity $\mathbf{v}$ is given by the two formulas

$$\mathbf{w} = \Psi(\mathbf{v}) = \frac{\mathbf{v}/c}{1+\sqrt{1-|\mathbf{v}|^2/c^2}} \tag{22}$$

and

$$\mathbf{v} = \Psi^{-1}(\mathbf{w}) = \frac{2c\mathbf{w}}{1+|\mathbf{w}|^2}. \tag{23}$$

The set of all relativistically admissible s-velocities form a unit ball

$$D_s = \{\mathbf{w} \in \mathbb{R}^3 : \ |\mathbf{w}| < 1\}. \tag{24}$$

The physical meaning of symmetric velocity is as follows. Consider two inertial systems with relative velocity $\mathbf{v}$ between them. Place two objects of equal mass (test masses) at the origin of each inertial system. The center of mass of the two objects will be called the *center of the two inertial systems*. The symmetric velocity is the velocity of each system with respect to the center of the systems, and the s-velocity is the dimensionless velocity of the systems with respect to their center (see Figure 5).

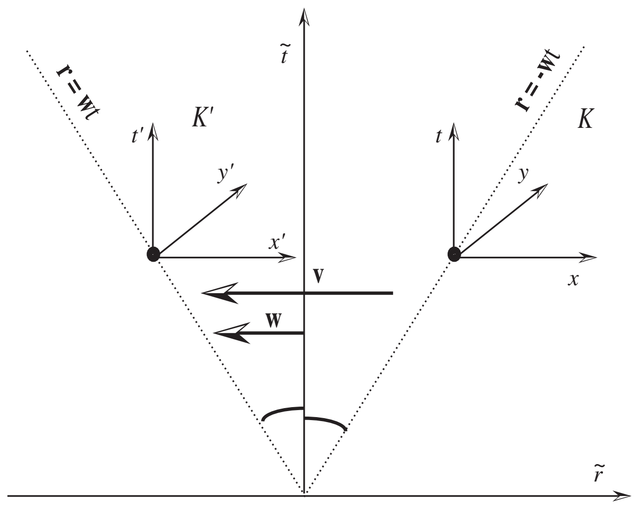

Figure 5. *The physical meaning of symmetric velocity.*
The physical meaning of symmetric velocity. Two inertial systems K and K' with relative velocity $\mathbf{v}$ between them are viewed from the system connected to their center. In this system, K and K' are each moving with velocity $\pm\mathbf{w}$.

The physical definition of s-velocity addition is as follows. Consider three inertial systems K_1, K_2 and K_3. We choose the space axes of K_2 to be parallel to the axes of K_1 and the axes of K_3 to be parallel to those of K_2. Denote their origins by O_1, O_2 and O_3, respectively. Denote by $\mathbf{a}$ the s-velocity of system K_2 with respect to K_1 and by $\mathbf{w}$ the s-velocity of system K_3 with respect to K_2. Then the s-velocity $\mathbf{w}_3$ of system K_3 with respect to K_1 (*i.e.*, the velocity of K_3 with respect to the center of systems K_1 and K_3) is called the *sum of the s-velocities* $\mathbf{a}$ and $\mathbf{w}$ and is denoted by $\mathbf{a} \oplus_s \mathbf{w}$ (see Figure 6). In other words, if $c\mathbf{a} \oplus_E c\mathbf{a} = \mathbf{v}$ and $c\mathbf{w} \oplus_E c\mathbf{w} = \mathbf{u}$, then $\mathbf{a} \oplus_s \mathbf{w}$ is $1/c$ times the relativistic half of $\mathbf{v} \oplus_E \mathbf{u}$.

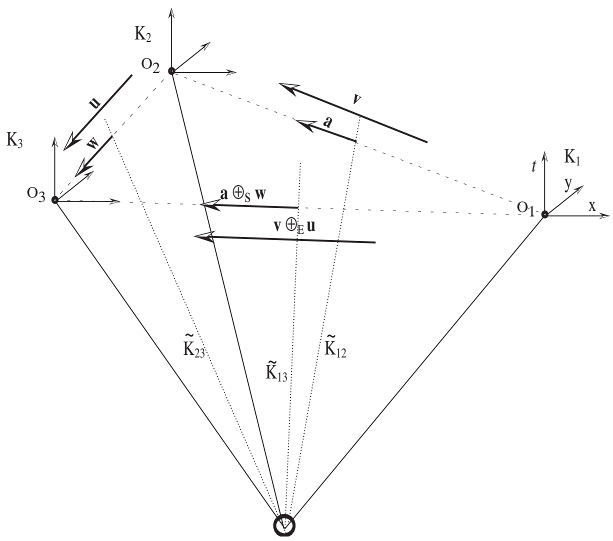

Figure 6. *The sum of s-velocities.*

The sum of s-velocities. Inertial systems K_1, K_2 and K_3, with origins O_1, O_2 and O_3, respectively, had a common origin at time $t = 0$. The line $\widetilde{K}_{12}$ is the world-line of the center of the two inertial systems K_1 and K_2. Similarly, the lines $\widetilde{K}_{23}$ and $\widetilde{K}_{13}$ represent the world-lines of the centers of the systems K_2, K_3 and K_1, K_3, respectively. The velocity of system K_2 with respect to system K_1 is $\mathbf{v}$, and its s-velocity $\mathbf{a}$ is the velocity of K_2 with respect to $\widetilde{K}_{12}$. Similarly, the velocity of system K_3 with respect to system K_2 is $\mathbf{u}$, and its s-velocity $\mathbf{w}$ is the velocity of K_3 with respect to $\widetilde{K}_{23}$. By definition of Einstein velocity addition, the velocity of system K_3 with respect to system K_1 is $\mathbf{v} \oplus_E \mathbf{u}$. The s-velocity of K_3 with respect to K_1, meaning the dimensionless velocity of K_3 with respect to $\widetilde{K}_{13}$, is called the sum of symmetric velocities $\mathbf{a}$ and $\mathbf{w}$ and is denoted by $\mathbf{a} \oplus_s \mathbf{w}$.

Using the above definition and formula (7) for Einstein velocity addition, we obtain the s-velocity-addition formula:

$$\mathbf{a} \oplus_s \mathbf{w} = \frac{(1 + |\mathbf{w}|^2 + 2\langle \mathbf{a} \mid \mathbf{w}\rangle)\mathbf{a} + (1 - |\mathbf{a}|^2)\mathbf{w}}{1 + |\mathbf{a}|^2|\mathbf{w}|^2 + 2\langle \mathbf{a} \mid \mathbf{w}\rangle}. \tag{25}$$

As in the case of Einstein velocity addition, it can be shown that $\mathbf{a} \oplus_s \mathbf{w} = \mathbf{w} \oplus_s \mathbf{a}$ if and only if $\mathbf{a}$ and $\mathbf{w}$ are parallel.

Note that $\mathbf{a} \oplus_s \mathbf{w}$ is a linear combination of $\mathbf{a}$ and $\mathbf{w}$ and therefore belongs to the plane Π generated by $\mathbf{a}$ and $\mathbf{w}$. This allows us to obtain a two-dimensional version of s-velocity addition. It is often sufficient (and easier!) to work with the two-dimensional version. Moreover, we obtain a new method of solving relativistic dynamic equations. If the motion under investigation has an invariant plane, then the relativistic dynamic equation for the symmetric velocity becomes a first-order *analytic* differential equation in one complex variable.

We obtain the two-dimensional version of s-velocity addition by imposing a complex structure on the plane Π. In other words, we treat the disk $\Delta = D_s \cap \Pi$ as a copy of the unit disk $|z| < 1$ of the complex plane $\mathbb{C}$. Denote by a the complex number corresponding to the vector $\mathbf{a}$ and by w the complex number corresponding to the vector $\mathbf{w}$. We use the identities

$$Re\,\langle a \mid w\rangle = \frac{\overline{a}w + a\overline{w}}{2},\ |w|^2 = w\overline{w}, \tag{26}$$

where the bar denotes complex conjugation, to convert (25) into our two-dimensional version:

$$a \oplus_s w = \frac{(1 + w\overline{w} + \overline{a}w + a\overline{w})a + (1 - a\overline{a})w}{1 + a\overline{a}w\overline{w} + \overline{a}w + a\overline{w}} \tag{27}$$

$$= \frac{(a + w)(1 + a\overline{w})}{(1 + \overline{a}w)(1 + a\overline{w})} = \frac{a + w}{1 + \overline{a}w}. \tag{28}$$

This is the well-known Möbius transformation of the complex unit disk. Thus, s-velocity addition is a generalization of the Möbius addition of complex numbers (see Figure 7).

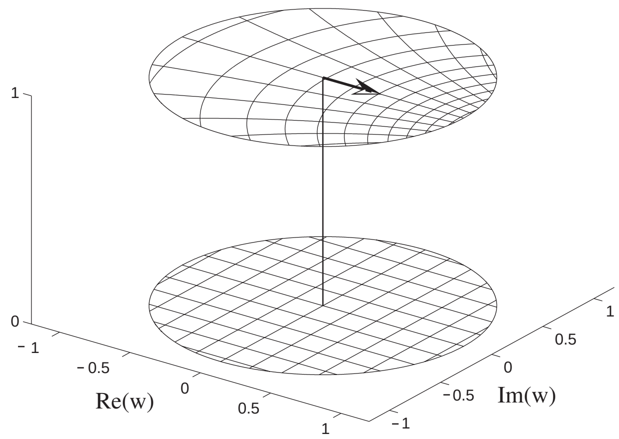

Figure 7. Symmetric velocity addition on D_s.

Symmetric velocity addition $a \oplus_s w$ for $a = 0.4$. The lower circle in the figure is the unit disc of the complex plane, representing a two-dimensional section of the s-velocity ball D_s. The upper circle is the image of the lower circle under the transformation $w \to \frac{a+w}{1+\bar{a}w}$. Each circle is enhanced with a grid to highlight the effect of this transformation. Notice how a typical square of the lower grid is deformed and changes in size under the transformation.

For each s-velocity $\mathbf{a} \in D_s$, we define the left translation $L_{\mathbf{a}} : D_s \to D_s$ by

$$L_{\mathbf{a}}(\mathbf{w}) = \mathbf{a} \oplus_s \mathbf{w}. \tag{29}$$

Each left translation $L_{\mathbf{a}}$ is a *conformal* automorphism of D_s. In addition, the inverse of a left translation is again a left translation. In fact $L_{\mathbf{a}}^{-1} = L_{-\mathbf{a}}$. As a result, the same argument as that in Section 3 shows that the group $Aut_c(D_s)$ of all conformal automorphisms of D_s has the following characterization:

$$Aut_c(D_s) = \{L_{\mathbf{a}}U : \mathbf{a} \in D_s,\ U \in O(3)\}. \tag{30}$$

We write $L_{\mathbf{a},U}$ instead of $L_{\mathbf{a}}U$.

The group $Aut_c(D_s)$ is a representation of the Lorentz group by conformal maps. It is a real Lie group of dimension 6, since any element of the group is determined by an element $\mathbf{a}$ of the three-dimensional open unit ball in $\mathbb{R}^3$ and an element U of the three-dimensional orthogonal group $O(3)$.

The two groups $Aut_p(D_s)$ and $Aut_c(D_s)$ are isomorphic. In fact, the isomorphism is given by

$$L_{\mathbf{v},U} \longleftrightarrow \Psi L_{\mathbf{v},U} \Psi^{-1}. \tag{31}$$

Nevertheless, they lead to different dynamics, as we will see.

The relativistic dynamics equation for symmetric velocities will be constructed from the elements of the Lie algebra $aut_c(D_s)$ of $Aut_c(D_s)$. To define the elements of $aut_c(D_s)$, consider differentiable curves $g(s)$ from a neighborhood I_0 of zero into $Aut_c(D_s)$, with $g(0) = L_{\mathbf{0},I}$, the identity of $Aut_c(D_s)$. Then

$$g(s) = L_{\mathbf{a}(s),U(s)}, \tag{32}$$

where $\mathbf{a} : I_0 \to D_s$ is a differentiable function satisfying $\mathbf{a}(0) = \mathbf{0}$ and $U(s) : I_0 \to O(3)$ is differentiable and satisfies $U(0) = I$. We denote by ξ the element of $aut_c(D_s)$ generated by $g(s)$. For any fixed $\mathbf{w} \in D_s$, $g(s)(\mathbf{w})$ is a smooth curve in D_s, with $g(0)(\mathbf{w}) = \mathbf{w}$, and $\xi(\mathbf{w})$ is a tangent vector to this line. Thus, the elements of $aut_c(D_s)$ are vector fields $\xi(\mathbf{w})$ on D_s defined by

$$\xi(\mathbf{w}) = \frac{d}{ds} g(s)(\mathbf{w})\Big|_{s=0}. \tag{33}$$

The explicit form of $\xi(\mathbf{w})$ is calculated in [1]. There it is shown that

$$aut_c(D_s) = \{\xi_{\mathbf{b},A} : \mathbf{b} \in \mathbb{R}^3, A \text{ is a } 3 \times 3 \text{ antisymmetric matrix}\}, \tag{34}$$

where

$$\xi_{\mathbf{b},A}(\mathbf{w}) = \mathbf{b} + A\mathbf{w} - 2\langle \mathbf{b}|\mathbf{w}\rangle \mathbf{w} + |\mathbf{w}|^2 \mathbf{b}. \tag{35}$$

It is useful to express the generator $\xi_{\mathbf{b},A}$ in terms of the triple product

$$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} = \langle \mathbf{a}|\mathbf{b}\rangle \mathbf{c} + \langle \mathbf{c}|\mathbf{b}\rangle \mathbf{a} - \langle \mathbf{a}|\mathbf{c}\rangle \mathbf{b}, \tag{36}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. This product is called the *spin triple product*. The bounded symmetric domain D_s endowed with the spin triple product is called the *spin factor* and is a domain of type IV in Cartan's classification. See Chapters 2 and 3 of [1] for a full treatment of the spin triple product in the theory of Bounded Symmetric Domains and Special Relativity.

Rewriting the generators (35) using the triple product, we find that

$$aut_c(D_s) = \{\xi_{\mathbf{b},\mathbf{B}} : \mathbf{b}, \mathbf{B} \in \mathbb{R}^3\}, \tag{37}$$

where $\xi_{\mathbf{b},\mathbf{B}} : D_s \to \mathbb{R}^3$ is the vector field defined by

$$\xi_{\mathbf{b},\mathbf{B}}(\mathbf{w}) = \mathbf{b} + \mathbf{w} \times \mathbf{B} - \{\mathbf{w}, \mathbf{b}, \mathbf{w}\}. \tag{38}$$

See Figures 8 and 9 for two examples of these vector fields.

The similarities between Figures 3 and 8 and between Figures 4 and 9 can be misleading. For example, the flow generating Figure 4 is elliptical, while the trajectories in Figure 9 are circles.

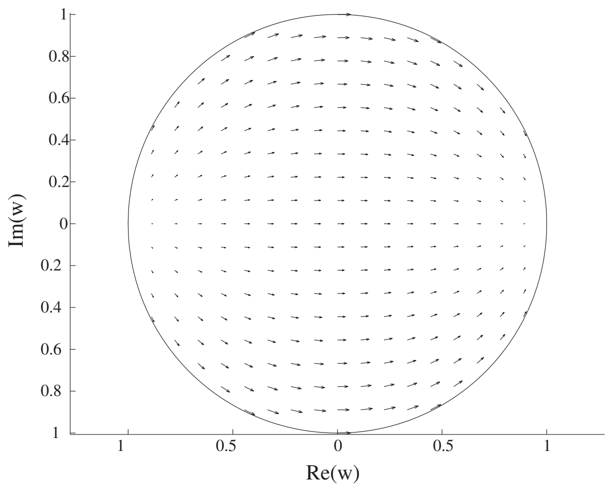

Figure 8. The vector field of the electric field E on D_s.

The vector field $\xi_{\mathbf{b},\mathbf{B}}$, with $\mathbf{b} = (0.07, 0, 0)$ and $\mathbf{B} = 0$, on a two-dimensional section of the s-velocity ball D_s. Note that this vector field is similar to the corresponding one for the Lie algebra $aut_p(D_v)$ of the velocity ball (see Figure 3).

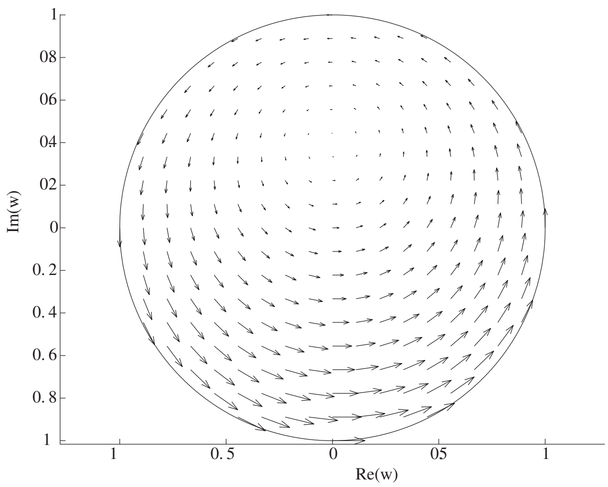

Figure 9. The vector field of the electromagnetic field $E, \mathbf{B}$ on D_s.

The vector field $\xi_{\mathbf{b},\mathbf{B}}$ with $\mathbf{b} = (0.07, 0, 0)$ and $\mathbf{B} = (0, 0, 0.1)$, on a two-dimensional section of the s-velocity ball D_s. Note that this vector field is similar to the corresponding one for the Lie algebra $aut_p(D_v)$ of the velocity ball (see Figure 4).

To obtain the relativistic dynamics equation for symmetric velocities, we start with

$$m_0 \frac{d(\gamma \mathbf{v})}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{39}$$

and change variables from velocity $\mathbf{v}$ to s-velocity $\mathbf{w}$. We obtain

$$m_0 c \frac{d\mathbf{w}(\tau)}{d\tau} = q(\mathbf{E}/2 + \mathbf{w}(\tau) \times c\mathbf{B} - \{\mathbf{w}(\tau), \mathbf{E}/2, \mathbf{w}(\tau)\}) = q\xi_{\mathbf{E}/2, c\mathbf{B}}(\mathbf{w}(\tau)). \tag{40}$$

Thus the left translations yield a nice development of relativistic dynamics also in the case of symmetric velocity. The right translations, on the other hand, are again problematic. Even in the ostensibly simpler two-dimensional version of symmetric velocity addition, the inverses of right translations are rather unwieldy. Recall that the two-dimensional version of s-velocity addition is

$$a \oplus_s b = \frac{a + b}{1 + \overline{a}b}. \tag{41}$$

The left inverses are well-behaved, and we have

$$L_a^{-1}(b) = \frac{-a + b}{1 - \overline{a}b} = L_{-a}(b). \tag{42}$$

Compare this to the right inverse, which is not even analytic:

$$R_a^{-1}(b) = \frac{b(1 - |a|^2) - a(1 - |b|^2)}{1 - |b|^2|a|^2}. \tag{43}$$

Again the right translations have lead to an apparent dead end.

6. Why are the left and right translations so different?

Why are the *left* translations so well-suited for relativistic dynamics, while the *right* translations are not? Who told forces that they have to act on the left?

Let's take another look at the physical definition of the Einstein velocity addition. Suppose an observer is at rest in an inertial system K. For any velocity $\mathbf{a} \in D_v$, let $K_{\mathbf{a}}$ denote the inertial system whose axes are parallel to those of K and moves with relative velocity $\mathbf{a}$ with respect to K. Then the inverse functions $L_{\mathbf{a}}^{-1}$ and $R_{\mathbf{a}}^{-1}$ now have the following physical interpretation. The question "What is the value of $L_{\mathbf{a}}^{-1}(\mathbf{u})$?" is equivalent to the question "Which *velocity* measured in the system $K_{\mathbf{a}}$ will be measured by our observer as $\mathbf{u}$?" whereas the question "What is the value of $R_{\mathbf{a}}^{-1}(\mathbf{u})$?" is equivalent to the question "In which *system* K' will the velocity $\mathbf{a}$ be measured by our observer as $\mathbf{u}$?" In other words, the two inverse functions are answering fundamentally different questions.

This might explain why the left and right translations behave *differently*. But it still does not explain the preferred status of left over right.

7. Suggestions for further research

As mentioned previously, we believe that the triple product might be helpful in overcoming the difficulties inherent in using the right translations. In the two-dimensional version of s-velocity addition, for example, the triple product is derived from (36) and takes the form

$$\{z, b, w\} = z\bar{b}w, \tag{44}$$

where $z, b, w \in \mathbb{C}$. Then, although neither the right translation R_a nor its inverse is analytic, each of these functions does have a power series expansion. More explicitly, for the right translation we have

$$R_a(b) = \frac{b+a}{1+\bar{b}a} = \sum_{n=0}^{\infty}(-1)^n D(a,b)^n(a+b), \tag{45}$$

where $D(a,b)x = \{a, b, x\}$ and $D(a,b)^0 = \mathrm{Id}$. For the inverse, we have

$$R_a^{-1}(b) = \frac{b(1-|a|^2) - a(1-|b|^2)}{1-|b|^2|a|^2} = \sum_{n=0}^{\infty} Q(a,b)^n(-a+b), \tag{46}$$

where $Q(a,b)x = \{a, x, b\}$ and $Q(a,b)^0 = \mathrm{Id}$. It will be interesting to see if these power series make the right translations more amenable to relativistic dynamics.

Another line of investigation involves the *Equivalence Principle*. This principle has several versions. The classical version ([3], p. 244) states that motion in a uniformly accelerated system is the same as that in an inertial system in the presence of a gravitational field. According to the *generalized Equivalence Principle*, motion in a uniformly accelerated system is the same as that in an inertial system in the presence of *any* force. We believe that using the left translations of either the Einstein or the symmetric velocity loop is equivalent to the generalized Equivalence Principle. In other words, the right translations are the wrong ones to use because they contradict the generalized Equivalence Principle. Moreover, if our belief is correct, then, once we succeed in developing relativistic dynamics from the right translations, we will have a way of testing the correctness of the Equivalence Principle itself. These ideas will be taken up in a forthcoming paper.

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