Quasigroups and Related Systems **15** (2007), 303 – 308

Decomposition of AG*-groupoids

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Abstract

We have shown that an AG*-groupoid S has associative powers, and S/ρ, where $a\rho b$ if and only if $ab^n = b^{n+1}$, $ba^n = a^{n+1}$ $\forall\ a, b \in S$, is a maximal separative commutative image of S.

An *Abel-Grassmann's groupoid* [9], abbreviated as an *AG-groupoid*, is a groupoid S whose elements satisfy the invertive law:

$$(ab)c = (cb)a. \tag{1}$$

It is also called a *left almost semigroup* [3, 4, 5, 7]. In [1], the same structure is called a *left invertive groupoid*. In this note we call it an AG-groupoid. It is a useful non-associative algebraic structure, midway between a groupoid and a commutative semigroup, with wide applications in the theory of flocks.

An AG-groupoid S is *medial* [2], i.e., it satisfies the identity

$$(ab)(cd) = (ac)(bd). \tag{2}$$

It is known [3] that if an AG-groupoid contains a left identity then it is unique. It has been shown in [3] that an AG-groupoid contains a left identity then it is unique. It has been proved also that an AG-groupoid with right identity is a commutative monoid, that is, a semigroup with identity element.

If an AG-groupoid satisfy one of the following equivalent identities:

$$(ab)c = b(ca) \tag{3}$$

$$(ab)c = b(ac) \tag{4}$$

2000 Mathematics Subject Classification: 20M10, 20N99
Keywords: LA-semigroup, AG^*-groupoid, invertive law, medial law, congruence.

then it is called an *AG^*-groupoid* [10].

Let S be an AG*-groupoid and a relation ρ be defined in S as follows. For a positive integer n, $a\rho b$ if and only if $ab^n = b^{n+1}$ and $ba^n = a^{n+1}$, for all a and b in S.

In this paper, we have shown that ρ is a *separative congruence* in S, that is, $a^2\rho ab$ and $ab\rho b^2$ implies that $a\rho b$ when $a, b \in S$.

The following four propositions have been proved in [10].

Proposition 1. *Every AG^*-groupoid has associative powers, i.e., $aa^n = a^n a$ for all a.*

Proposition 2. *In an AG^*-groupoid S, $a^m a^n = a^{m+n}$ for all $a \in S$ and positive integers m, n.*

Proposition 3. *In an AG^*-groupoid S, $(a^m)^n = a^{mn}$ for all $a \in S$ and positive integers m, n.*

Proposition 4. *If S is an AG^*-groupoid, then for all $a, b \in S$, $(ab)^n = a^n b^n$ and positive integer $n \geqslant 1$ and $(ab)^n = b^n a^n$ for $n > 1$.*

Theorem 1. *Let S be an AG^*-groupoid. If $ab^m = b^{m+1}$ and $ba^n = a^{n+1}$ for $a, b \in S$ and positive integers m, n then $a\rho b$.*

Proof. For the sake of definiteness assume that $m < n$ and $m > 1$. Then by multiplying, $ab^m = b^{m+1}$ by b^{n-m} and successively applying Proposition 1, identities (1) and (2), we obtain

$$\begin{aligned} b^{m+1}b^{n-m} &= (ab^m)b^{n-m} = a(b^{m-1}b)b^{n-m} = (b^{m-1}a)bb^{n-m} \\ &= (b^{n-m}b)(b^{m-1}a) = (bb^{n-m})(b^{m-1}a) = b^{n-m}(b(b^{m-1}a)) \\ &= b^{n-m}((ab)b^{m-1}) = ((ab)b^{n-m})b^{m-1} = (b^{n-m+1}a)b^{m-1} \\ &= a(b^{n-m+1}b^{m-1}) = ab^n. \end{aligned}$$

Thus $ab^n = b^{n+1}$, $ba^n = a^{n+1}$ and so $a\rho b$. □

Theorem 2. *The relation ρ on an AG^*-groupoid is a congruence relation.*

Proof. Evidently ρ is reflexive and symmetric. For transitivity we may proceed as follows.

Let $a\rho b$ and $b\rho c$ so that there exist positive integers n, m such that,

$$ab^n = b^{n+1}, \quad ba^n = a^{n+1} \quad \text{and} \quad bc^m = c^{m+1}, \quad cb^m = b^{m+1}.$$

Let $k=(n+1)(m+1)-1$, that is, $k=n(m+1)+m$. Using identities (1), (2) and Propositions 2 and 3, we get

$$\begin{aligned}
ac^k &= ac^{n(m+1)+m} = a(c^{n(m+1)}c^m) = a((c^{m+1})^n c^m) = a((bc^m)^n c^m)\\
&= a((b^n c^{mn})c^m) = a(c^{m(n+1)}b^n) = (b^n a)c^{m(n+1)} = (b^n a)(c^{m(n+1)-1}c)\\
&= (b^n c^{m(n+1)-1})(ac) = ((ac)c^{m(n+1)-1})b^n = (c(ac^{m(n+1)-1}))b^n\\
&= (b^n(ac^{m(n+1)-1}))c = ((ab^n)c^{m(n+1)-1})c = (b^{n+1}c^{m(n+1)-1})c\\
&= ((bb^n)c^{m(n+1)-1})c = (b^n(bc^{m(n+1)-1}))c = (c(bc^{m(n+1)-1}))b^n\\
&= ((bc)c^{m(n+1)-1})b^n = (b^n c^{m(n+1)-1})(bc) = (b^n b)(c^{m(n+1)-1}c)\\
&= b^{n+1}c^{m(n+1)} = (bc^m)^{n+1} = c^{(m+1)(n+1)} = c^{k+1}.
\end{aligned}$$

Similarly, $ca^k = a^{k+1}$. Thus ρ is an equivalence relation. To show that ρ is compatible, assume that $a\rho b$ such that for some positive integer n,

$$ab^n = b^{n+1} \quad \text{and} \quad ba^n = a^{n+1}.$$

Let $c \in S$. Then by identity (2) and Propositions 4 and 1, we get

$$(ac)(bc)^n = (ac)(b^n c^n) = (ab^n)(cc^n) = b^{n+1}c^{n+1}.$$

Similarly, $(bc)(ac)^n = (ac)^{n+1}$. Hence ρ is a congruence relation on S. $\square$

Theorem 3. *The relation ρ is separative.*

Proof. Let $a, b \in S$, $ab\rho a^2$ and $ab\rho b^2$. Then by definition of ρ there exist positive integers m and n such that,

$$\begin{aligned}
(ab)(a^2)^m &= (a^2)^{m+1}, & a^2(ab)^m &= (ab)^{m+1},\\
(ab)(b^2)^n &= (b^2)^{n+1}, & b^2(ab)^n &= (ab)^{n+1}.
\end{aligned}$$

Now using identities (3), (2), (1) and Proposition 1, we get

$$\begin{aligned}
ba^{2m+1} &= b(a^{2m}a) = (ab)a^{2m} = (ab)(a^m a^m) = (aa^m)(ba^m)\\
&= a^{m+1}(ba^m) = (ba^{m+1})a^m = (b(a^m a))a^m = ((a^m b)a)a^m\\
&= (a^m a)(a^m b) = (aa^m)(a^m b) = a^m(a(a^m b))\\
&= a^m((ba)a^m) = ((ba)a^m)a^m = ((a^m a)b)a^m\\
&= (a^{m+1}b)a^m = b(a^{m+1}a^m) = ba^{2m+1} = b(a^{2m}a)\\
&= (ab)a^{2m} = (ab)(a^2)^m = (a^2)^{m+1} = a^{2m+2}.
\end{aligned}$$

Using identities (3), (2) and (1) and Theorem 2, 3, we get

$$\begin{aligned} ab^{2n+1} &= a(b^{2n}b) = (ba)b^{2n} = (ba)(b^n b^n) = (bb^n)(ab^n) \\ &= (b^n(bb^n))a = ((b^n b^n)b)a = (ab)(b^n b^n) \\ &= (ab)(b^{2n}) = (ab)(b^2)^n = (b^2)^{n+1} = b^{2n+2}. \end{aligned}$$

Now by Theorem 1, $a\rho b$. Hence ρ is separative. □

The following Lemma has been proved in [10]. We re-state it without proof for use in our later results.

Lemma 1. *Let σ be a separative congruence on an AG^*-groupoid S, then for all $a, b \in S$ it follows that $ab\sigma ba$.*

Theorem 4. *Let S be an AG^*-groupoid. Then S/ρ is a maximal separative commutative image of S.*

Proof. By Theorem 3, ρ is separative, and hence S/ρ is separative. We now show that ρ is contained in every separative congruence relation σ on S. Let $a\rho b$ so that there exists a positive integer n such that,

$$ab^n = b^{n+1} \quad \text{and} \quad ba^n = a^{n+1}.$$

We need to show that $a\sigma b$, where σ is a separative congruence on S. Let k be any positive integer such that,

$$ab^k \sigma b^{k+1} \quad \text{and} \quad ba^k \sigma a^{k+1}.$$

Suppose $k \geqslant 2$. Putting $ab^0 = a$ in the next term (if $k = 2$)

$$\begin{aligned} (ab^{k-1})^2 &= (ab^{k-1})(ab^{k-1}) = a^2 b^{2k-2} = (aa)(b^{k-2}b^k) \\ &= (ab^{k-2})(ab^k) = (ab^{k-2})b^{k+1}, \end{aligned}$$

i.e., $ab^{k-2})(ab^k)\sigma(ab^{k-2})b^{k+1}$.

Using identity (1) and Proposition 2 we get

$$\begin{aligned} (ab^{k-2})b^{k+1} &= (b^{k+1}b^{k-2})a = b^{2k-1}a = (b^k b^{k-1})a = (ab^{k-1})b^k, \\ (ab^{k-1})b^k &= (b^k b^{k-1})a = b^{2k-1}a = (b^{k-1}b^k)a = (ab^k)b^{k-1}. \end{aligned}$$

Thus $(ab^{k-1})^2 \sigma (ab^k)b^{k-1}$.

Since $ab^k \sigma b^{k+1}$ and $(ab^k)b^{k-1} \sigma b^{k+1}b^{k-1}$, hence $(ab^{k-1})^2 \sigma (b^k)^2$. It further implies that, $(ab^{k-1})^2 \sigma (ab^{k-1})b^k \sigma (b^k)^2$. Thus $ab^{k-1} \sigma b^k$. Similarly, $ba^{k-1} \sigma a^k$.

Thus if (1) holds for k, it holds for $k+1$. By induction down from k, it follows that (1) holds for $k=1$, $ab\sigma b^2$ and $ba\sigma a^2$. Hence by Lemma 1 and separativity of σ it follows that $a\sigma b$. $\square$

Lemma 2. *If $xa = x$ for some x and for some a in an AG^*-groupoid, then $x^n a = x^n$ for some positive integer n.*

Proof. Let $n=2$, then identity (3) implies that

$$x^2a = (xx)a = x(xa) = xx = x^2.$$

Let the result be true for k, that is $x^k a = x^k$. Then by identity (3) and Proposition 1, we get

$$x^{k+1}a = (xx^k)a = x^k(xa) = x^k x = x^{k+1}.$$

Hence $x^n a = x^n$ for all positive integers n. $\square$

Theorem 5. *Let a be a fixed element of an AG^*-groupoid S, then*
$$Q = \{x \in S \mid xa = x \text{ and } a = a^2\}$$
is a commutative subsemigroup.

Proof. As $aa = a$, we have $a \in Q$. Now if $x, y \in Q$ then by identity (2),

$$xy = (xa)(ya) = (xy)(aa) = (xy)a.$$

To prove that Q is commutative and associative, assume that x and y belong to Q. Then by using identity (1), we get $xy = (xa)y = (ya)x = yx$, and commutativity gives associativity. Hence Q is a commutative subsemigroup of S. $\square$

Theorem 6. *Let η and ξ be separative congruences on an AG^*-groupoid S and $x^2a = x^2$, for all $x \in S$. If $\eta \cap (Q \times Q) \subseteq \xi \cap (Q \times Q)$, then $\eta \subseteq \xi$.*

Proof. If $x\eta y$ then,

$$(x^2(xy))^2\eta(x^2(xy)(x^2y^2)\eta(x^2y^2)^2.$$

It follows that $(x^2(xy))^2, (x^2y^2)^2 \in Q$. Now by identities (2), (1), (3), respectively and Lemma 2, it means that,

$$\begin{aligned}(x^2(xy))(x^2y^2) &= (x^2x^2)((xy)y^2) = (x^2x^2)(y^3x)\\ &= x^4(y^3x) = (xx^4)y^3 = x^5y^3,\\ (x^5y^3)a = (x^5y^3)(aa) &= (x^5a)(y^3a) = x^5y^3.\end{aligned}$$

So, $x^2(xy)(x^2y^2) \in Q$. Hence $(x^2(xy))^2\xi(x^2(xy)(x^2y^2)\xi(x^2y^2)^2$ implies that $x^2(xy)\xi x^2y^2$.

Since $x^2y^2\eta x^4$ and $x^2a = x^2$ for all $x \in S$, so $(x^2y^2), x^4 \in Q$. Thus $x^2y^2\xi x^4$ and it follows from Proposition 4 that $x^2y^2 = (xy)^2$. So $(x^2)^2\xi x^2(xy)\xi(xy)^2$ which means that $x^2\xi xy$. Finally, $x^2\eta y^2$ and $x^2, y^2 \in Q$, means that $x^2\xi y^2, x^2\xi xy\xi y^2$. As ξ is separative so $x\xi y$. Hence $\eta \subseteq \xi$ and by Lemma 1, S/η is the maximal separative commutative image of S. $\square$

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Received October 9, 2006
Revised December 18, 2006

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