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Subdirectly irreducible sloops and SQS-skeins

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Abstract

It was shown in [2] that there is 8 classes of nonsimple subdirectly irreducible SQS-skeins of cardinality 32 (SK(32)s). Now, we present the same classification for sloops of cardinality 32 (SL(32)s) and unify this classification for both SL(32)s and SK(32)s in one table. Next, some recursive construction theorems for subdirectly irreducible SL($2n$)s and SK($2n$)s which are not necessary to be nilpotent are given. Further, we construct an SK($2n$) with a derived SL($2n$) such that SK($2n$) and SL($2n$) are subdirectly irreducible and have the same congruence lattice. We also construct an SK($2n$) with a derived SL($2n$) such that the congruence lattice of SK($2n$) is a proper sublattice of the congruence lattice of SL($2n$).

1. Introduction

A *Steiner quadruple* (*triple*) *system* is a pair $(S;B)$ where S is a finite set and B is a collection of 4-subsets (3-subsets) called *blocks* of S such that every 3-subset (2-subset) of S is contained in exactly one block of B (cf. [13] and [17]). Let SQS(m) denotes a Steiner quadruple system (briefly quadruple system) of cardinality m and STS(n) be a Steiner triple system (briefly: triple system) of the cardinality n. It is well known that SQS(m) exists iff $m \equiv 2$ or $4 \pmod 6$, and STS(n) exists iff $n \equiv 1$ or $3 \pmod 6$ (cf. [13] and [17]). Let $(S;B)$ be an SQS. If $S_a = S - \{a\}$ for some point $a \in S$, then deleting a from all blocks which contain it we obtain the triple system $(S_a; B(a))$, where

$$B(a) = \{b' = b - \{a\} : b \in B \text{ and } a \in b\}.$$

The system $(S_a; B(a))$ is called a *derived triple system* (or briefly DTS) of $(S;B)$ (cf. [13] and [17]).

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There is one-to-one correspondence between STSs and sloops. A *sloop* (briefly SL) $L = (L; \cdot, 1)$ is a groupoid with a neutral element 1 satisfying the identities:

$$x \cdot y = y \cdot x, \quad 1 \cdot x = x, \quad x \cdot (x \cdot y) = y.$$

A sloop L is called *Boolean* if it satisfies the associative law.

Also, there is one-to-one correspondence between SQSs and SQS-skeins (cf. [13] and [17]). An SQS-*skein* (briefly: SK) $(Q; q)$ is an algebra with a ternary operation q such that

$$q(x, y, z) = q(x, z, y) = q(z, x, y), \quad q(x, x, y) = y, \quad q(x, y, q(x, y, z)) = z$$

is valid for all $x, y, z \in Q$. An SQS-skein $(Q; q)$ satisfying the identity:

$$q(a, x, q(a, y, z)) = q(x, y, z)$$

is called *Boolean*. Any sloop associated with a given derived triple system is also called *derived*. A sloop $(Q_a; \cdot, a)$ with the binary operation "$\cdot$" defined by $x \cdot y = q(a, y, z)$, where $a \in Q$, is called *derived sloop* of an SQS-skein $(Q; q)$ *with respect to* $a \in Q$.

A subsloop N of L is called *normal* if and only if $N = [1]\theta$ for a congruence θ on L. Similarly, a sub-SQS-skein of Q is called *normal* if and only if $N = [a]\theta$ for a congruence θ of Q (cf. [13] and [18]). The congruence θ associated with the normal subsloop (sub-SQS-skein) N is given by:

$$\theta = \{(x, y) : x \cdot y \in N\}.$$

All congruences of sloops (SQS-skeins) are permutable, regular and uniform (cf. [1] and [18]). Congruence lattices of sloops and SQS-skeins are modular.

Theorem 1. *Every subsloop (sub-SQS-skein) M of a finite sloop $(L; \cdot, 1)$ (SQS-skein $(Q; q)$) such that $|L| = 2\,|M|$ (resp. $|Q| = 2\,|M|$) is normal.* □

If $(G; +)$ is a Boolean sloop or, equivalently, a Boolean group, then $(G; q)$ with $q(x, y, z) = x + y + z$ is a Boolean SQS-skein [1]. The class $\mathbf{A}_0$ of all Boolean sloops (SQS-skeins) is the smallest non-trivial subvariety of the variety of all sloops (SQS-skeins).

A congruence θ on a sloop L or on an SQS-skein Q is called *central*, if the diagonal relation Δ_L (resp. Δ_Q) is a normal subsloop (sub-SQS-skein) of L (resp. Q). The largest central congruence is called the *center* of L (resp. Q) and is denoted by $\zeta(L)$ (resp. $\zeta(Q)$) (cf. [1] and [11]). A series of congruences $1 = \theta_0 \supseteq \theta_1 \supseteq \theta_2 \supseteq \ldots \supseteq \theta_n = 0$ (or Δ) is called *central series* if $\theta_i/\theta_{i+1} \subseteq \zeta(L/\theta_{i+1})$ (resp. $\theta_i/\theta_{i+1} \subseteq \zeta(Q/\theta_{i+1})$). If L (resp. Q) contains a central series, then L (resp. Q) is called *nilpotent*. If in nilpotent L (resp. Q) the smallest length of a central series is n, then it is called *nilpotent of class* n (cf. [4] and [5]).

Lemma 2. (cf. [4] and [15]) *If θ is a congruence on a sloop L or on an SQS-skein Q and $|[x]\theta| = 2$, then θ is a central congruence. Moreover, if L (resp. Q) is subdirecly irreducible, then $\theta = \zeta(L)$ (resp. $\theta = \zeta(Q)$).* $\square$

2. Subdirectly irreducible SL(32)s and SK(32)s

For any congruence θ on a sloop L or on an SQS-skein Q we may define the dimension $d(\theta)$ as the length of the maximal chain between the smallest congruence 0 (the diagonal relation) and θ in $\mathrm{C}(L)$ or $\mathrm{C}(Q)$. All maximal chains in a finite modular lattice have the same length [16].

All SL(16)s (also SK(16)s) can be divided into 5 classes according to the shape of its congruence lattice or, equivalently, to the number of sub-SL(8)s (sub-SK(8)s) (cf. [8] and [9]). Let L_* (resp. Q_*) be an SL(16) (resp. SK(16)) and let θ_* be an atom in $\mathrm{C}(L_*)$ (resp. $\mathrm{C}(Q_*)$), then $\mathrm{C}(L_*/\theta_* \cong S(\mathbb{Z}_2^r))$ (resp. $\mathrm{C}(Q_1/\theta_*)) \cong S(\mathbb{Z}_2^r)$ (the lattice of all subgroups of the Boolean group $\mathbb{Z}_2^r$). Consequently, for the length of the maximal chain in $\mathrm{C}(L)$ or $\mathrm{C}(Q)$) we have $d(1) = r + 1$ with $r = 0, 1, 2, 3, 4$. So, there are 5 classes for each of SL(16)s and SK(16)s which are presented in Table 1. Examples for each class of SL(16)s and SK(16)s and for an SK(16)s with a derived SL(16) for all possible congruence lattices of SK(16) and its derived SL(16), can be found in [8] and [9].

Armanious gave in [2] all 8 classes of nonsimple subdirectly irreducible SK(32)s. The same classification holds for nonsimple subdirectly irreducible SL(32)s.

If in modular lattice two elements θ and φ cover $\theta \wedge \varphi$, then $\theta \vee \varphi$ covers θ and φ [16]. Moreover, $\theta \vee \varphi = \theta \circ \varphi$ in permutable varieties. This implies that if θ and φ are atoms in the congruence lattice $\mathrm{C}(L)$ (resp. $\mathrm{C}(Q)$) of a finite sloop (SQS-skein), then the congruence $\theta \vee \varphi = \theta \circ \varphi$ covers θ and φ. Also, the dimensions d(1) of the largest congruence 1 of both L/θ and L/φ (resp. Q/θ and Q/φ) are the same.

d(1)	C(L_*) and C(Q_*) are isomorphic to	Algebraic properties of SL(16) $= L_*$ and SQ(16) $= Q_*$	Properties of the STS(15) and SQS(16)
1	$\theta_* = 1$ $\mid [x] \mid \theta_* = 16$	L_* and Q_* are simple.	STS(15) has no sub-STS(7)s and SQS(16) has no sub-SQS(8)s.
2	$\|[x]\|\ \theta_* = 8$	$\mathbf{C}(L_*)$ and $\mathbf{C}(Q_*)$ have one proper congruence. L_* and Q_* are subdirectly irreducible, but not nilpotent.	STS(15) has one sub-STS(7) and SQS(16) has two disjoint sub-SQS(8)s.
3	$\mid [x]\theta_* \mid = 4$	$\mathbf{C}(L_*)$ and $\mathbf{C}(Q_*)$ have 3 co-atoms. L_* and Q_* are subdirectly irreducible, but not nilpotent.	STS(15) has 3 sub-STS(7)s and SQS(16) has 6 sub-SQS(8)s.
4	$\|[x]\theta_* \| = 2$	$\mathbf{C}(L_*)$ and $\mathbf{C}(Q_*)$ have 7 co-atoms. The atom θ_* is the center of L_* (resp. Q_*). L_* and Q_* are subdirectly of nilpotence calss 2.	STS(15) has exactly 7 sub-STS(7)s and SQS(16) has exactly 14 sub-SQS(8)s.
5	It has more than one atom.	Both L_* and Q_* are Boolean. So $L_* \cong \mathrm{SL}(2)^4$ and $Q_* \cong \mathrm{SK}(2)^4$. It has $2^4 - 1$ atoms and $2^4 - 1$ co-atoms.	STS(15) has exactly 15 sub-STS(7)s and SQS(16) has exactly 30 sub-SQS(8)s.

Table 1. All classes of subdirectly irreducible sloops and SQS-skeins of cardinality 16.

So, $d(1) = 1$ iff L (resp. Q) is simple and $d(1) = n$ if L (resp. Q) is Boolean of the cardinality 2^n. In general, $1 \leqslant d(1) \leqslant n$ for each of SL(2^n) and SK(2^n).

Consider a sloop $L = \mathrm{SL}(32)$ and an SQS-skein $Q = SK(32)$, in which both L and Q are subdirectly irreducible with a monolith θ_0. It is well-known that there are simple SK(n)s and simple SL(n)s for each $n \equiv 2$ or 4 (mod 6) (see [1], [7], [10] and [18]). Except the case $d(1) = 1$, when SK(n)s and SL(n)s are simple, we have four other cases $d(1) = 2, 3, 4, 5$. For each $d(1) = r$, we have two different classes of L/θ_0 (resp. Q/θ_0). In the first class L/θ_0 (resp. Q/θ_0) is Boolean and has 2^{r-1} elements. In the second L/θ_0 (resp. $Q\theta_0$) is an SL(16) (resp. SK(16)) and belongs to the class $r - 1$ of Table 1. This means that the congruence lattice $C(L)$ (resp. $C(Q)$) is isomorphic to one of the following two lattices:

In the following table, we review the algebraic and combinatoric properties of each class of nonsimple subdirectly irreducible SL(32)s and SK(32)s.

d(1)	The lattices C(L_*) and C(Q_*)	Properties of $\mathrm{SL}(32) = L$ and $\mathrm{SQ}(32) = Q$	Properties of the associated STS(31) and SQS(32)
2 (a)	$\mid [x]\theta_0 \mid = 2$	Normal subalgebras of L and Q have 2 elements. Has no subalgebras of cardinality > 8. Only homomorphic images of L/θ_0 and Q/θ_0 are simple of cardinality 16.	STS(31) has $(15 \cdot 14)/6$ sub-STS(7)s. SQS(32) has $(16 \cdot 15 \cdot 14)/24$ sub-SQS(8)s.
2 (b)	$\mid [x]\theta_0 \mid = 16$	L has one sub-SL(16) and Q has two disjoint sub-SK(16)s. Only proper homomorphic images of L/θ_0 and Q/θ_0 are of cardinality 2.	STS(31) has only one sub-STS(15). SQS(32) has two disjoint sub-SQS(16)s. These 3 subsystems belongs to the classes from Table 1.

3 (a)	$\mid [x]\theta_1 \mid= 16$ $\mid [x]\theta_0 \mid= 2$	$\mid L/\theta_0 \mid=\mid Q/\theta_0 \mid= 16$, $\mid L/\theta_1 \mid=\mid Q/\theta_1 \mid= 2$. L/θ_0 and Q/θ_0 belong to the class 2 from Table 1. $L(Q)$ has only one normal sub-SL(16) (two disjoint normal sub-SK(16)s). These subsystems belong to the class $4(a)$ or $4(b)$ of Table 1.	STS(31) has only one sub-STS(15) and at least $(15 \cdot 14)/6$ sub-STS(7)s. The SQS(32) has two disjoint sub-SQS(16)s and at least $(16 \cdot 15 \cdot 14)/24$ sub-SQS(8)s.
3 (b)	$\mid [x]\theta_0 \mid= 8$	$\mid L/\theta_0 \mid=\mid Q/\theta_0 \mid= 4$. $L(Q)$ has 3 normal sub-SL(16)s (6 normal sub-SK(16)s) and only one normal sub-SL(8) (4 disjoint normal sub-SK(8)s). Sub-SL(16)s and sub-SK(16)s are not simple and belong to some nonsimple class from Table 1.	The STS(31) has exactly three sub-STS(15)s and the SQS(32) has six sub-SQS(16)s.
4 (a)	$\mid [x]\theta_1 \mid= 8$ $\mid [x]\theta_0 \mid= 2$	$\mid L/\theta_1 \mid=\mid Q/\theta_1 \mid= 4$, $\mid L/\theta_0 \mid=\mid Q/\theta_0 \mid= 16$. L/θ_0 and Q/θ_0 belong to the class 3 of Table 1. $L(Q)$ has three normal sub-SL(16)s (6 normal sub-SK(16)s) and only one normal sub-SL(8) (4 disjoint normal sub-SK(8)s). Sub-SL(16)s and sub-SK(16)s belong to the class $4(a)$ or $4(b)$ of Table 1.	STS(31) has only 3 sub-STS(15)s. The associated SQS(32) has 6 sub-SQS(16)s.

4 (b)	$\mid [x]\theta_0 \mid = 4$	$\mid L/\theta_0 \mid = \mid Q/\theta_0 \mid = 8$. $L(Q)$ has 7 normal sub-SL(16)s (14 normal sub-SK(16)s) and only one normal sub-SL(4) (8 disjoint normal sub-SK(4)s). Sub-SL(16)s and sub-SK(16)s belong to the class 3 or 4 of Table 1.	The STS(31) has exactly 7 sub-STS(15)s. The associated SQS(32) has 14 sub-SQS(16)s.
5 (a)	$\mid [x]\theta_1 \mid = 4$ $\mid [x]\theta_0 \mid = 2$	$\mid L/\theta_1 \mid = \mid Q/\theta_1 \mid = 8$, $\mid L/\theta_0 \mid = \mid Q/\theta_0 \mid = 16$. L/θ_0 and Q/θ_0 belong to the class $4(a)$ of Table 1. θ_0 is the center of $L(Q)$ and θ_1/θ_0 is the center of $L/\theta_0(Q/\theta_0)$. $L(Q)$ is of nilpotence class 3 and has 7 normal sub-SL(16)s (14 normal sub-SK(16)s) and exactly one normal sub-SL(4) (8 disjoint normal sub-SK(4)s) and one normal sub-SL(2) (16 disjoint normal sub-SK(2)s).	The STS(31) has exactly 7 sub-STS(15)s and the associated SQS(32) has exactly 14 sub-SQS(16)s. All sub-STS(16)s and sub-SQS(16)s belong to the class $4(a)$ or $4(b)$ of Table 1.
5 (b)	$\mid [x]\theta_0 \mid = 2$	$\mid L/\theta_0 \mid = \mid Q/\theta_0 \mid = 16$. $L(Q)$ is nilpotent of the class 2 and θ_0 is its center. $L(Q)$ has 15 normal sub-SL(16)s (30 normal sub-SK(16)s) and exactly one normal sub-SL(2) (16 disjoint normal sub-SK(2)s). Sub-SL(16)s and sub-SK(16)s belong to the class $4(a)$ or $4(b)$ of Table 1.	STS(31) has exactly 15 sub-STS(15)s and the associated SQS(32) has exactly 30 sub-SQS(16)s.

3. Subdirectly irreducible SL(2n)s and SK(2n)s

In this section, we find recursive constructions for subdirectly irreducible sloops and SQS-skeins, i.e., for subdirectly irreducible SK(n) $= Q_*$ and SL(n) $= L_*$ with a monolith θ_*, we construct subdirectly irreducible $Q = SK(2n)$ (resp. $L = SL(2n)$) having a homomorphic image which congruent to Q_* (resp. to L_*).

For a given subdirectly irreducible SK(n) and SL(n) of nilpotence class $k > 1$ Guelzow (cf. [14], [15]) and Armanious (cf. [3], [4], [5]) constructed a subdirectly irreducible SK($2n$) (resp. SL($2n$)) of nilpotence class $k+1$. Below, basing on results of [15] and [4], we present three recursive constructions for subdirectly irreducible SQS-skeins and sloops. Namely, for a given subdirectly irreducible SK(n) and SL(n) (not necessary nilpotent or simple) with a monolith, we construct a subdirectly irreducible SK($2n$) (resp. SL($2n$)).

Construction. Let $Q_* = (Q_*; q_*)$ be an SK(n) and $L_* = (L_*; *, 1)$ be an SL(n). Let $L_* = Q_* = \{x_0, x_1, \ldots, x_{n-1}\}$ and R be a set of sub-SK(4)s of Q_* (sub-SL(4)s of L_*), where x_0 denotes the unit 1 of sloops. Consider the binary operation $\bullet$ on $L = L_* \times GF(2)$ and the ternary operation q on $Q = Q_* \times GF(2)$ defined as follows:

$$q((x, i_x), (y, i_y), (z, i_z)) = (q_*(x, y, z), i_x + i_y + i_z + \chi_R\langle x, y, z\rangle_{Q_*}),$$
$$(x, i_x) \bullet (y, i_y) = (x * y, i_x + i_y + \chi_R\langle x, y\rangle_{L_*}),$$

where χ_R is the characteristic function such that $\chi_R\langle x, y, z\rangle_{Q_*} = 1$ if $\langle x, y, z\rangle_{Q_*}$ generates a sub-SK(4) $\in R$, and 0 otherwise; $\chi_R\langle x, y\rangle_{L_*} = 1$ if $\langle x, y\rangle_{L_*}$ generates a sub-SL(4) $\in R$, and 0 otherwise.

It easy to prove that $Q = (Q; q)$ is an SK($2n$) and $L = (L; \bullet)$ is an SL($2n$) (for details see [15] and [4]). In the sequel, the SQS-skein Q and the sloop L will be denoted by $2 \times_R Q_*$ and $2 \times_R L_*$, respectively.

If R is empty, then $\chi_R\langle x, y, z\rangle_{Q_*} = 0$ for $x, y, z \in Q_*$ and $\chi_R\langle x, y\rangle_{L_*} = 0$ for $x, y \in L_*$. Thus $(Q; q) = Q_* \times SK(2)$ and $(L; \bullet, (1, 0)) = L_* \times SL(2)$. If R is the set of all sub-SK(4)s of Q_* (resp. sub-SL(4)s of L_*), then $Q(L)$ is Boolean or of nilpotence class $k+1$ if and only if $Q_* L_*)$ is Boolean or of nilpotence class $k > 1$, respectively. Moreover, Q is semi-boolean if and only if Q_* is semi-boolean (see [15]).

Lemma 3. *Let Q_* (resp. L_*) be a subdirectly irreducible SQS-skein (sloop) with monolith θ_* and let R be the set of sub-*SK(4)*s (sub-*SL(4)*s). The con-*

structed SKS-*skein* $2\times_R Q_*$ (*resp. sloop* $2\times_R L_*$) *has a congruence* θ_1 *which covers all its minimal congruences.*

Proof. The projection π from $Q(L)$ into the first component is onto homomorphism and the congruence $\ker\pi = \theta_0$ on $Q(L)$ is determined by the relation $\{((x,i),(x,j)) : \forall x \in Q_*(L_*), \forall i,j \in \{0,1\}\}$. Now $Q/\theta_0 \cong Q_*$ and $L/\theta_0 \cong L_*$. Since θ_* is the monolith of Q_* and L_*, Q/θ_0 (resp. L/θ_0) has a monolith θ_1/θ_0 for a congruence θ_1 on Q (resp. on L). Thus θ_1 is the unique congruence in $\mathrm{C}(Q)$ (resp. $C(L)$) which covers θ_0. If δ is another atom of $\mathrm{C}(Q)$ (resp. $C(L)$), then $\delta \circ \theta_0 = \theta_1$ covers δ and θ_0. Therefore, θ_1 covers all atoms of $\mathrm{C}(Q)$ (resp. $C(L)$). ☐

Moreover, since $\mid [(x_0,0)]\theta_0 \mid= 2$, it follows that if $\mid [(x_0)]\theta_* \mid= m$, then $\mid [(x_0,0)]\theta_1 \mid= 2m$.

Guelzow [15] and Armanious [4] for a given subdirectly irreducible $\mathrm{SK}(n) = Q_*$ ($\mathrm{SL}(n) = L_*$) of nilpotence class k with a minimal congruence θ_* such that $\mid [x]\theta_* \mid= 2$ constructed subdirectly irreducible $\mathrm{SK}(2n) = Q$ and $\mathrm{SL}(2n) = L$ of nilpotence class $k+1$.

Below we prove that for a subdirectly irreducible SK $(n) = Q_*$ (resp. $\mathrm{SL}(n) = L_*$) with a monolith θ_* for each possible cardinality of $|[x]\theta_* \mid$ the constructed $Q = 2 \times_R Q_*$ (resp. $L = 2 \times_R L_*$) is subdirectly irreducible. Note that Q_* and L_* are not nilpotent, in general.

In the following three theorems, let x_0 be the unit 1 of sloops, $*$ the binary operation on L_* and $\bullet$ the operation on L, i.e., $x * y = q_*(x_0,x,y)$ on the set Q_* and $(x,i)\bullet(y,j) = q((x_0,\,0),\,(x,\,i),\,(y,\,j))$ on the set Q.

The proof of the theorem presented below is analogous to the proof of the corresponding theorems for nilpotent SQS-skeins and sloops from [3] and [4].

Theorem 4. *Let* $n > 8$. *If* $\mathrm{SK}(n) = Q_*$ (*resp.* $\mathrm{SL}(n) = L_*$) *is subdirectly irreducible with a monolith* $\theta_* = \cup\{\{x_i,x_{i+1}\}^2 : i = 0,2,\ldots,n-2\}$ *and* $R = \{x_0,x_1,x_2,x_3\}$, *then the constructed SQS-skein* $Q = 2\times_R Q_*$ (*resp. sloop* $L = 2\times_R L_*$) *is also subdirectly irreducible.*

Proof. As in Lemma 3, $\theta_0 = \{((x,i),(x,j)) : x \in Q_*,\, i,j \in \{0,1\}\}$ (resp. $x \in L_*$) is an atom of $\mathrm{C}(Q)$($resp.\mathrm{C}(L)$) and $\theta_1 \in \mathrm{C}(Q)$ (resp. $\mathrm{C}(L)$) is the unique congruence covering all atoms of $\mathrm{C}(Q)$ (resp. $\mathrm{C}(L)$). The theorem will be proved if we show that the congruence θ_0 is the unique atom in the congruence lattice $\mathrm{C}(Q)$ (resp. $\mathrm{C}(L)$). If there is another atom $\delta \neq \theta_0$ in the congruence lattice $\mathrm{C}(Q)$ (resp. $\mathrm{C}(L)$), then $\delta \circ \theta_0 = \theta_1$ covers both

δ and θ_0. Since $[x_0]\theta_* = \{x_0,\ x_1\}$ and $\mid [(x_0,0)]\theta_0 \mid= 2$, it follows that $\mid [(x_0,0)]\theta_1 \mid= 4$. Then $[(x_0,0)]\theta_1 = \{(x_0,0),\ (x_0,1),\ (x_1,0),\ (x_1,1)\}$. This means that if there is another atom δ in $C(Q)$ (resp. $C(L)$), then

$$[(x_0,0)]\delta = \{(x_0,0),(x_1,0)\} \quad \text{or} \quad [(x_0,0)]\delta = \{(x_0,0),(x_1,1)\},$$

which is impossible.Indeed, each 2-element subset $\{x,y\}$ of an SK is a sub-SK(2) and for each element x of an SL the set $\{1,x\}$ is a sub-SL(2). Also, if θ is a congruence on an SK (resp. SL), then $[x]\theta\cup[y]\theta([1]\theta\cup[x]\theta)$ is a sub-SK (sub-SL). In addition, $x\theta y$ if and only if $q(a,y,z)\theta a$ (resp. $x \cdot y\theta 1$). Moreover, for 3 distinct elements $x,\ y,\ z$, we have $q(x,y,z) \notin \{x,y,z\}$ because, for example, $q(x,y,z) = z$ implies $y = q(x,z,q(x,z,y)) = q(x,z,z) = x$.

In the case of SQS-skeins, according to the definition of θ_*, we see that $\{x_0,x_1\}\cup\{x_2,x_3\}$ and $\{x_0,x_1\}\cup\{x_4,x_5\}$ are sub-SK(4). Thus $q_*(x_0,x_1,x_2) = x_3$ and $q_*(x_0,x_1,x_4) = x_5$. So, $q_*(x_0,x_2,x_4) = x_k$ and $[x_k]\theta_* = \{x_k,x_{k+1}\}$. Therefore, $q_*(x_0,x_k,x_{k+1}) = x_1$.

For $[(x_0,0)]\delta = \{(x_0,0),(x_1,0)\}$, we have $q((x_0,0),(x_2,0),(x_3,1)) = (x_1,0)$ and $q((x_0,0),(x_4,0),(x_5,0)) = (x_1,0)$. Thus $(x_2,0)\delta(x_3,1)$ and $(x_4,0)\delta(x_5,0)$. But $(x_0,0)\delta(x_0,0)$, so, $(q_*(x_0,x_2,x_4),0)\delta(q_*(x_0,x_3,x_5),1)$, i.e., $(x_k,0)\delta(q_*(x_0,x_3,x_5),1)$. This means that $(q_*(x_0,x_3,x_5),1) \in [(x_k,0)]\delta$, which is a contradiction because $[(x_k,0)]\delta = q((x_0,0),(x_k,0),[(x_0,\ 0)]\delta) = \{(x_k,0),(x_{k+1},0)\}$, where $q_*(x_0,x_k,x_{k+1}) = x_1$.

For $[(x_0,0)]\delta = \{(x_0,0),(x_1,1)\}$, we have $q((x_0,0),(x_2,0),(x_3,0)) = (x_1,1)$ and $q((x_0,0),(x_4,0),(x_5,1)) = (x_1,1)$. Whence $(x_2,0)\delta(x_3,0)$ and $(x_4,0)\delta(x_5,1)$. This implies that $(q_*(x_0,x_2,x_4),0)\delta(q_*(x_0,x_3,x_5),1)$. Thus $(x_k,0)\delta(q_*(x_0,x_3,x_5),1)$. From this, as a simple consequence, we obtain $q((x_0,0),(x_5,1),(x_k,0))\delta q((x_0,0),(x_5,1),(q_*(x_0,x_3,x_5),1))$. This means that $(q_*(x_0,x_5,x_k),1)\delta(x_3,0)$, i.e., $(q_*(x_0,x_5,x_k),1) = (x_2,0)$ or $(x_3,0)$, which is impossible.

Therefore, the congruence θ_0 is the unique atom of C(Q). $\square$

Note that for each positive integers n and k there exists a subdirectly irreducible SK(2^n) (resp. SL(2^n)) of nilpotence class k with a monolith θ_* such that $\mid [x]\theta_* \mid= 2$ (cf. [15] and [4]).

The above results we can summarize in the following table:

If Q_* (resp. L_*) is a subdirectly irreducible SK(n) (resp. SL(n)) for $n \geq 16$ with a monolith θ_* such that $\|[x]\theta_*\|= 2$, then the constructed SQS-skein Q (sloop L) is a subdirectly irreducible with a congruence lattice C(Q) (resp. C(L)) isomorphic to the lattice Γ and C(Q_*) (resp. C(L_*)) is equal to $[\theta_0 : 1]$. Note that in general Q_* and L_* are not nilpotent.	
In particular, Q_* (resp. L_*) may be of nilpotence class $t \geqslant 1$ of cardinality $n = 2^{r+t}$, $r \geqslant 3$, with C(Q_*) (resp. C(L_*)) isomorphic to Γ_1. Then $Q(L)$ is of nilpotence class $t + 1$ having a congruence lattice C(Q) (resp. C(L)) isomorphic to Γ_2 with $\| [x]\theta_i \|= 2^{i+1}$ $(i = 0, 1, \dots, t-1)$ and $Q/\theta_0 \cong Q_*$ (resp. $L/\theta_0 \cong L_*$). Also, θ_{i+1}/θ_i is the center and in the same time is the monolith of Q/θ_i (resp. L/θ_i). For example, let $r = 3$ and $t = 1$, then Q_* =SK(16) (resp. L_* =SL(16)) belongs to the class 4(a) of Table 1 and Q =SK(32) (resp. L =SL(32)) belongs to the class 5(a) of Table 2.	

Theorem 5. *Let* $(Q_*; q_*)$ *(resp.* $L_*; *, 1)$*) be a subdirectly irreducible SQS-skein (resp. sloop) of cardinality* $n > 8$ *with a minimum congruence* θ_* *such that* $\mid [x]\theta_* \mid= 4$*. If* $R = [x_0]\theta_*$ *is a sub-SK*(4) *(resp. sub-SL*(4)*), then the constructed SQS-skein* $Q = 2 \times_R Q_*$ *(sloop* $L = 2 \times_R L_*$*) is also subdirectly irreducible.*

Proof. As in Lemma 3, θ_0 is an atom and θ_1 is the unique congruence covering θ_0 in C(Q) (resp. in C(L)). Similar to the above theorem, it is suffices to show that θ_0 is the unique atom in the congruence lattice C(Q) (resp. C(L)).

If there is another atom δ in C(Q)(**C**(L)), then θ_1 covers δ and θ_0, and also $\delta \circ \theta_0 = \theta_1$. If $[x_0]\theta_* = \{x_0, x_1, x_2, x_3\}$, then

$$[(x_0, 0)]\theta_1 = \{(x_0, 0), (x_1, 0), (x_2, 0), (x_3, 0), (x_0, 1), (x_1, 1), (x_2, 1), (x_3, 1)\}.$$

This means that the class $[(x_0, 0)]\theta_1$ is divided in to two subclasses $[(x_0, 0)]\delta$ and $[(x_0, 1)]\delta$ such that both $(x, 0)$ and $(x, 1)$ can not be in the same subclass. Indeedd, if $(x,0)\delta(x,1)$, then

$$q((x_0,0),(x,0),(x,1)) = (q_*(x_0,x,x),1) = (x_0,1) \in [(x_0,0)]\delta,$$

which implies $[(x_0,0)]\delta \supseteq [(x_0,0)]\theta_0$. But this is impossible.

If $(x_1,0)$ and $(x_2,0) \in [(x_0,0)]\delta$, then

$$q((x_1,0),(x_2,0),(x_0,0)) = (q_*(x_1,x_2,x_0),1) = (x_3,1).$$

Thus, $(x_3,1) \in [(x_0,0)]\delta$. If $(x_1,1),(x_2,1) \in [(x_0,0)]\delta$, then

$$q((x_1,1),(x_2,1),(x_0,0)) = (q_*(x_1,x_2,x_0),1) = (x_3,1),$$

which gives $(x_3,1) \in [(x_0,0)]\delta$. This means that $[(x_0,0)]\delta$ contains exactly 3-element subset of the set $\{x_0,x_1,x_2,x_3\} \times \{0,1\}$ with the same second component.

We have $|[(x_0,0)]\delta| = 4$ and $|[(x_0,0)]\theta_1| = 8$. Without loss of generality, we can assume that

(i) $[(x_0,0)]\delta = \{(x_0,0),(x_1,0),(x_2,0),(x_3,1)\}$ or

(ii) $[(x_0,0)]\delta = \{(x_0,0),(x_1,1),(x_2,1),(x_3,1)\}$.

Case (i) for SQS-skeins: Assume that $(x,0) \in Q$ such that $x \notin \{x_0,x_1,x_2,x_3\}$, then:

$$\begin{aligned}[(x,0)]\delta &= q((x,0),(x_0,0),[(x_0,0)]\delta)\\ &= \{q((x,0),(x_0,0),(x_0,0)), q((x,0),(x_0,0),(x_1,0)),\\ &\qquad q((x,0),(x_0,0),(x_2,0)), q((x,0),(x_0,0),(x_3,1))\}\\ &= \{(x,0),(q_*(x,x_0,x_1),0),(q_*(x,x_0,x_2),0),(q_*(x,x_0,x_3),1)\}\end{aligned}$$

and

$$\begin{aligned}[(q_*(x,x_0,x_3),1)]\delta &= q((q_*(x,x_0,x_3),1),(x_0,0),[(x_0,0)]\delta)\\ &= \{q((q_*(x,x_0,x_3),1),(x_0,0),(x_0,0)),\\ &\quad q((q_*(x,x_0,x_3),1),(x_0,0),(x_1,0)),\\ &\quad q((q_*(x,x_0,x_3),1),(x_0,0),(x_2,0)),\\ &\quad q((q_*(x,x_0,x_3),1),(x_0,0),(x_3,1))\}\\ &= \{(q_*(x,x_0,x_3),1),(q_*(q_*(x,x_0,x_3),x_0,x_1),1),\\ &\quad (q_*(q_*(x,x_0,x_3),x_0,x_2),1),(x,0)\}.\end{aligned}$$

This means that $[(x,0)]\delta \cap [(q_*(x,\ x_0,\ x_3),\ 1)]\delta \neq \emptyset$ and $[(x,0)]\delta$ is not identical with $[(q_*(x,x_0,x_3),1)]\delta$, which contradicts the fact that δ is a congruence. So, this case is impossible.

In a similar way we can prove that also the second case is impossible. $\square$

Theorem 6. *Let $(Q_*; q_*)$ (resp. $(L_*; *, 1)$) be a subdirectly irreducible SQS-skein (resp. sloop) of cardinality $n > 8$ with a minimum congruence θ_* such that $|[x]\theta_*| > 4$. If $R = \{x_0, x_1, x_2, x_3\}$ is a sub-SK(4) of Q_* (resp. sub-SL(4) of L_*) contained in $[x_0]\theta_*$, then the constructed SQS-skein $Q = 2 \times_R Q_*$ (sloop $L = 2 \times_R L_*$) is also subdirectly irreducible with a monolith θ_0 and $Q/\theta_0 \cong Q_*$ (resp. $L/\theta_0 \cong L_*$).*

Proof. As in Lemma 3, θ_1 is the unique congruence covering the atom θ_0 and all other atoms in C(Q) (resp. C(L)). We need only prove that θ_0 is the unique atom in the congruence lattice C(Q) (resp. C(L)). Assume that there is another atom δ of C(Q) (resp. C(L)). Then $\theta_1 = \delta \circ \theta_0$ covers both δ and θ_0. Since $| \, [(x, i_x)]\theta_0 \, |= 2$ and θ_1 covers δ, it follows that if $| \, [(x, i_x)]\theta_1 \, |= 2m$, then $| \, [(x, i_x)]\delta \, |= m$.

Let $[x_0]\theta_* = \{x_0, x_1, x_2, x_3, \dots, x_{m-1}\}$, then

$$[(x_0, 0)]\theta_1 = \{(x_0, 0), (x_1, 0), (x_2, 0), \dots, (x_0, 1), (x_1, 1), (x_2, 1), \dots\}.$$

This means that the class $[(x_0, 0)]\theta_1$ is divided in to two disjoint subclasses $[(x_0, 0)]\delta$ and $[(x_0, 1)]\delta$. In the same manner as in the previous proof, we can prove that $[(x_0, 0)]\delta$ contains exactly 3-element subset of the set $\{x_0, x_1, x_2, x_3\} \times \{0, 1\}$ with the same second component.

Now, $|[(x_0, 0)]\delta| > 4$, i.e., $|[(x_0, 0)]\theta_1| > 8$ and $R = \{x_0, x_1, x_2, x_3\}$. So,

$$[(x_0, 0)]\delta = \{(x_0, 0), (x_1, 0), (x_2, 0), (x_3, 1), (a_1, 0), \\ \dots, (a_i, 0), (b_1, 1), \dots, (b_j, 1)\}$$

or

$$[(x_0, 0)]\delta = \{(x_0, 0), (x_1, 1), (x_2, 1), (x_3, 1), (a_1, 0), \\ \dots, (a_i, 0), (b_1, 1), \dots, (b_j, 1)\},$$

where $x_0, x_1, x_2, x_3, a_1, a_2, \dots, a_i, b_1, b_2, \dots, b_j$ are m distinct elements.

In the first case for SQS-skeins for all $(a_h, 0) \in [(x_0, 0)]\delta$ with $a_h \notin \{x_0, x_1, x_2, x_3\}$, we have $q((x_3, 1), (a_h, 0), (x_0, 0)) = (b_k, 1) \in [(x_0, 0)]\delta$ and $(b_k, 1) \neq (x_3, 1)$. Moreover, if $(a_{h_1}, 0) \neq (a_{h2}, 0)$, then $(b_{k_1}, 1) \neq (b_{k_2}, 1)$. Also, for all $(b_h, 1) \in [(x_0, 0)]\delta$ with $b_h \notin \{x_0, x_1, x_2, x_3\}$, we can see that $q((x_3, 1), (b_h, 1), (x_0, 0)) = (a_l, 0) \in [(x_0, 0)]\delta$ for $(a_l, 0) \neq (x_0, 0), (x_1, 0)$ or $(x_2, 0)$. Also, if $(b_{h_1}, 1) \neq (b_{h_2}, 1)$, then $(a_{k_1}, 0) \neq (a_{k_2}, 0)$. This implies that the sets $\{a_1, a_2, \dots, a_i\}$ and $\{b_1, b_2, \dots, b_j\}$ have the same cardinality. Then $i = j \geqslant 1$. Let $i = j = r$, then

$$[(x_0, 0)]\delta = \{(x_0, 0), (x_1, 0), (x_2, 0), (x_3, 1), (a_1, 0), ..., (a_r, 0), (b_1, 1), ..., (b_r, 1)\}.$$

Hence the class $[(x_0, 0)]\delta$ is a sub-SQS-skein having $r + 3$ elements with the second component 0 and $r+1$ elements with the second component 1.

But $q((b_1,1),(x_0,0),\{(x_0,0),(x_1,0),(x_2,0),(a_1,0),\ldots,(a_r,0)\}$ gives $r+3$ distinct elements with second component 1, which is impossible.

In the second case we obtain a similar contradiction. This proves that θ_0 is the unique minimal congruence on Q (resp. L). □

In view of Theorems 5 and 6 we get the following constructions:

If Q_* (resp. L_*) is a subdirectly irreducible SK(n) (resp. SL(n)) for $n \geq 16$ with a monolith θ_* satisfying $\mid [x]\theta_* \mid > 2$, then Q_* (resp. L_*) is not nilpotent. The constructed SQS-skein Q (sloop L) is a subdirectly irreducible having a congruence lattice C(Q) (resp. C(L)) isomorphic to $\mathbf{\Gamma}_2$ and C$(Q_*) = [\theta_0 : 1]$ (resp. C$(L_*) = [\theta_0 : 1]$) isomorphic to $\mathbf{\Gamma}_1$. The sublattice $[\theta_* : 1]$ of Γ_1 is not necessary to be isomorphic to S$(\mathbb{Z}_2^r)$.	
In particular, if Q_* (resp. L_*) is a subdirectly irreducible SK(2^n) (resp. SL(2^n)) for $n \geqslant 4$ with a monolith θ_* such that $\|[x]\theta_*\| = 2^r$, then then the constructed SQS-skein Q (resp. sloop L) is a subdirectly irreducible and has a congruence lattice C(Q) (resp. C(L) isomorphic to $\mathbf{\Gamma}_2$ and C$(Q_*) = [\theta_0 : 1]$ (resp. C$(L_*) = [\theta_0 : 1]$) isomorphic to $\mathbf{\Gamma}_1$. Indeed, $\|[x]\theta_1\| = 2^{r+1}$ and $\|[x]\theta_0\| = 2$, for each $n \geqslant 4$ and $r = n, n-1, \ldots, 1$. Note that Q_* (resp. L_*) is not nilpotent for $r > 1$ and Q_* (resp. L_*) is simple for $r = n$.	

Examples. 1. For $n = 4$ and $r = 3, 2$ or 1, we may choose an SK$(2^4) = Q_*$ (resp. SL$(2^4) = L_*$) belonging to the classes 2, 3 or 4(a) of Table 1, respectively. Applying Theorems 6, 5 and 4 to Q_* (resp. L_*), we get three examples of a subdirectly irreducible SK$(2^5) = Q$ (resp. SL$(2^5) = L$) belonging to classes 3(a), 4(a) and 5(a) of Table 2.

2. For $n > 3$ and $r = n$, we observe that Q_* (resp. L_*) is simple of cardinality 2^n and the congruence lattice of C(Q) (resp. C(L)) is a chain of length 2, i.e., θ_1 is the largest congruence in C(Q) (resp. C(L)) and θ_0 is the monolith. For instance, take $r = n = 4$ and choose a simple SK$(2^4) = Q_*$ (resp. SL$(2^4) = L_*$) as in the class 1 of Table 1. In view of Theoerem 6, we get a subdirectly irreducible $\mathrm{SK}(2^5) = Q$ (resp. $\mathrm{SL}(2^5) = L$) belonging to the class $2(a)$ of Table 2.

3. In [6] and [7] Armanious has shown that if we have a simple $\mathrm{SK}(n) = Q_*$ (resp. SL $(n) = L_*$), then there is a subdirectly irreducible $\mathrm{SK}(2n)$ (resp. $\mathrm{SL}(2n)$) having only one proper congruence. In particular, for $n = 16$, choose a simple $\mathrm{SK}(16) = Q_*$ (resp. $\mathrm{SL}(16) = L_*$) the construction $\mathrm{SK}(32)= 2 \otimes_\alpha Q_*$ [7] (resp. $\mathrm{SL}(32) = 2 \otimes_\alpha L_*$ [6]) is an example of a subdirectly irreducible SK(32) (resp. SL(32)) belonging to the class $2(b)$ of Table 2.

4. For $n \geqslant 3$ and $r = 0$, Q_* (resp. L_*) is Boolean of cardinality 2^n. According to the constructions given in [15] and [5], we may say that there is a subdirectly irreducible $\mathrm{SK}(2^{n+1}) = Q$ (resp. $\mathrm{SL}(2^{n+1}) = L$) with a monolith θ_0 such that Q/θ_0 (resp. (L/θ_0) is a Boolean $\mathrm{SK}(2^n)$ (resp. $\mathrm{SL}(2^n)$). For instance, let $n = 4$ and $r = 0$, then the constructed $\mathrm{SK}(2^5) = Q$ (resp. $\mathrm{SL}(2^5) = L$) is an example of $5(b)$ of Table 2. $\square$

In fact, these theorems permit us to construct examples for 6 classes of Table 2, but it is not enough to construct examples for classes $3(b)$ and $4(b)$.

3.1. The SQS-skein $\mathbf{2} \times_R Q_*$ having $\mathbf{2} \times_R L_*$ as a derived sloop

In [4] Armaniuous has constructed a nilpotent SQS-skein of whose all derived sloops are nilpotent of the same class and both have the same congruence lattice. Also, he has constructed [7] a subdirectly irreducible $\mathrm{SK}(2n)$ having a derived subdirectly irreducible $\mathrm{SL}(2n)$ for $n > 8$, in which the congruence lattice of each of $\mathrm{SL}(2n)$ and $\mathrm{SK}(2n)$ are isomorphic to a chain of length 2.

Let L_* be a derived sloop of the SQS-skein Q_* with respect to the element x_0, in which L_* and Q_* are subdirectly irreducible with the same monolith θ_* (Theorems 4, 5 and 6). Let R be the same 4-element subalgebra $\{x_0, x_1, x_2, x_3\}$ in both Q_* and L_* such that $R = [x_0]\theta_* \cup [x_2]\theta_*$ (as in Theorem 4), $R = [x_0]\theta_*$ as in Theorem 5 and $R \subseteq [x_0]\theta_*$ as in Theorem 6. Therefore, $x * y = q_*(x_0, x, y)$, and consequently $\chi_R\langle x, y\rangle_{L_*} = \chi_R\langle x_0, x, y\rangle_{Q_*}$ for all $x, y \in L_* = Q_*$. Hence $(x, i_x) \bullet (y, i_y) = q((x_0, 0), (x, i_x), (y, i_y))$ for all $(x, i_x), (y, i_y) \in L = Q$, this means directly that the constructed sloop $L = 2 \times_R L_*$ is derived sloop of the constructed SQS-skein $Q = 2 \times_R Q_*$. Therefore, we have the following result.

Corollary 7. *Let L_* be a derived sloop of the SQS-skein Q_* with respect to the element x_0 and let both Q_* and L_* be subdirectly irreducible having a* *monolith θ_*. If θ_* and $R = \{x_0, x_1, x_2, x_3\}$ are defined as stated in Theorem 4, 5 and 6, then the sloop $L = 2 \times_R L_*$ is always a derived sloop of the SQS-skein $Q = 2 \times_R Q_*$ with respect to $(x_0, 0)$ for each case of θ_*.* |Box

We may choose an SQS-skein Q_* with a derived sloop L_* and both Q_* and L_* are subdirectly irreducible of cardinality n (or in particular of cardinality 2^n). In view of Theorems 4, 5, 6 and Corollary 7, we may say that:

There is an SQS-skein $Q = 2 \times_R Q_*$ with a derived sloop $L = 2 \times_R L_*$ of cardinality n (or 2^n), in which both Q and L are subdirectly irreducible of cardinality $2n$ (or 2^{n+1}) having the same congruence lattice for each possible number n. In particular, there is always an $SK(32)$ with a derived $SL(32)$, both are subdirectly irreducible and have the same congruence lattices.

Note that the construction of a semi-Boolean SQS-skein Q (each derived sloop L of Q is Boolean) given in [14] guarantees that $\mathrm{C}(Q)$ is a proper sublattice of the congruence lattice $\mathrm{C}(L)$ of its derived sloop L. Also each nonsimple $\mathrm{SL}(16) = L$ can be extended to a nonsimple $\mathrm{SK}(16) = Q$ with all possible congruence lattice $\mathrm{C}(Q)$ as a sublattice of $\mathrm{C}(L)$ (for details see [8]).

We know that if L_* is a derived sloop from Q_*, then each congruence on Q_* is a congruence on L_*. If both Q_* and L_* are subdirectly irreducible, then the monolith φ_* of Q_* is a congruence on L_* containing the monolith θ_* of L_* (examples for $L_* = SL(16)$ and $Q_* = SK(16)$ one can find in [8]). This means that we can choose the sub-SL(4) $= R = [x_0]\theta_* \cup [x_0]\theta_*$, $R = [x_0]\theta_*$ or $R \subseteq [x_0]\theta_*$, if $|[x_0]\theta_*| = 2, 4$ or > 4, respectively. Note that x_0 represents the unit of L_*. Since $\varphi_* \supseteq \theta_*$, R is a sub-SK(4) such that

$$\begin{aligned}
&R = [x_0]\varphi_* \cup [x_0]\varphi_* \ \text{ if } \ |\, [x_0]\varphi_* \,|=|\, [x_0]\theta_* \,|= 2,\\
&R = [x_0]\varphi_* \ \text{ if } \ |\, [x_0]\varphi_* \,|= 4 \ \text{ and } \ |\, [x_0]\theta_* \,|= 2 \text{ or } 4,\\
&R \subseteq [x_0]\varphi_* \ \text{ if } \ |\, [x_0]\varphi_* \,|> 4 \ \text{ and } \ |\, [x_0]\theta_* \,|= 2, 4 \text{ or } > 4.
\end{aligned}$$

In view of Theorems 4, 5 and 6, $Q = 2 \times_R Q_*$ and $L = 2 \times_R L_*$ are subdirectly irreducible with a monolith θ_0 such that $L/\theta_0 \cong L_*$ and $Q/\theta_0 \cong Q_*$. And as a result of Corollary 7, L is a derived sloop of Q, which means that $\mathrm{C}(Q)$ is a proper sublattice of $\mathrm{C}(L)$, if φ_* properly contains θ_*. In particular, we may construct the following examples:

Examples. Let $L_* = SL(16)$ be a derived sloop of $Q_* = SK(16)$. The result obtained in [8] enables us to choose L_* belonging to class $4(a)$ and Q_* belonging to class $4(a)$, 3 or 2 of Table 1. Now, we may construct an

SQS-skein SK(32) $= 2 \times_R Q_*$ with a derived sloop SL(32) $= 2 \times_R L_*$, in which the SL(32) belongs to the class 5(a) of Table 2. The SK(32) will belong to the class 5(a) of Table 2, if Q_* belongs to the class 4(a) of Table 1. Also, the SK (32) will belong to 4(a) or 3(a) of Table 2, if Q_* belongs to the class 3 or 2 of Table 1, respectivly. For the last two cases the congruence lattice C(SK(32)) is a proper sublattice of C(SL(32)). □

A natural open problem for future investigations is a construction of an SQS-skein Q with a derived sloop L for which a congruence lattice C(Q) is a sublattice of C(L).

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