Biembeddings of Latin squares of side 8

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Abstract

Face 2-colourable triangular embeddings of complete tripartite graphs $K_{n,n,n}$ correspond to biembeddings of Latin squares of side n. We consider biembeddings that contain any of the five Latin squares derived from the Cayley tables of finite groups of order 8. Up to isomorphism, we determine all such biembeddings.

1. Background

In our paper [1] we discuss, in some detail, face 2-colourable topological embeddings of complete regular tripartite graphs $K_{n,n,n}$ in which all faces are triangular. Such embeddings are equivalent to biembeddings of Latin squares of side n and, as proved in [1], the supporting surfaces are necessarily orientable. Up to isomorphism, this earlier paper gives all such biembeddings for $n = 3, 4, 5$ and 6, and it summarizes the results for $n = 7$. For $n = 4, 5$ and 6, there are Latin squares which do not appear in any biembedding. Another interesting feature is the partitioning of the 147 main classes of Latin squares of side 7 into sub-classes of sizes 1, 1, 1, 2, 3, 3, 3, 6, 6, 8, 8, 9, 18, 19, 26 and 33, such that within each sub-class most of the squares biembed with one another, but there are no biembeddings of two squares taken from different sub-classes. We refer the reader to [1] for details of this and for items of terminology.

In the current paper we turn our attention to Latin squares of side 8, where there are 283 657 main classes [3]. It is computationally infeasible to determine all possible biembeddings of these squares and here we restrict

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ourselves to seeking biembeddings that contain at least one of those squares that arise from the Cayley tables of groups of order 8. Another reason for considering these particular squares is that, whilst squares which arise from the Cayley tables of cyclic groups always appear in biembeddings, those from the groups $C_2 \times C_2$ and D_3 do not. It is therefore appropriate to consider the Cayley tables of the groups of order 8. There are five such groups, usually denoted by $C_2^3 = C_2 \times C_2 \times C_2$, $C_4 \times C_2$, C_8, D_4 and Q. Here C_n denotes the cyclic group of order n, D_n is the dihedral group of order $2n$, and Q is the quaternion group. We take the corresponding Latin squares as shown in Table 1.

0	1	2	3	4	5	6	7
1	0	4	5	2	3	7	6
2	4	0	6	1	7	3	5
3	5	6	0	7	1	2	4
4	2	1	7	0	6	5	3
5	3	7	1	6	0	4	2
6	7	3	2	5	4	0	1
7	6	5	4	3	2	1	0

C_2^3

0	1	2	3	4	5	6	7
1	2	3	0	5	6	7	4
2	3	0	1	6	7	4	5
3	0	1	2	7	4	5	6
4	5	6	7	0	1	2	3
5	6	7	4	1	2	3	0
6	7	4	5	2	3	0	1
7	4	5	6	3	0	1	2

$C_4 \times C_2$

0	1	2	3	4	5	6	7
1	2	3	4	5	6	7	0
2	3	4	5	6	7	0	1
3	4	5	6	7	0	1	2
4	5	6	7	0	1	2	3
5	6	7	0	1	2	3	4
6	7	0	1	2	3	4	5
7	0	1	2	3	4	5	6

C_8

0	1	2	3	4	5	6	7
1	2	3	0	5	6	7	4
2	3	0	1	6	7	4	5
3	0	1	2	7	4	5	6
4	7	6	5	0	3	2	1
5	4	7	6	1	0	3	2
6	5	4	7	2	1	0	3
7	6	5	4	3	2	1	0

D_4

0	1	2	3	4	5	6	7
1	0	3	2	5	4	7	6
2	3	1	0	6	7	5	4
3	2	0	1	7	6	4	5
4	5	7	6	1	0	2	3
5	4	6	7	0	1	3	2
6	7	4	5	3	2	1	0
7	6	5	4	2	3	0	1

Q

Table 1. Group-based squares of side 8.

2. Results

There are 3 167 nonisomorphic biembeddings that contain at least one of the five group-based squares of side 8. Table 2 gives a breakdown of these by the individual squares and the size of the automorphism group Γ of the biembedding. The column sums given in the last line of the table exclude duplications arising from biembeddings that contain a pair of group-based squares.

$\lvert\Gamma\rvert$	1	2	3	4	6	8	12	16	>16	$\sum$
C_2^3	23	6	4	6	–	2	2	5	1	49
$C_4 \times C_2$	1 750	126	19	55	–	7	5	2	–	1 964
C_8	568	54	60	–	6	–	1	1	11	701
D_4	159	37	18	5	–	3	–	5	–	227
Q	183	16	20	12	–	2	2	1	–	236
$\sum$	2 683	235	120	75	6	14	10	12	12	3 167

Table 2. Biembeddings containing a group-based square.

As regards the biembeddings whose groups of automorphisms have orders greater than 16, there is one of C_2^3 with 48 automorphisms, while C_8 has one with 24 automorphisms (forming S_4), four with 32 automorphisms, one with 64 automorphisms, two with 128 automorphisms, one with 192 automorphisms, one with 256 automorphisms and one with 768 automorphisms. This last biembedding, which is of C_8 with a copy of itself, is the unique regular triangular embedding of $K_{8,8,8}$ in an orientable surface (see [1] and [2] for details). The biembedding of C_2^3 with an automorphism group of order 48 is with a non group-based Latin square, but all 11 biembeddings of C_8 are with copies of itself.

The method for obtaining these biembeddings was to select one of the five group-based squares and to regard its triples of row, column and entry symbols as triangles with the common clockwise orientation (row, column, entry). In any biembedding containing this Latin square, the rotation about each point contains 8 known ordered pairs; what remains unknown is the ordering of these pairs. By considering all possible orderings and rejecting those which give rise to pseudosurfaces, all biembeddings containing the given square may be determined. Working through the five squares, each new biembedding was checked for isomorphism with those found previously. The large number of biembeddings to be checked required the use of an effective invariant in order to establish the isomorphism classes. The invariant used was as follows.

Consider a fixed biembedding of Latin squares of side 8. Denote by ρ_z the rotation around a vertex z. Since ρ_z is a cyclic permutation of order 16, for each two neighbours x and y of z there are integers m_1 and m_2 such that $y = \rho_z^{m_1}(x)$ and $y = \rho_z^{-m_2}(x)$, where $1 \leqslant m_1, m_2 \leqslant 15$ and $m_1 + m_2 = 16$. Put

$$d(z; x, y) = \min\{m_1, m_2\}.$$

Now if $d(z; x, v) = d(z; v, y) = 1$, and $x \neq y$, then $d(v; x, y) = 2$. However if $d(z; x, v) = d(z; v, y) = 3$, and $x \neq y$, then $d(v; x, y)$ can be any even number from 2 to 8. (Note we cannot use $d(z; x, v) = d(z; v, y) = 2$ because then v is not adjacent to either x or y, being in the same vertex partition set.) Let I_v be the sum of the 16 numbers given by the formula

$$I_v = \sum_{vz \in E(G)} (d(v; x, y) : \text{where } d(z; x, v) = d(z; v, y) = 3 \text{ and } x \neq y).$$

Now the multiset of 24 elements I_v, together with the number of automorphisms, forms a satisfactory invariant for our biembeddings. There is just one pair of biembeddings for $C_4 \times C_2$ and two pairs for C_8, which represent nonisomorphic biembeddings, although their invariants coincide.

Up to isomorphism, there are 23 biembeddings where *both* the Latin squares are group-based. In Table 3, in each of these cases, we specify a representative biembedding from the isomorphism class by means of a vector (A, B, p_1, p_2, p_3) where A, B identify the two squares as in Table 1, and p_1, p_2, p_3 specify permutations applied respectively to the rows, columns and entries of the second square. From these, the biembedding may be constructed by taking the two squares exactly as in Table 1 and then applying the permutations to the second square, finally sewing the resulting triangular faces together along their common edges. A permutation entry such as $p_1 = 31267405$ is to be read as the permutation $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 6 & 7 & 4 & 0 & 5 \end{pmatrix}$, indicating that row 0 of the square from Table 1 is placed in row 3, row 3 is placed in row 6, and so on. We use I to denote the identity permutation. In no case do we need to permute rows, columns and entries with each other. We also give information about the automorphism group Γ of each biembedding with a second vector $(M; m_1, m_2, m_3, m_4)$ denoting that $|\Gamma| = M$ and that there are m_1 mappings which preserve orientation and colour classes, m_2 mappings which preserve orientation and reverse the colour classes, m_3 mappings which reverse orientation and preserve the colour classes, and m_4 mappings which reverse orientation and reverse the colour classes.

1. $(C_2^3, D_4, 31267405, 45203617, 35061427)$, $(3; 3, 0, 0, 0)$,
2. $(C_4 \times C_2, D_4, 64752103, 32104567, 21034567)$, $(16; 8, 0, 8, 0)$,
3. $(C_4 \times C_2, D_4, 53261407, 61204357, 41263057)$, $(4; 2, 0, 2, 0)$,
4. $(C_4 \times C_2, D_4, 51302647, 61250347, 40351267)$, $(2; 2, 0, 0, 0)$,
5. $(C_4 \times C_2, D_4, 24673105, 12306547, 23561047)$, $(2; 2, 0, 0, 0)$,
6. $(C_4 \times C_2, Q, 54670213, 13024657, 20134657)$, $(16; 8, 0, 8, 0)$,
7. $(C_4 \times C_2, Q, 53601472, 64310257, 03152647)$, $(4; 2, 0, 2, 0)$,
8. $(C_4 \times C_2, Q, 24601573, 64210357, 13254607)$, $(4; 2, 0, 2, 0)$,
9. $(C_4 \times C_2, Q, 21706354, 53420617, 20134657)$, $(2; 2, 0, 0, 0)$,
10. $(C_4 \times C_2, Q, 54601273, 64310257, 14253607)$, $(2; 2, 0, 0, 0)$,
11. $(C_8, C_8, 12345670, I, I)$, $(768; 192, 192, 192, 192)$, regular,
12. $(C_8, C_8, 52741630, I, I)$, $(256; 64, 64, 64, 64)$,
13. $(C_8, C_8, 56341270, 05634127, 45230167)$, $(192; 48, 48, 48, 48)$,
14. $(C_8, C_8, 16745230, I, I)$, $(128; 32, 32, 32, 32)$,
15. $(C_8, C_8, 52741630, I, 45230167)$, $(128; 32, 32, 32, 32)$,
16. $(C_8, C_8, 52741630, 05634127, 45230167)$, $(64; 16, 16, 16, 16)$,
17. $(C_8, C_8, 12367450, I, I)$, $(32; 8, 8, 8, 8)$,
18. $(C_8, C_8, 14763250, I, I)$, $(32; 8, 8, 8, 8)$,
19. $(C_8, C_8, 12547630, I, I)$, $(32; 8, 8, 8, 8)$,
20. $(C_8, C_8, 16347250, I, I)$, $(32; 8, 8, 8, 8)$,
21. $(C_8, C_8, 16345270, 01634527, 05234167)$, $(24; 12, 0, 12, 0)$,
22. $(C_8, C_8, 34561270, 05634127, 45230167)$, $(16; 4, 4, 4, 4)$,
23. $(C_8, C_8, 34561270, 03456127, 23450167)$, $(12; 3, 3, 3, 3)$.

Table 3. Biembeddings containing two group-based squares.

Table 4 summarizes these biembeddings where both squares are group-based. The entries give the number of biembeddings of square A with square B.

	C_2^3	$C_4 \times C_2$	C_8	D_4	Q
C_2^3	–	–	–	1	–
$C_4 \times C_2$	–	–	–	4	5
C_8	–	–	13	–	–
D_4	1	4	–	–	–
Q	–	5	–	–	–

Table 4. Numbers of mutual biembeddings of group-based squares.

It can be seen that there are, for example, no biembeddings of two squares both derived from C_2^3. A very recent result gives a partial explanation for the partitioning of the squares of side 7 described in our earlier paper and establishes the non-biembeddability of two copies of C_2^n for $n \geqslant 2$, as well as other non-biembeddability results. A paper describing these results is in preparation.

Finally we give the exceptional biembedding of C_2^3 with a non group-based square and having an automorphism group of order 48. The square C_2^3 is taken as in Table 1, and the other square is as follows.

7	0	1	4	2	3	5	6
6	4	5	2	3	7	1	0
1	2	7	5	0	6	4	3
4	6	0	7	1	2	3	5
5	3	6	1	4	0	2	7
2	5	3	6	7	1	0	4
0	1	4	3	6	5	7	2
3	7	2	0	5	4	6	1

The two squares generate triangular faces that are sewn together along common edges to form the embedding. The automorphism type is given by the vector $(48; 24, 0, 24, 0)$.

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