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Connected transversals and multiplication groups of loops

Markku Niemenmaa and Miikka Rytty

Abstract

Several properties of loops and their multiplication groups can be reduced to the properties of connected transversals in groups. We discuss these transversals and prove group theoretical results which have direct loop theoretical consequences. We are particularly interested in the case where the inner mapping group is abelian and we show that it can never be a finite nontrivial cyclic group.

1. Introduction

The purpose of this paper is to explore the connection between loops and groups. The left and right translations of a loop Q generate a group $M(Q)$ called the multiplication group of the loop. The multiplication group can be characterized in purely group theoretical terms (Theorem 5.1 of this paper) and the notion of connected transversals to a subgroup H in a group G is central to this characterization. Here G corresponds to $M(Q)$ and H is the inner mapping group $I(Q)$ of Q.

The first three sections are devoted to H-connected transversals in a group G. We consider their basic properties and after that we are particularly interested in the case where H is abelian (the subcase where H is cyclic gets a very thorough treatment in section four). One of our goals is to show how loop theory is a source of interesting group theoretical problems – some of which are not easy at all to solve. Our results are not necessarily new but some of the proofs are and, in some cases, we have added some new spice to the old proofs. The reader should not be worried about the

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amount of group theory in this paper. After all, groups are nothing but associative loops.

In sections five and six we go to the other direction: we introduce the loop theoretic interpretations of the results that we have proved in the group theory sections. We see that the inner mapping group $I(Q)$ can never be a finite nontrivial cyclic group and we also see that finite loops with abelian inner mapping groups are centrally nilpotent. We also discuss the recent interesting results and constructions where the inner mapping group $I(Q)$ is an abelian p-group and the nilpotency class of Q equals either two or three.

As pointed out earlier, our approach is based on abstract group theory and the efficient use of connected transversals. Naturally, it is possible to deal with these problems by using permutation group theory combined with elementary (or advanced) loop theory (see Drápal [7]). We shall not go into the details of this approach in this paper and we also omit some other important questions like the relation between solvable loops and solvable multiplication groups or the structure of multiplication groups in the case of Moufang loops. The reader interested in these topics should consult Vesanen [16] and the excellent survey by Nagy and Vojtěchovský [13].

Some words about our notation. We bring with us some bad habits from abstract group theory: we write maps to the left of their arguments. If G is a group and x, y are two elements from G, the *commutator* $x^{-1}y^{-1}xy$ is denoted by $[x, y]$. If X, Y are nonempty subsets of G, then $[X, Y] = \langle [x, y] \mid x \in X, y \in Y \rangle$, the subgroup generated by all commutators $[x, y]$. The subgroup $G' = [G, G]$ is the *derived group* (or *commutator subgroup*) of G.

If H is a subgroup of G, then the largest normal subgroup of G contained in H is said to be the *core of* H *in* G and we denote it by H_G (thus $H_G = \bigcap_{x \in G} H^x$). The *conjugate of* H is the subgroup $x^{-1}Hx$ which we denote by H^x. The subgroup $N_G(H) = \{x \in G \mid H^x = H\}$ is the *normalizer of* H *in* G. A subgroup H is *subnormal in* G, if there are subgroups $H_0, H_1, \ldots, H_n$ of G such that $H_0 = H$, $H_n = G$ and H_{i-1} is normal in H_i for every $i = 1, 2, \ldots, n$. We say that H is a *characteristic subgroup of* G, if H is invariant under every automorphism of G. Naturally, if N is a normal subgroup of G and M is a characteristic subgroup of N, then M is normal in G. Finally, we assume that the reader is familiar with the Sylow theorems.

2. Connected transversals in groups

Let G be a group and $H \leq G$. A subset A of G is said to be *a left transversal to* H *in* G if it contains exactly one element from each left coset of H. A *right transversal* is defined similarly. If A and B are two left transversals to H in G and $[A,B] \leq H$, then we say that these two transversals are *H-connected*. In the case that $[A,A] \leq H$, we say that A is *H-selfconnected*. If A and B are H-connected transversals, then A and B are both left and right transversals to H in G (see [14], Lemma 2.1).

We shall now prove some elementary results about connected transversals. These results turn out to be very useful when we prove more substantial results which have interesting interpretations in loop theory. In the following lemmas A and B are H-connected transversals to H in G. Thus $a^{-1}b^{-1}ab \in H$ for every $a \in A$ and $b \in B$.

Lemma 2.1. *If $H_G = 1$, then $Z(G) \subseteq A \cap B$.*

Proof. Let $z \in Z(G)$ and assume that $z = ah$, where $a \in A$ and $h \in H$. Then $b^{-1}hb = b^{-1}a^{-1}zb = b^{-1}a^{-1}bz = b^{-1}a^{-1}bah \in H$ for every $b \in B$. Thus $h \in \bigcap_{b \in B} H^{b^{-1}} = 1$, hence $z = a \in A$. In similar way, we can show that $z \in B$. □

Remark 2.2. If $H_G = 1$, then by Lemma 2.1, $1 \in A \cap B$.

Lemma 2.3. *Let $C \subseteq A \cup B$ and $K = \langle H, C \rangle$. Then $C \subseteq K_G$.*

Proof. Let $c \in C$ and assume that $c \in A$ and $x = bh$, where $b \in B$ and $h \in H$. Now $x^{-1}c^{-1}x = h^{-1}b^{-1}c^{-1}bh = h^{-1}b^{-1}c^{-1}bcc^{-1}h$. As $b^{-1}c^{-1}bc \in H$, we may conclude that $x^{-1}c^{-1}x \in K$, hence $x^{-1}cx \in K$ and $c \in K_G$. If $c \in B$, then the same conclusion holds. □

In the proof of our following lemma, we need two results on commutators:

1. $[xy, z] = [x, z]^y \, [y, z]$ and
2. if $H \leq G$, then $[H, G]$ is a normal subgroup of G.

For the proofs, see [10], p. $253 - 255$.

Lemma 2.4. *If $H_G = 1$, then $N_G(H) = H \times Z(G)$.*

Proof. Let $K = N_G(H) = A_1H = B_1H$, where $A_1 \subseteq A$ and $B_1 \subseteq B$. As H is normal in K and K/H is abelian, we may conclude that $K' \leq H$. By Lemma 2.3, $\langle A_1, B_1 \rangle \leq K_G$. Thus $[A_1, B_1] \leq K'_G \leq K' \leq H$. Now K'_G is normal in G and since $H_G = 1$, it follows that $K'_G = [A_1, B_1] = 1$.

If $g = ah \in G$ (here $a \in A$ and $b \in H$) and $b \in B_1$, then $[b, g] = b^{-1}h^{-1}a^{-1}bah = kb^{-1}a^{-1}bah$, where $k \in H$. Thus $[b, g] \in H$. As K_G is abelian and $a^{-1}b^{-1}a \in K_G$, we have $\left[b^{-1}, g\right] = bh^{-1}a^{-1}b^{-1}ah = dba^{-1}b^{-1}ah$ $= da^{-1}b^{-1}abh \in H$ (here $d \in H$). Further, if $b, c \in B_1$, then $[bc, g] = [b, g]^c\,[c, g] \in H$. Thus $D = [\langle B_1 \rangle, G] \leq H$ and since D is normal in G and $H_G = 1$, it follows that $[\langle B_1 \rangle, G] = 1$ and $\langle B_1 \rangle \leq Z(G)$. Now it is clear that $N_G(H) = H \times Z(G)$. $\square$

Lemma 2.5. *If $H_G = 1$ and $[A, B] = 1$, then A and B are isomorphic subgroups of G.*

Proof. If we write $C = \langle A \rangle \cap H$, then $bc = cb$ for every $c \in C$ and $b \in B$. If $x \in G$, then $x = bh$, where $b \in B$ and $h \in H$. Thus $x^{-1}cx = (bh)^{-1}cbh = h^{-1}b^{-1}cbh = h^{-1}ch \in H$ whenever $c \in C$. This means that $c \in H^x$ for every $x \in G$ and, in fact, $c \in H_G = 1$. We have shown that $\langle A \rangle \cap H = 1$ and therefore $\langle A \rangle = A$. It is also clear that $\langle B \rangle = B$. For every $a \in A$ there exists a unique $f(a) \in B$ such that $a^{-1}H = f(a)H$. If $a, d \in A$, then $f(ad)H = (ad)^{-1}H = d^{-1}f(a)H = f(a)d^{-1}H = f(a)f(d)H$ and we see that $A \cong B$. $\square$

We conclude this section by proving a result which deals with simple groups.

Lemma 2.6. *If G is a simple group and H is a proper subgroup of G, then H is maximal in G.*

Proof. Let $a \in (G \setminus H) \cap A$ and write $K = \langle a, H \rangle$. By Lemma 2.3, $a \in K_G$. As $K_G > 1$ and G is simple, it follows that $K_G = G$. But then $K = G$, and thus H is a maximal subgroup of G. $\square$

3. Connected transversals to abelian subgroups

In this section we assume that $H \leq G$ is an abelian p-group (for a prime number p) and there exist H-connected transversals A and B in G. For the proof of our next theorem we need the following well-known result by Burnside (see [10], p. $419 - 420$).

Lemma 3.1. *Let G be a finite group and P a Sylow p-subgroup of G such that $P \leq Z(N_G(P))$ (or $N_G(P) = C_G(P)$). Then there exists a normal subgroup K of G such that $G = KH$ and $K \cap H = 1$.* □

Now we are ready to prove

Theorem 3.2. *Let G be a finite group and $H \leq G$ an abelian p-group. Assume further that $H_G = 1$ and $G = \langle A, B\rangle$. Then $Z(G) > 1$.*

Proof. Assume that the claim is not true and $Z(G) = 1$. As $H_G = 1$, we can apply Lemma 2.4 and thus $N_G(H) = H \times Z(G) = H$. If $H < P \leq G$, where P is a p-group, then $H < N_P(H)$, a contradiction. We conclude that H is a Sylow p-subgroup of G. Now $N_G(H) = C_G(H)$ and by Lemma 3.1 there exists a normal subgroup K of G such that $G = KH$ and $K \cap H = 1$. As $G/K \cong H$ is abelian, it follows that $G' \leq K$. Thus $a^{-1}b^{-1}ab \in G' \cap H \leq K \cap H = 1$ and we get $ab = ba$ for every $a \in A$ and $b \in B$.

The subgroup $L = H \cap \langle A\rangle$ is normal in H and as $N_G(L) \supseteq B$, it follows that $N_G(L) \geq \langle H, B\rangle = G$. Since $H_G = 1$, we conclude that $L = 1$. This means that $\langle A\rangle = A$ is a normal subgroup of G. Similarly, B is a normal subgroup of G. Thus $K = A = B$ and as $G = \langle A, B\rangle$, we have a contradiction. We conclude that $Z(G) > 1$. □

Corollary 3.3. *Assume that the conditions of Theorem 3.2 hold for G and H. Then H is subnormal in G.* □

Remark 3.4. By using a similar but somewhat more complicated argumentation we could prove that the result of Theorem 3.2 also holds in the case that H is an abelian subgroup. Naturally, in this case H would be subnormal in G, too.

4. Connected transversals to cyclic subgroups

We first consider the situation that H is cyclic of order p (here p is a prime number) and then we proceed to more general cases. Naturally A and B are connected transversals to H in G.

Lemma 4.1. *Let H be a cyclic subgroup of order p. If $G = \langle A, B\rangle$, then $G' \leq H$.*

Proof. If $H_G > 1$, then $H = H_G$ is normal in G and $G' \leq H$. Thus we assume that $H_G = 1$. By Lemma 2.4 we know that $N_G(H) = H \times Z(G)$.

Let $a \in A$ and $b \in B$ such that $aH = bH$ and $a^{-1}b \neq 1$. Then $H = \langle a^{-1}b \rangle$ and $(a^{-1}b)^a = a^{-1}b^a = a^{-1}bb^{-1}a^{-1}ba \in H$. This means that $a \in N_G(H) = H \times Z(G)$, hence $a \in Z(G)$. By Lemma 2.1, it follows that $A = B$.

Let a and d be two elements from A. If $a, d \in Z(G)$, then $ad \in Z(G) \subseteq A$. Now assume that $a \notin Z(G)$ and write $ad = ch$, where $c \in A$ and $h \in H$. It follows that $d^{-1}a^{-1}da = d^{-1}a^{-1}a^{-1}ada = h^{-1}c^{-1}a^{-1}cha = h^{-1}c^{-1}a^{-1}caa^{-1}ha \in H$ and thus $h^a \in H$. If $h \neq 1$, then $a \in N_G(H)$, hence $a \in Z(G)$, a contradicting the choise of a. Thus $h = 1$ and $ad = c \in A$. Furthermore, let $a^{-1} = bh$ where $b \in A$ and $h \in H$. Then $h = b^{-1}a^{-1}$ and $h^{-1} = ab \in A \cap H = 1$. Thus $a^{-1} \in A$ and $A = B$ is a subgroup of G and this contradicts the condition $G = \langle A, B \rangle$. □

Now we proceed to the situation where H is a cyclic group of prime power order. In the following lemma we can very efficiently use the result of Theorem 3.2.

Lemma 4.2. *Let G be a finite group and $H \leq G$ a cyclic p-group. If $G = \langle A, B \rangle$, then $G' \leq H$.*

Proof. Let G be a minimal counterexample. If $H_G > 1$, then we consider the group G/H_G and the subgroup H/H_G. Then $(G/H_G)' \leq H/H_G$, hence $G' \leq H$.

Thus we may assume that $H_G = 1$. By Theorem 3.2, $Z(G) > 1$. Let $z \in Z(G)$ such that $|z| = q$, where q is a prime number. We now consider the groups $G/\langle z \rangle$ and $H\langle z \rangle/\langle z \rangle$ and conclude that $G' \leq H\langle z \rangle$. This means that $H\langle z \rangle$ is normal in G. If $p \neq q$, then H is a Sylow p-subgroup of $H\langle z \rangle$. As H is characteristic in $H\langle z \rangle$, it follows that H is normal in G and $G' \leq H$. Thus we may assume that $q = p$. We write $E = \langle x^p \mid x \in H\langle z \rangle \rangle$. Clearly, $E \leq H$ and as E is characteristic in $H\langle z \rangle$, it follows that E is normal in G. Since $H_G = 1$, we conclude that $E = 1$ and thus $|H| = p$. Now the claim follows from Lemma 4.1. □

We are now ready to prove our main result on connected transversals to cyclic subgroups.

Theorem 4.3. *Let G be a finite group and H a cyclic subgroup. If $G = \langle A, B \rangle$, then $G' \leq H$.*

Proof. Let G be a minimal counterexample. Clearly, we can assume that $H_G = 1$ and H is not of prime power order. If $N_G(H) > H$, then $Z(G) > 1$

by Lemma 2.1. Let $z \in Z(G)$ and $|z| = q$, where q is a prime number. Then $G' \leq H\langle z\rangle$ and H contains a Sylow p-subgroup P (with $p \neq q$) such that P is normal in G, a contradiction.

Thus we may assume that $N_G(H) = H$. Let P be a Sylow p-subgroup of H such that $N_G(P) > H$. Now $C_G(P)$ is normal in $N_G(P)$ and therefore $C_G(P) > H$. If $C_G(P) = G$, then P is normal in G, which is not possible. Thus G has a subgroup T such that $H < T \leq C_G(P) < G$. By Lemma 2.3, $T_G > 1$. We consider the groups G/T_G and $HT_G/T_G = T/T_G$ and get $G' \leq T$. It follows that T is normal in G. Now $P \leq Z(T)$ and $Z(T) \leq H$. Since $Z(T)$ is characteristic in T, we conclude that $Z(T)$ is normal in G. As $H_G = 1$, this is not possible.

Thus we may assume that $N_G(P) = C_G(P) = H$ for every Sylow subgroup P of H. All Sylow subgroups of H are also Sylow subgroups of G and by applying Lemma 3.1, we conclude that there exist a normal subgroup K of G such that $G = KH$ and $K \cap H = 1$. By standard arguments (as in the proof of Theorem 3.2), it follows that $K = A = B$ is a normal subgroup of G. But this contradicts $G = \langle A, B\rangle$ and our proof is ready. □

We shall next prove that the result of Theorem 4.3 also holds in the case that G is infinite. We first introduce a useful lemma (which was introduced to the first author by Tomáš Kepka some thirteen years ago).

Lemma 4.4. *Let H be a finite subgroup of G, $H_G = 1$ and $G = \langle A, B\rangle$. Then $G/Z(G)$ is finite.*

Proof. Let a be a fixed element of A, h fixed element of H and write $F(a, h) = \{b \in B \mid a^{-1}b^{-1}ab = h\}$. If $b, c \in F(a, h)$, then $bc^{-1} \in C_G(a)$ and $b \in C_G(a)c$. Thus $F(a, h) \subseteq C_G(a)b(h)$, where $b(h)$ is a fixed element from $F(a, h)$. Further, $B = \bigcup F(a, h)$, where h goes through all the elements of H. Now $G = BH \subseteq C_G(a)\{b(h) \mid h \in H\}H$, hence $[G : C_G(a)] \leq |H|^2$. As H is a finite subgroup of $\langle A, B\rangle$, we may conclude that $[G : C_G(H)]$ is finite. Then $[G : N_G(H)]$ is finite and since $N_G(H) = H \times Z(G)$, we have $G/Z(G)$ finite. □

Theorem 4.5. *Let H be a finite cyclic subgroup of G and let $G = \langle A, B\rangle$. Then $G' \leq H$.*

Proof. We proceed by induction on $|H|$. It is obvious that we may assume that $H_G = 1$. By using Lemma 4.4, we consider the finite group $G/Z(G)$ and its cyclic subgroup $HZ(G)/Z(G)$. By Theorem 4.3, $G' \leq HZ(G)$.

Let $p \mid |H|$ be a prime number and $E = \langle x \in HZ(G) \mid x^p = 1\rangle$. Now E is characteristic in $HZ(G)$, hence E is normal in G. As $|HE/E| < |H|$, we apply induction and get $G' \leq HE$. Thus HE is normal in G. The group $L = \langle x^p \mid x \in HE\rangle$ is characteristic in HE and $L \leq H$ is normal in G. Since $H_G = 1$, it follows that $|H| = p$. But now the result follows from Lemma 4.1. $\square$

Remark 4.6. By using Zorn's lemma and the result of the previous theorem it is possible to prove that $G' \leq H$ also in the case that H is an infinite cyclic group (for the details, see [12]).

Remark 4.7. Drápal [7] uses elementary loop theory combined with permutation group theory and proves results which are basicly the same as the preceding results of this section. In Drápal's article it also remains an open question whether it is necessary to use Zorn's lemma when proving the result of Theorem 4.5 for an infinite cyclic subgroup H.

We now have a very good understanding of the situation when H is a finite cyclic subgroup of G and $G = \langle A, B\rangle$. How does the situation change if $H \cong C_p \times C_p$?

Theorem 4.8. *Let $H \cong C_p \times C_p$ and $G = \langle A, B\rangle$. Then $G' \leq N_G(H)$.*

Proof. If $H_G > 1$, then $G' \leq H$ by Lemma 4.1. Thus we may assume that $H_G = 1$. By Lemma 2.4, $N_G(H) = H \times Z(G)$ and from Lemma 4.4 we conclude that $G/Z(G)$ is finite. Consider the subgroup $HZ(G)/Z(G)$ of $G/Z(G)$. If the core of $HZ(G)$ in G properly contains $Z(G)$, then $G' \leq HZ(G) = N_G(H)$ (again we use Lemma 4.1). We next assume that the core of $HZ(G)$ in G is $Z(G)$. By Lemma 2.4,

$$N_{G/Z(G)}(HZ(G)/Z(G)) = HZ(G)/Z(G) \times Z(G/Z(G)).$$

We write $M/Z(G) = Z(G/Z(G))$. Then $N_G(HZ(G)) = HM$, where M is normal in G and $H \cap M = 1$. By Theorem 3.2, $Z(G)$ is a proper subgroup of M. Then we write $HM = CH = DH$, where $C \subseteq A$ and $D \subseteq B$. By Lemma 2.1, $M/Z(G) \subseteq AZ(G)/Z(G) \cap BZ(G)/Z(G)$, which means that $M \subseteq CZ(G) \cap DZ(G)$. If $m \in M$, then $m = cz_1 = dz_2$, where $c \in C$, $d \in D$ and $z_1, z_2 \in Z(G)$. If $x \in A \cup B$, then $[x, m] \in M \cap H = 1$. Thus $C_G(m) \geq \langle A, B\rangle = G$ and consequently $m \in Z(G)$. But then $M = Z(G)$, a contradiction. $\square$

If $H \cong C_p \times C_p \times C_p$, then things get more complicated. However, in 2006 Csörgő [5] managed to prove the following

Theorem 4.9. *If G is finite group, $H \cong C_p \times C_p \times C_p$ and $G = \langle A, B\rangle$, then $G' \leq N_G(H)$.* $\square$

5. Multiplication groups of loops

Let Q be a loop (a groupoid with unique division and neutral element e). For each $a \in Q$ we have two permutations L_a (left translation) and R_a (right translation) on Q defined by $L_a(x) = ax$ and $R_a(x) = xa$ for every $x \in Q$. The set of all left and right translations generates a subgroup $M(Q)$ of S_Q called the *multiplication group of the loop Q*. The stabilizer of the neutral element e is called the *inner mapping group of the loop Q* and we denote it by $I(Q)$. The concept of multiplication groups was introduced by Albert in [1] and [2] and in his famous article [3], Bruck laid the foundation for the theory of multiplication and inner mapping groups. If Q is a group, then $I(Q)$ consists of the inner automorphims of Q. It is well-known that the inner mapping group is generated by the set

$$\{R_{yx}^{-1} R_x R_y, L_{xy}^{-1} L_x L_y, L_x^{-1} R_x \mid x, y \in Q\}.$$

If we write $A = \{L_a \mid a \in Q\}$ and $B = \{R_a \mid a \in Q\}$, then A and B are transversals to $I(Q)$ in $M(Q)$ and as $L_a^{-1} R_b^{-1} L_a R_b(e) = e$, we see that A and B are $I(Q)$-connected transversals in $M(Q)$. Now $M(Q)$ is transitive on Q and therefore, if $1 < N \leq I(Q)$, N is not normal in $M(Q)$ (thus the core of I(Q) in M(Q) is trivial). As a matter of fact, we have now introduced all the properties which completely characterize multiplication groups of loops. We state this characterization that was proved by Kepka and Niemenmaa [14] in 1990 as

Theorem 5.1. *A group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H of G satisfying $H_G = 1$ and H-connected transversals A and B such that $G = \langle A, B\rangle$.*

Proof. Assume that the group G has a subgroup H and H-connected transversals A and B satisfying the conditions of the theorem. For each $x \in G$ there exists exactly one $f(x)$ of A such that $xH = f(x)H$. Let K be the set of left cosets of H. Now we define a binary operation $(*)$ on the set K by $(xH) * (yH) = f(x)yH$.

If $xH = uH$ and $yH = vH$, then $f(x) = f(u)$ and $f(x)yH = f(x)vH = f(u)vH$. We conclude that $(*)$ is well-defined. Now we shall show that the groupoid $(K, *)$ is a loop. By Lemma 2.1, we have that $1 \in A$. Therefore $(1H)*(yH) = f(1)yH = yH$ and $(xH)*(1H) = f(x)H = xH$, which means that $1H$ is the neutral element of K. If xH and yH are fixed elements in $(xH) * (yH) = zH$, then $yH = f(x)^{-1}zH$ is a unique element from the set K. Respectively let yH and zH be known elements in K and consider the equation $(xH) * (yH) = zH$. For every $y \in G$ there exists exactly one $g(y)$ of B such that $yH = g(y)H$. Since A and B are H-connected, we have $(xH) * (yH) = f(x)g(y)H = g(y)f(x)H = g(y)xH$. Thus $xH = g(y)^{-1}zH$ is the unique solution for the equation $(xH) * (yH) = zH$, so the groupoid $(K, *)$ is a loop.

Now we consider the action of G on K by left multiplication as its permutation representation is a homomorphism from G to $M(K)$ with the kernel $H_G = 1$. Since $G = \langle A, B\rangle$ and the left and right translations are of the form $L_{xH}(yH) = (xH) * (yH) = f(x)yH$ and $R_{xH}(yH) = (yH) * (xH) = f(y)g(x)H = g(x)f(y)H = g(x)yH$ where $f(x) \in A$ and $g(x) \in B$, we conclude that the image of the permutation representation is the whole $M(K)$. Therefore G is isomorphic to $M(K)$. $\square$

When we combine Theorems 4.5 and 5.1, we immediately have

Theorem 5.2. *Let Q be a loop. If $I(Q)$ is a finite cyclic group, then $I(Q) = 1$ and Q is an abelian group.* $\square$

From the previous result we see that a nontrivial finite cyclic group can never be in the role of $I(Q)$. On the other hand, there are finite abelian groups which are isomorphic to inner mapping groups of loops. Thus we pose

Problem 1. *Classify those finite abelian groups which are (are not) isomorphic to inner mapping groups of loops.*

6. Centrally nilpotent loops

The *centre $Z(Q)$ of a loop Q* consists of all elements a, which satisfy the equations $(ax)y = a(xy)$, $(xa)y = x(ay)$, $(xy)a = x(ya)$ and $ax = xa$ for all $x, y \in Q$. Thus $a \in Z(Q)$ if and only if $U(a) = a$ for every $U \in I(Q)$. Clearly, $Z(Q)$ is an abelian group and normal in Q. The following well-known result was first proved by Albert [1].

Lemma 6.1. *We have $Z(Q) \cong Z(M(Q))$.*

Proof. Let $T \in Z(M(Q))$. Thus $L_xT(e) = TL_x(e)$ and it follows that $xT(e) = T(x)$ for every $x \in Q$. We see that $T = R_{T(e)}$. If $U \in I(Q)$, then $UT(e) = TU(e) = T(e)$ and so $T(e) \in Z(Q)$. We conclude that $Z(M(Q)) = \{R_c \mid c \in Z(Q)\}$. □

If we put $Z_0 = 1$, $Z_1 = Z(Q)$ and $Z_i/Z_{i-1} = Z(Q/Z_{i-1})$, then we obtain a series of normal subloops of Q. If Z_{n-1} is a proper subloop of Q and $Z_n = Q$, then Q is centrally nilpotent of class n.

Now the mapping $f : I(Q) \to I(Q/Z(Q))$ defined by $f(P)(xZ(Q)) = P(x)Z(Q)$ is a surjective homomorphism and

$$\mathrm{Ker}(f) = \{P \in I(Q) \mid P(x)Z(Q) = xZ(Q) \text{ for every } x \in Q\}.$$

We thus get

Lemma 6.2. *If $K = \{P \in I(Q) \mid P(x) \in xZ(Q)$ for every $x \in Q\}$, then K is a normal subgroup of $I(Q)$ and $I(Q/Z(Q)) \cong I(Q)/K$.* □

We combine the preceding lemma with Theorem 3.2.

Theorem 6.3. *Let Q be a finite loop and $I(Q)$ an abelian group of prime power order. Then Q is centrally nilpotent.*

Proof. By Theorem 3.2, $Z(M(Q)) > 1$ and thus $Z(Q) > 1$, by Lemma 6.1. If K is as in Lemma 6.2, we have $I(Q/Z(Q)) \cong I(Q)/K$. Again, $Z(Q/Z(Q)) > 1$. We continue like this and it follows that Q is centrally nilpotent. □

Remark 6.4. The result of Theorem 6.3 also holds if $I(Q)$ is abelian without any restrictions on the order of $I(Q)$ (for the details see [11] and [15]).

In the light of Theorem 6.3 is quite natural to pose the following problem.

Problem 2. *Assume that Q is a finite loop and $I(Q)$ is an abelian p-group whose structure is known. What can we say about the nilpotency class of a loop Q?*

We now recall a nilpotency criterion given by Bruck [3]. First write $I_0 = I(Q)$ and $I_i = N_{M(Q)}(I_{i-1})$ for each $i \geq 1$.

Theorem 6.5. *A necessary and sufficient condition that Q be centrally nilpotent of class n is that $I_n = M(Q)$ but $I_{n-1} \neq M(Q)$.* □

If Q is centrally nilpotent of class ≤ 2, then $N_{M(Q)}(I(Q)) = I(Q) \times Z(M(Q))$ is normal in $M(Q)$. It follows that $I(Q)'$ is normal in $M(Q)$, hence $I(Q)' = 1$ and $I(Q)$ is an abelian group.

The results given in Theorems 4.8 and 4.9 can now easily be interpreted in loop theory.

Theorem 6.6. *If Q is a finite loop and $I(Q) \cong C_p \times C_p$ or $I(Q) \cong C_p \times C_p \times C_p$, then Q is centrally nilpotent of class 2.* □

One is tempted to think that if $I(Q)$ is an elementary abelian p-group, then Q is centrally nilpotent of class 2. However, in a recent article Csörgő [4] has constructed an example of a finite group G of order 2^{13} such that G has an elementary abelian subgroup H of order 2^6 with H-connected transversals A and B, $G = \langle A, B \rangle$ and $G' \not\leq N_G(H)$. These conditions naturally imply the existence of a loop Q of order 2^7 with elementary abelian $I(Q)$ of order 2^6 and with nilpotency class greater than two.

Remark 6.7. Drápal and Vojtěchovský [9] have also constructed examples of loops Q with $I(Q)$ an abelian 2-group and Q centrally nilpotent of class 3 by means of a special group modification. It is interesting to note that in the case of a left conjugacy closed loop Q, it is centrally nilpotent of class 2 if and only if its inner mapping group is a nontrivial abelian group. This result is due to Csörgő and Drápal [6]. Finally, Drápal and Kinyon [8] have constructed a Buchsteiner loop of order 128 whose inner mapping group is abelian and nilpotency class is three.

We shall put an end to this article with the following

Problem 3. *Let Q be a loop such that $I(Q)$ is an abelian p-group. Is it possible that the nilpotency class of Q is greater than three?*

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Department of Mathematical Sciences
University of Oulu
Pentti Kaiteran katu 1
PL 3000
90014 University of Oulu
Finland
E-mail: Markku.Niemenmaa@oulu.fi

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