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On decomposition of commutative Moufang groupoids

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Abstract

We prove that every commutative Moufang groupoid is a semilattice of Archimedean subgroupoids.

It is well-known that the multiplicative groupoid of an alternative/Jordan algebra satisfies Moufang identities [1, 4]. Therefore it seems interesting to study the structure of such groupoids. In this note we apply to Moufang groupoids an approach which is widespread in semigroup theory – decomposition into a semilattice of subsemigroups [3].

We shall call a groupoid with the identity

$$(xy)(zx) = (x(yz))x \tag{1}$$

a Moufang groupoid. **Everywhere in this article M denotes a commutative Moufang groupoid.**

Theorem 1. *If M consists of idempotents, then it is a semilattice.*

Proof. Under assumption of the theorem it follows from (1) for $y = z$

$$(xy)x = xy \tag{2}$$

Applying (2) to the right part of (1), we get:

$$x(yz) = (xy)(xz) \tag{3}$$

Now define a binary relation $\leqslant$ on M :

$$a \leqslant b \Longleftrightarrow ab = a$$

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and show that it is a partial order.

Indeed, the reflexivity follows from idempotentness, the antisymmetry follows from commutativity. Let $a \leqslant b \leqslant c$. Then

$$ac = (ab)c = (ac)(bc) = (ac)b = (ab)(bc) = ab = a,$$

i.e., $a \leqslant c$.

Further, ab is a greatest lower bound for the pair $\{a, b\}$. Really, $ab \leqslant a$, $ab \leqslant b$ by (2). Suppose that $x \leqslant a$, $x \leqslant b$. Then $(ab)x = (ax)(bx) = x \cdot x = x$, i.e., $x \leqslant ab$. □

Lemma 2. *M is a groupoid with associative powers.*

Proof. For $a \in M$ we shall denote by $a^{(n)}$ an arbitrary term of the length $n \geq 1$, all letters of which are a. If all such terms coincide in M, we denote them by a^n.

We use the induction on length of the term. Let $a^{(k)} = a^k$ for any $k < n$ (for $k = 3$ this follows from commutativity). Consider some term $a^{(n)}$. It can be written in the form $a^{(n)} = a^{(k)}a^{(l)}$, where $k, l \geqslant 1$ and $k + l = n$; in view of commutativity one can assume that $k \leqslant l$.

Suppose that $k \geqslant 2$. Then under hypothesis of the induction

$$a^{(n)} = a^k a^l = (aa^{k-1})(aa^{l-1}) = (a(a^{k-1}a^{l-1}))a = (aa^{n-2})a = aa^{n-1}.$$

Hence all terms of the form $a^{(n)}$ are equal. □

We denote by L_a the left translation corresponding to an element a: $L_a b = ab$. From (1) we have:

$$(xy)^2 = L_x^2 y^2.$$

We generalize this identity:

Lemma 3. $(ab)^{2^n} = L_a^{2^n} b^{2^n}$ *for any* $a, b \in M$, $n \geqslant 0$ (*here powers are defined correctly in view of Lemma* 2).

Proof. Assume that for n the statement is faithful and prove it for $n+1$:

$$(ab)^{2^{n+1}} = [(ab)^2]^{2^n} = [a(ab^2)]^{2^n} = L_a^{2^n}(ab^2)^{2^n} = L_a^{2^n} L_a^{2^n} b^{2^{n+1}} = L_a^{2^{n+1}} b^{2^{n+1}}$$

□

Corollary 4. $(L_{a_1} \dots L_{a_{k-1}} a_k)^{2^n} = L_{a_1}^{2^n} \dots L_{a_{k-1}}^{2^n} a_k^{2^n}$.

Further we shall need one more equality for translations:

Lemma 5. *$L_a^{2n}L_b = L_{L_a^n b}L_a^n$ for any $a, b \in M$, $n \geqslant 1$.*

Proof. For $n = 1$ this statement coincides with (1). The general case is obtained by induction on n. □

Let I_a be denoted the principal ideal, generated by $a \in M$. It is clear that each element from I_a can be written in the form $L_{x_1} \dots L_{x_{k-1}} L_{x_k} a$.

Define relations ρ and σ:

$$a\rho b \Longleftrightarrow \exists n \geqslant 1 \quad a^n \in I_b, \tag{4}$$

$$a\sigma b \Longleftrightarrow a\rho b \quad \& \quad b\rho a. \tag{5}$$

Lemma 6. *σ is a congruence.*

Proof. Reflexivity and symmetry are obvious, it is enough to check transitivity and stability of ρ. Note that one can assume in the definition of ρ that n is the power of the two.

Let $a\rho b$, $b\rho c$, i.e.,

$$a^{2^m} = L_{x_1} \dots L_{x_k} b, \quad b^{2^n} = L_{y_1} \dots L_{y_l} c.$$

By Corollary 4

$$a^{2^{m+n}} = L_{x_1}^{2^n} \dots L_{x_k}^{2^n} b^{2^n} = L_{x_1}^{2^n} \dots L_{x_k}^{2^n} L_{y_1} \dots L_{y_l} c \in I_c,$$

so ρ is transitive.

Now let $a\rho b$, i.e., $a^{2^n} = L_{x_1} \dots L_{x_k} b$, and $c \in M$.

1) Suppose that $k \leqslant n$. Then using several times Lemma 5, we get for some $u_1, \dots, u_k \in M$:

$$(ca)^{2^n} = L_c^{2^n} a^{2^n} = L_c^{2^n} L_{x_1} \dots L_{x_k} b = L_{u_1} \dots L_{u_k} L_c^{2^{n-k}} b \in I_{cb}.$$

2) Let $k > n$. Then $a^{2^{n+k+1}} = L_y L_{x_1} \dots L_{x_k} b$, where $y = a^{2^{n+k+1} - 2^n}$. Since $k + 1 < n + k + 1$, we get the case 1). Consequently, $ca\rho cb$. □

Lemma 7. *M/σ is a semilattice.*

Proof. Obviously, $a\sigma a^2$ for any $a \in M$. So M/σ is an idempotent groupoid. By Theorem 1 it is a semilattice. □

Now let us to consider the structure of σ-classes (of course, they are subgroupoids).

Like to theory of semigroups, we call a groupoid M *Archimedean* if $a\sigma b$ for any $a, b \in M$, where σ is defined by the conditions (4) and (5). It is clear that an Archimedean groupoid is indecomposable into a semilattice of subgroupoids.

Lemma 8. *Let σ be a congruence on M, defined by conditions* (4) *and* (5). *Then each σ-class is Archimedean.*

Proof. Let N is a σ-class, $a, b \in N$. Then

$$a^n = L_{x_1} \dots L_{x_k} b \tag{6}$$

for some $n > 0$, $x_1, \dots, x_k \in M$. We need to prove that in the equality (6) elements $x_1, \dots, x_k$ can be chosen from N.

From (6) and Lemma 5 we have:

$$a^{n+2^k} = L_a^{2^k} L_{x_1} \dots L_{x_k} b = L_{L_a^{2^{k-1}} x_1} L_{L_a^{2^{k-2}} x_2} \dots L_{L_a x_k} b.$$

Show that for any $i \leqslant k$ the element $y_i = L_a^{2^{k-i}} x_i$ is contained in N. Indeed, since $y_i = a(L_a^{2^{k-i}-1} x_i)$ then $y_i \rho a$. On the other hand,

$$a^{n+2^k} = L_{y_1} \dots L_{y_k} b = L_{y_1} \dots L_{y_{i-1}} [(L_{y_{i+1}} \dots L_{y_k} b) y_i],$$

whence $a \rho y_i$. Thereby, $a \sigma y_i$, i.e., $y_i \in N$. $\square$

The final result:

Theorem 9. *A commutative Moufang groupoid is a semilattice of Archimedean groupoids.*

Example. Let a finite semigroup S satisfy the identity $ab = a$ (*a left zero semigroup*), F be a field, $\operatorname{char} F \neq 2$, $A = FS$ be the semigroup algebra. A is a Jordan algebra with respect to the operation $x * y = \frac{1}{2}(xy + yx)$. Denote by A^* its multiplicative groupoid (as is well-known it is Moufang and commutative [4]).

The operation in A^* can be written as follows. For $x = \sum_{a \in S} \alpha_a a \in A^*$, $\alpha_a \in F$, denote $|x| = \sum_{a \in S} \alpha_a$. Then

$$x * y = \frac{1}{2}(|x| y + |y| x)$$

From here $x^{*n} = |x|^{n-1}x$ and $|xy| = |x||y|$. In particular, $x \in \operatorname{Rad} A^*$ iff $|x| = 0$.

Evidently, all elements from $\operatorname{Rad} A^*$ constitute one σ-class. On the other hand, if $x, y \notin \operatorname{Rad} A^*$ then they divide one another. To make sure that, it is enough to put

$$t = \frac{1}{|x|^2}(2|x|y - |y|x);$$

then $y = x * t$. Thus $A^* = \operatorname{Rad} A^* \cup (A^* \setminus \operatorname{Rad} A^*)$ is the decomposition of A^* into Archimedean components.

Finally we discuss some problems which arise here.

1. For loops the identity (1) (*central Moufang identity*) is equal to each of ones $x(y(xz)) = ((xy)x)z$ and $((zx)y)x = z(x(yx))$ (*left and right Moufang identities*). This is valid for multiplicative groupoids of Jordan algebras as well, but not in the general case. So we can consider left and right Moufang (commutative) groupoids. Are there similar decompositions for them?

2. Is there Archimedean decomposition in noncommutative situation? This is the case for semigroups [2].

3. What can one say about the structure of an Archimedean component? For instance, can it contain more than one idempotent (cf. [3], Ex.4.3.2)?

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