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Fuzzy ideals in ordered semigroups I

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Abstract

We prove that: a regular ordered semigroup S is left simple if and only if every fuzzy left ideal of S is a constant function. We also show that an ordered semigroup S is left (resp. right) regular if and only if for every fuzzy left(resp. right) ideal f of S we have, $f(a) = f(a^2)$ for every $a \in S$. Further, we characterize some semilattices of ordered semigroups in terms of fuzzy left(resp. right) ideals. In this respect, we prove that an ordered semigroup S is a semilattice of left (resp. right) simple semigroups if and only if for every fuzzy left(resp. right) ideal f of S we have, $f(a) = f(a^2)$ and $f(ab) = f(ba)$ for all $a, b \in S$.

1. Introduction

A fuzzy subset f of a given set S is described as an arbitrary function $f : S \longrightarrow [0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. This fundamental concept of a fuzzy set, was first introduced by Zadeh in his pioneering paper [24] of 1965, provides a natural frame-work for the generalizations of some basic notions of algebra, e.g. logic, set theory, group theory, ring theory, groupoids, real analysis, measure theory, topology, and differential equations etc. Rosenfeld (see [21]) was the first who considered the case when S is a groupoid. He gave the definition of a fuzzy subgroupoid and the fuzzy left (right, two-sided) ideal of S and justified these definitions by showing that a subset A of a groupoid S is a subgroupoid or a left (right, or two-sided) ideal of S if and only if the characteristic mapping $f_A : S \to \{0, 1\}$ of A defined by

$$x \longmapsto f_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

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is, respectively, a fuzzy subgroupoid or a fuzzy left (right or two-sided) ideal of S. The concept of a fuzzy ideal in semigroups was first developed by Kuroki (see [12-17]). Fuzzy ideals and Green's relations in semigroups were studied by McLean and Kummer in [18]. Dib and Galhum in [2], introduced the definitions of a fuzzy groupoid, and a fuzzy semigroups and studied fuzzy ideals and fuzzy bi-ideals of a fuzzy semigroups. Ahsan et. al in [1] characterized semisimple semigroups in terms of fuzzy ideals. A systematic exposition of fuzzy semigroups by Mordeson, Malik and Kuroki appeared in [20], where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. The monograph by Mordeson and Malik [19] deals with the applications of fuzzy approach to the concepts of automata and formal languages. Fuzzy sets in ordered semigroups/ordered groupoids were first introduced by Kehayopulu and Tsingelis in [8]. They also introduced the concepts of fuzzy bi-ideals and fuzzy quasi-ideals in ordered semigroups in (see [9] and [10]).

In [22], Shabir and Khan, introduced the concept of a fuzzy generalized bi-ideal of ordered semigroups and characterized different classes of ordered semigroups by using fuzzy generalized bi-ideals. They also gave the concept of fuzzy left (resp. bi-) filters in ordered semigroups and gave the relations of fuzzy bi-filters and fuzzy bi-ideal subsets of ordered semigroups in [23].

In this paper, which is a continuation of the work carried out by Kehayopulu-Tesingelis [11] for ordered semigroups in terms of fuzzy ideals, we characterize regular, left and right simple ordered semigroups and completely regular ordered semigroups in terms of fuzzy left (resp. right) ideals. In this respect, we prove that: A regular ordered semigroup S left simple if and only if every fuzzy left ideal f of S is a constant function. We also prove that S is left regular if and only if for every fuzzy left ideal f of S we have $f(a) = f(a^2)$ for every $a \in S$. Next we characterize semilattices of left simple ordered semigroups in terms of fuzzy left ideals of S. We prove that an ordered semigroup S is a semilattice of left simple semigroups if and only if for every fuzzy left ideal f of S we have, $f(a) = f(a^2)$ and $f(ab) = f(ba)$ for all $a, b \in S$.

2. Preliminaries

By an *ordered semigroup* (or *po-semigroup*) we mean a structure $(S, \cdot, \leqslant)$ in which

$(OS1)$ $(S, \cdot)$ is a semigroup,

$(OS2)$ $(S, \leqslant)$ is a poset,
$(OS3)$ $(\forall a, b, x \in S)(a \leqslant b \longrightarrow ax \leqslant bx$ and $xa \leqslant xb)$.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A non-empty subset A of S is called a *left* (resp. *right*) *ideal* of S (see [7-10]) if:
(i) $SA \subseteq A$ (resp. $AS \subseteq A$) and
(ii) $(\forall a \in A)(\forall b \in S)\,(b \leqslant a \longrightarrow b \in A)$.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then A is called a *subsemigroup* of S (see [9]) if $A^2 \subseteq A$. A subsemigroup A of S is called a *bi-ideal* of S if:
(i) $ASA \subseteq A$ and
(ii) $(\forall a \in A)(\forall b \in S)\,(b \leqslant a \longrightarrow b \in A)$.

A subsemigroup A of S is called a $(1, 2)$-*ideal* of S if:
(i) $ASA^2 \subseteq A$ and
(ii) $(\forall a \in A)(\forall b \in S)\,(b \leqslant a \longrightarrow b \in A)$.

By a *fuzzy subset* f of S we mean a *mapping* $f : S \longrightarrow [0, 1]$.

Definition 2.1. Let $(S, \cdot, \leqslant)$ be an ordered semigroups and f a fuzzy subset of S. Then f is called a *fuzzy left* (resp. *right*) *ideal* of S if:
(1) $(\forall x, y \in S)(x \leqslant y \longrightarrow f(x) \geqslant f(y))$.
(2) $(\forall x, y \in S)(f(xy) \geqslant f(y)$(resp. $f(xy) \geqslant f(x))$.

A fuzzy left and right ideal f of S is called a *fuzzy two-sided ideal* of S.

For any fuzzy subset f of S and $t \in (0, 1]$, the set

$$U(f; t) := \{x \in S \mid f(x) \geqslant t\}$$

is called the *level subset* of f.

Theorem 2.2. (cf. [8]) *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A fuzzy subset f of S is a fuzzy left (resp. right) ideal of S if and only if for every $t \in (0, 1]$ $U(f; t) \neq \emptyset$ is a left (resp. right) ideal.* $\square$

Example 2.3. Let $S = \{a, b, c, d, e, f\}$ be an ordered semigroup defined by the multiplication and the order below:

$\cdot$	a	b	c	d	e	f
a	a	a	a	d	a	a
b	a	b	b	d	b	b
c	a	b	c	d	e	e
d	a	a	d	d	d	d
e	a	b	c	d	e	e
f	a	b	c	d	e	f

$$\leqslant := \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,e),(f,f)\}$$

Right ideals of S are: $\{a,d\},\{a,b,d\}$ and S. Left ideals of S are: $\{a\},\{d\},\{a,b\},\{a,d\},\{a,b,d\},\{a,b,c,d\},\{a,b,d,e,f\}$ and S(see [7]).

Define $f : S \longrightarrow [0,1]$ by $f(a) = 0.8$, $f(b) = 0.5$, $f(d) = 0.6$ and $f(c) = f(e) = f(f) = 0.4$. Then

$$U(f;t) := \begin{cases} S & \text{if} \quad t \in (0.2, 0.4], \\ \{a,b,d\} & \text{if} \quad t \in (0.4, 0.5], \\ \{a,d\} & \text{if} \quad t \in (0.5, 0.6], \\ \emptyset & \text{if} \quad t \in (0.8, 1]. \end{cases}$$

and $U(f;t)$ is a right ideal of S, By Theorem 2.2, f is a fuzzy right ideal of S.

Let $\emptyset \neq A \subseteq S$. The *characteristic mapping* $f_A : S \longrightarrow \{0,1\}$ of A is defined by:

$$f_A : S \longrightarrow [0,1],\ x \longmapsto f_A(x) := \begin{cases} 1 & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \notin A. \end{cases}$$

Lemma 2.4. (cf. [4, 5]) *A non-empty subset A of an ordered semigroup $(S,\cdot,\leqslant)$ is a left (resp. right and bi-) ideal of S if and only if its characteristic function f_A is a fuzzy left (resp. right and bi-) ideal of S.* □

A subset T of an ordered semigroup S is called *semiprime* (see [9]) if for every $a \in S$ from $a^2 \in T$ it follows $a \in T$, or equivalently, if for each subset A of S $A^2 \subseteq T$ implies $A \subseteq T$.

3. Characterizations of regular semigroups

An ordered semigroup S is *regular* (see [5]) if for every $a \in S$, there exists $x \in S$ such that $a \leqslant axa$ or, equivalently, if $a \in (aSa]$ for every $a \in S$, and $A \subseteq (ASA]$ for every $A \subseteq S$.

An ordered semigroup S is *left* (resp. *right*) *simple* (see [9]) if for every left (resp. right) ideal A of S, we have $A = S$. S is called *simple* if it is left simple and right simple.

Theorem 3.1. *A regular ordered semigroup S is left simple if and only if every fuzzy left ideal of S is a constant map.*

Proof. Let S be a left simple ordered semigroup, f a fuzzy left ideal of S and $a \in S$. We consider the set,

$$E_S := \{e \in S \mid e^2 \geqslant e\}.$$

Then $E_S \neq \emptyset$. In fact, since S is regular and $a \in S$, there exists $x \in S$ such that $a \leqslant axa$. It follows from $(OS3)$ that

$$(ax)^2 = (axa)x \geqslant ax,$$

and so $ax \in E_S$ and hence $E_S \neq \emptyset$.

(1) Let $t \in E_S$. Then $f(e) = f(t)$ for every $e \in E_S$. Indeed, since S is left simple and $t \in S$ we have $(St] = S$. Since $e \in S$, then $e \in (St]$ and there exists $z \in S$ such that $e \leqslant zt$. Hence $e^2 \leqslant (zt)(zt) = (ztz)t$. Since f is a fuzzy left ideal of S, we have

$$f(e^2) \geqslant f((ztz)t) \geqslant f(t).$$

Since $e \in E_S$, we have $e^2 \geqslant e$. Then $f(e) \geqslant f(e^2)$ and we have $f(e) \geqslant f(t)$. Besides, since S is left simple and $e \in S$, we have $(Se] = S$. Since $t \in E_S$, exactly on the previous case$-$by symmetry$-$ we get $f(t) \geqslant f(e)$. Hence $f(t) = f(e)$, i.e., f is constant on E_S.

(2) Let $a \in S$, then $f(a) = f(t)$ for every $t \in S$. Indeed, since S is regular there exists $x \in S$ such that $a \leqslant axa$. We consider the element $xa \in S$. Then it follows by $(OS3)$ that,

$$(xa)^2 = x(axa) \geqslant xa,$$

then $xa \in E_S$ and by (1), we have $f(xa) = f(t)$. Besides, f is fuzzy left ideal of S, we have $f(xa) \geqslant f(a)$. Then $f(t) \geqslant f(a)$. On the other hand, since S is left simple and $t \in S$ then $S = (St]$. Since $a \in S$, we have $a \leqslant st$ for some $s \in S$. Since f is fuzzy left ideal of S, we have $f(a) \geqslant f(st) \geqslant f(t)$. Thus $f(t) = f(a)$, i.e., f is constant on S.

Conversely, let $a \in S$. Then the set $(Sa]$ is a left ideal of S. In fact, $S(Sa] = (S](Sa] \subseteq (SSa] \subseteq (Sa]$. If $x \in (Sa]$ and $S \ni y \leqslant x$, then $y \in ((Sa]] = (Sa]$. Since $(Sa]$ is a left ideal of S. By Lemma 2.4, the characteristic mapping

$$f_{(Sa]} : S \longrightarrow \{0,1\}, \quad x \longmapsto f_{(Sa]}(x)$$

is a fuzzy left ideal of S. By hypothesis $f_{(Sa]}$ is a constant mapping, that is, there exists $c \in \{0,1\}$ such that

$$f_{(Sa]}(x) = c \quad \text{for every } x \in S.$$

Let $(Sa] \subset S$ and let $t \in S$ such that $t \notin (Sa]$ then $f_{(Sa]}(t) = 0$. On the other hand, since $a^2 \in (Sa]$, then we have $f_{(Sa]}(a^2) = 0$, a contradiction to the fact that $f_{(Sa]}$ is a constant mapping. Hence $S = (Sa]$. $\square$

From left–right dual of Theorem 3.1, we have the following:

Theorem 3.2. *A regular ordered semigroup S is right simple if and only if every fuzzy right ideal of S is a constant mapping.* $\square$

An ordered semigroup $(S, \cdot, \leqslant)$ is *left* (resp. *right*) *regular* [4, 6], if for every $a \in S$ there exists $x \in S$ such that $a \leqslant xa^2$ (resp. $a \leqslant a^2x$) or, equivalently, if $a \in (Sa^2]$ (resp. $a \in (a^2S]$) for all $a \in S$, and $A \subseteq (SA^2]$ (resp. $A \subseteq (A^2S]$) for all $A \subseteq S$.

An ordered semigroup S is called *completely regular* (see [6]) if it is regular, left regular and right regular.

Lemma 3.3. (cf. [9]) *An ordered semigroup S is completely regular if and only if $A \subseteq (A^2SA^2]$ for every $A \subseteq S$ or, equivalently, if and only if $a \in (a^2Sa^2]$ for every $a \in S$.* $\square$

Theorem 3.4. *An ordered semigroup $(S, \cdot, \leqslant)$ is left regular if and only if for each fuzzy left ideal f of S, we have $f(a) = f(a^2)$ for all $a \in S$.*

Proof. Suppose that f is a fuzzy left ideal of S and let $a \in S$. Since S is left regular, there exists $x \in S$ such that $a \leqslant xa^2$. Since f is a fuzzy left ideal of S, we have

$$f(a) \geqslant f(xa^2) \geqslant f(a^2) \geqslant f(a).$$

Conversely, let $a \in S$. We consider the left ideal $L(a^2) = (a^2 \cup Sa^2]$ of S, generated by a^2. Then by Lemma 2.4, the characteristic mapping

$$f_{L(a^2)} : S \longrightarrow \{0, 1\}, \quad x \longmapsto f_{L(a^2)}(x)$$

is a fuzzy left ideal of S.

By hypothesis we have $f_{L(a^2)}(a) = f_{L(a^2)}(a^2)$. Since $a^2 \in L(a^2)$, we have $f_{L(a^2)}(a^2) = 1$ and $f_{L(a^2)}(a) = 1$. Then $a \in L(a^2) = (a^2 \cup Sa^2]$ and $a \leqslant y$ for some $y \in a^2 \cup Sa^2$. If $y = a^2$, then $a \leqslant y = a^2 = aa = aa^2 \in Sa^2$ and $a \in (Sa^2]$. If $y = xa^2$ for some $x \in S$, then $a \leqslant y = xa^2 \in Sa^2$, and $a \in (Sa^2]$. $\square$

From left$-$right dual of Theorem 3.4, we have the following:

Theorem 3.5. *An ordered semigroup $(S, \cdot, \leqslant)$ is right regular if and only if for each fuzzy right ideal f of S, we have $f(a) = f(a^2)$ for all $a \in S$.* $\square$

From ([9, Theorem 3]) and Theorems 3.1 and 3.4, and by Lemma 3.3, we have the following characterization theorem for completely regular ordered semigroups.

Theorem 3.6. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Then the following are equivalent:*

(*i*) *S is completely regular,*

(*ii*) *for each fuzzy bi-ideal f of S we have $f(a) = f(a^2)$ for all $a \in S$,*

(*iii*) *for each fuzzy left ideal g and each fuzzy right ideal h of S we have $g(a) = g(a^2)$ and $h(a) = h(a^2)$ for all $a \in S$.* $\square$

An ordered semigroup $(S, \cdot, \leqslant)$ is called *left* (resp. *right*) *duo* if every left (resp. right) ideal of S is a two-sided ideal of S, and *duo* if every its ideal is both left and right duo.

Definition 3.7. An ordered semigroup $(S, \cdot, \leqslant)$ is called *fuzzy left* (resp. *right*) *duo* if every fuzzy left (resp. right) ideal of S is a fuzzy two-sided ideal of S. An ordered semigroup S is called *fuzzy duo* if it is both fuzzy left and fuzzy right duo.

Theorem 3.8. *A regular ordered semigroup is left* (*right*) *duo if and only if it is fuzzy left* (*right*) *duo.*

Proof. Let S be left duo and f a fuzzy left ideal of S. Let $a, b \in S$. Then the set $(Sa]$ is a left ideal of S. In fact, $S(Sa] = (S](Sa] \subseteq (SSa] \subseteq (Sa]$ and if $x \in (Sa]$ and $S \ni y \leqslant x$ then $y \in ((Sa]] = (Sa]$. Since S is left duo, then $(Sa]$ is a two-sided ideal of S. Since S is regular there exists $x \in S$ such that $a \leqslant axa$ then

$$ab \leqslant (axa)b \in (aSa)b \subseteq (Sa)S \subseteq (Sa]S \subseteq (Sa].$$

Then $ab \in ((Sa]] = (Sa]$ and $ab \leqslant xa$ for some $x \in S$. Since f is a fuzzy left ideal of S, we have

$$f(ab) \geqslant f(xa) \geqslant f(a).$$

Let $x, y \in S$ be such that $x \leqslant y$. Then $f(x) \geqslant f(y)$, because f is a fuzzy left ideal of S. Thus f is a fuzzy right deal of S and S is fuzzy left duo.

Conversely, if S is fuzzy left duo and A a left ideal of S, then the characteristic function f_A of A is a fuzzy left ideal of S. By hypothesis f_A is a fuzzy right ideal of S and by Lemma 2.4, A is a right ideal of S. Thus S is left duo. □

Theorem 3.9. *In a regular ordered semigroup every bi-ideal is a right (left) ideal if and only if every its fuzzy bi-ideal is a fuzzy right (left) ideal.*

Proof. Let $a, b \in S$ and f a fuzzy bi-ideal of S. Then $(aSa]$ is a bi-ideal of S. In fact, $(aSa]^2 \subseteq (aSa](aSa] \subseteq (aSa]$, $(aSa]S(aSa] = (aSa](S](aSa] \subseteq (aSa]$ and if $x \in (aSa]$ and $S \ni y \leqslant x \in (aSa]$ then $y \in ((aSa]] = (aSa]$. Since $(aSa]$ is a bi-ideal of S, by hypothesis $(aSa]$ is right ideal of S. Since $a \in S$ and S is regular there exists $x \in S$ such that $a \leqslant axa$ then

$$ab \leqslant (axa)b \in (aSa)S \subseteq (aSa]S \subseteq (aSa].$$

Hense $ab \leqslant aza$ for some $z \in S$. Since f is a fuzzy bi-ideal of S, we have

$$f(ab) \geqslant f(aza) \geqslant \min\{f(a), f(a)\} = f(a).$$

Let $x, y \in S$ be such that $x \leqslant y$. Then $f(x) \geqslant f(y)$ because f is a fuzzy bi-ideal of S. Thus f is a fuzzy right ideal of S.

Conversely, if A is a bi-ideal of S, then by Lemma 2.4, f_A is a fuzzy bi-ideal of S. By hypothesis f_A is a fuzzy right ideal of S. By Lemma 2.4, A is a right ideal of S. □

Definition 3.10. Let $(S, \cdot, \leqslant)$ be an ordered semigroup and f a fuzzy subsemigoup of S. Then f is called a *fuzzy* $(1,2)$-*ideal* of S if:

(*i*) $x \leqslant y \longrightarrow f(x) \geqslant f(y)$,

(*ii*) $f(xa(yz)) \geqslant \min\{f(x), f(y), f(z)\}$

for all $x, y, z, a \in S$.

Proposition 3.11. *Every fuzzy bi-ideal of an ordered semigroup S is a fuzzy $(1,2)$-ideal of S.*

Proof. Let f be a fuzzy bi-ideal of S and let $x, y, z, a \in S$. Then

$$\begin{aligned} f(xa(yz)) = f((xay)z) &\geqslant \min\{f(xay), f(z)\} \\ &\geqslant \min\{\min\{f(x), f(y)\}, f(z)\} = \min\{f(x), f(y), f(z)\}. \end{aligned}$$

Now, let $x, y \in S$ be such that $x \leqslant y$. Then $f(x) \geqslant f(y)$, because f is a fuzzy bi-ideal of S. □

Corollary 3.12. *Every fuzzy left (resp. right) ideal f of an ordered semigroup S is a fuzzy $(1,2)$-ideal of S.*

The converse of the Proposition 3.11, is not true in general. However, if S is a regular ordered semigroup then we have the following Proposition:

Proposition 3.13. *A fuzzy $(1,2)$-ideal of a regular ordered semigroup is a fuzzy bi-ideal.*

Proof. Assume that S is regular ordered semigroup and let f be a fuzzy $(1,2)$-ideal of S. Let $x,y,a \in S$. Since S is regular and $(xSx]$ is a bi-ideal of S, so it is a right ideal of S, by Theorem 3.9. Thus

$$xa \leqslant (xSx)a \in (xSx)S \subseteq (xSx]S \subseteq (xSx],$$

whence $xa \leqslant xyx$ for some $y \in S$. Thus $xay \leqslant (xyx)y$ and we have

$$\begin{aligned} f(xay) &\geqslant f((xyx)y) \geqslant \min\{f(xyx), f(y)\} \\ &\geqslant \min\{\min\{f(x), f(x)\}, f(y)\} = \min\{f(x), f(y)\}. \end{aligned}$$

Let $x,y \in S$ be such that $x \leqslant y$. Then $f(x) \geqslant f(y)$, because f is a fuzzy $(1,2)$-ideal of S. Thus f is a fuzzy bi-ideal of S. □

4. Semilattices of left simple ordered semigroups

A subsemigroup F of an ordered semigroup $(S,\cdot,\leqslant)$ is called a *filter* of S if:

(1) $ab \in F \longrightarrow a \in F$ and $b \in F$,

(2) $c \geqslant a \in F \longrightarrow c \in F$

for all $a,b,c \in S$.

For $x \in S$, we denote by $N(x)$ the filter of S generated by x. $\mathcal{N}$ denotes the equivalence relation on S defined by

$$\mathcal{N} := \{(x,y) \in S \times S \mid N(x) = N(y)\}.$$

Definition 4.1. (cf. [7]) An equivalence relation σ on S is called *congruence* if $(a,b) \in \sigma$ implies $(ac,bc) \in \sigma$ and $(ca,cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called *semilattice congruence* if $(a^2,a) \in \sigma$ and $(ab,ba) \in \sigma$ for each $a,b \in S$. If σ is a semilattice congruence on S then the σ-class $(x)_\sigma$ of S containing x is a subsemigroup of S for every $x \in S$.

Lemma 4.2. (cf. [9]) *Let $(S, \cdot, \leqslant)$ be an ordered semigroup. Then $(x)_{\mathcal{N}}$ is a left simple subsemigroup of S, for every $x \in S$ if and only every left ideal of S is a right ideal of S and it is semiprime.* □

An ordered semigroup S is called a *semilattice of left simple semigroups* if there exists a semilattice congruence σ on S such that the σ-class $(x)_\sigma$ of S containing x is a left simple subsemigroup of S for every $x \in S$ or, equivalently, if there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left simple subsemigroups of S such that

(1) $S_\alpha \cap S_\beta = \emptyset \ \ \forall \alpha, \beta \in Y, \quad \alpha \neq \beta,$

(2) $S = \bigcup\limits_{\alpha \in Y} S_\alpha,$

(3) $S_\alpha S_\beta \subseteq S_{\alpha\beta} \ \forall \alpha, \beta \in Y.$

In ordered semigroups the semilattice congruences are defined exactly same as in the case of semigroups −without order− so the two definitions are equivalent (see [7]).

Lemma 4.3. *An ordered semigroup $(S, \cdot, \leqslant)$ is a semilattice of left simple semigroups if and only if for all left ideals A, B of S we have*

$$(A^2] = A \quad \textit{and} \quad (AB] = (BA].$$

Proof. $(\rightarrow)$ Let S be a semilattice of left simple semigroups and A, B are left ideals of S. Then there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left simple subsemigroups of S satisfying all conditions mentioned in the definition of a semilattice of left simple semigroups.

Let $a \in A$. Since $a \in S = \bigcup\limits_{\alpha \in Y} S_\alpha$, there exists $\alpha \in Y$ such that $a \in S_\alpha$. Since S_α is left simple, we have

$$S_\alpha = (S_\alpha b] = \{c \in S \mid \exists x \in S_\alpha : c \leqslant xb\}$$

for all $b \in S_\alpha$.

Since $a \in S_\alpha$, we have $S_\alpha = (S_\alpha a]$ that is $a \leqslant xa$ for some $x \in S_\alpha$. Since $x \in S_\alpha = (S_\alpha a]$, we have $x \leqslant ya$ for some $y \in S_\alpha$. Thus we have $a \leqslant xa \leqslant (ya)a \in (SA)A \subseteq AA = A^2$ and $a \in (A^2]$. Hence $A \subseteq (A^2]$. On the other hand, since A is a subsemigroup of S, hence $A^2 \subseteq A$ and we have $(A^2] \subseteq (A] = A$. Let $x \in (AB]$, then $x \leqslant ab$ for some $a \in A$ and $b \in B$. Since $a, b \in S = \bigcup\limits_{\alpha \in Y} S_\alpha$, there exist $\alpha, \beta \in Y$ such that $a \in S_\alpha$, $b \in S_\beta$. Then $ab \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$ and $ba \in S_\beta S_\alpha \subseteq S_{\beta\alpha} = S_{\alpha\beta}$(since $\alpha, \beta \in Y$

and Y is a semilattice). Since $S_{\alpha\beta}$ is left simple, we have $S_{\alpha\beta} = (S_{\alpha\beta}c]$ for each $c \in S_{\alpha\beta}$. Then $ab \in (S_{\alpha\beta}ba]$ and $ab \leqslant yba$ for some $y \in S_{\alpha\beta}$. Since B is a left ideal of S, we have $yba \in (SB)A \subseteq BA$, then $x \in (BA]$. Thus $(AB] \subseteq (BA]$. By symmetry we have $(BA] \subseteq (AB]$.

($\leftarrow$) Since $\mathcal{N}$ is a semilattice congruence on S, which is equivalent to the fact that $(x)_{\mathcal{N}}$ $\forall x \in S$, is a left simple subsemigrup of S. By Lemma 4.2, it is enough to prove that every left ideal is right ideal and semiprime. Let L be a left ideal of S. Then

$$LS \subseteq (LS] = (SL] \subseteq (L] = L.$$

If $x \in L$, $S \ni y \leqslant x \in L$, then $y \in L$, since L is a left ideal of S. Thus L is a right ideal of S. Let $x \in S$ be such that $x^2 \in L$. We consider the bi-ideal $B(x)$ of S generated by x. Then

$$\begin{aligned} B(x)^2 &= (x \cup x^2 \cup xSx](x \cup x^2 \cup xSx] \subseteq ((x \cup x^2 \cup xSx)(x \cup x^2 \cup xSx)] \\ &= (x^2 \cup x^3 \cup xSx^2 \cup x^4 \cup xSx^3 \cup x^2Sx \cup x^3Sx \cup xSx^2Sx]. \end{aligned}$$

Since $x^2 \in L,\ x^3 \in SL \subseteq L,\ (xS)x^2 \subseteq SL \subseteq L,\ x^4 \in SL \subseteq L$. Then

$$B(x)^2 \subseteq (L \cup LS] = (L] = L.$$

Thus $(B(x)^2] \subseteq (L] = L$ and $x \in L$. Hence L is semiprime. □

Theorem 4.4. *An ordered semigroup $(S, \cdot, \leqslant)$ is a semilattice of left (right) simple semigroups if and only if for every fuzzy left (right) ideal f of S and all $a, b \in S$, we have*

$$f(a^2) = f(a) \quad \text{and} \quad f(ab) = f(ba).$$

Proof. Let S be a semilattice of left simple semigroups. By hypothesis, there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left simple subsemigroups of S such that:

(1) $S_\alpha \cap S_\beta = \emptyset$ $\forall \alpha, \beta \in Y$, $\alpha \neq \beta$,

(2) $S = \bigcup_{\alpha \in Y} S_\alpha$,

(3) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ $\forall \alpha, \beta \in Y$.

Let f be a fuzzy left ideal of S and $a \in S$. Then $f(a) = f(a^2)$. In fact, by Theorem 3.4, it is enough to prove that $a \in (Sa^2]$ for every $a \in S$. Let $a \in S$, then there exists $\alpha \in Y$ such that $a \in S_\alpha$. Since S_α is left simple, we have $S_\alpha = (S_\alpha a]$ and $a \leqslant xa$ for some $x \in S$.

Since $x \in S_\alpha$, we have $x \in (S_\alpha a]$ and $x \leqslant ya$ for some $y \in S_\alpha$. Thus we have

$$a \leqslant xa \leqslant (ya)a = ya^2,$$

which for $y \in S$, implies $a \in (Sa^2]$.

Let $a, b \in S$. Then, by the above, we have

$$f(ab) = f((ab)^2) = f(a(ba)b) \geqslant f(ba).$$

By symmetry we can prove that $f(ba) \geqslant f(ab)$. Hence $f(ab) = f(ba)$.

Conversely, assume that for every fuzzy left ideal f of S, we have

$$f(a^2) = f(a) \quad \text{and} \quad f(ab) = f(ba)$$

for all $a, b \in S$.

Then by condition (1) and by Theorem 3.4, we see that S is left regular. Let A be a left ideal of S and let $a \in A$. Then $a \in S$, since S is left regular, there exists $x \in S$ such that

$$a \leqslant xa^2 = (xa)a \in (SA)A \subseteq AA = A^2.$$

Hence $a \in (A^2]$ and $A \subseteq (A]$. On the other hand, since A is a left ideal of S, we have $A^2 \subseteq SA \subseteq A$, then $(A^2] \subseteq (A] = A$. Let A and B be left ideals of S and let $x \in (BA]$ then $x \leqslant ba$ for some $a \in A$ and $b \in B$. We consider the left ideal $L(ab)$ generated by ab. That is, the set $L(ab) = (ab \cup Sab]$. Then by Lemma 3.4, the characteristic function $f_{L(ab)}$ of $L(ab)$ is a fuzzy left ideal of S. By hypothesis, we have $f_{L(ab)}(ab) = f_{L(ab)}(ba)$. Since $ab \in L(ab)$, we have $f_{L(ab)}(ab) = 1$ and $f_{L(ab)}(ba) = 1$ and hence $ba \in L(ab) = (ab \cup Sab]$. Then $ba \leqslant ab$ or $ba \leqslant yab$ for some $y \in S$. If $ba \leqslant ab$ then $x \leqslant ab \in AB$ and $x \in (AB]$. If $ba \leqslant yab$ then $x \leqslant yab \in (SA)B \subseteq AB$ and $x \in (AB]$. Thus $(BA] \subseteq (AB]$. By symmetry we can prove that $(AB] \subseteq (BA]$. Therefore $(AB] = (BA]$ and by Lemma 4.3, it follows that S is a semilattice of left simple semigroups. □

Proposition 4.5. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup and f a fuzzy left (resp. right) ideal of S, $a \in S$ such that $a \leqslant a^2$. Then $f(a) = f(a^2)$.*

Proof. Since $a \leqslant a^2$ and f is a fuzzy left ideal of S, we have

$$f(a) \geqslant f(a^2) = f(aa) \geqslant f(a),$$

and so $f(a) = f(a^2)$. □

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