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Kernel normal system of inverse AG^{**}-groupoids

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Abstract. Abel-Grassmann's groupoids or shortly AG-groupoids have been studied in a number of papers, although under different names. In some papers they are named LA-semigroups [3], in other left invertive groupoids [2]. In this paper we introduce a notion of kernel normal system of inverse AG^{**}-groupoids and introduce congruences, in a similar way as that have been defined for inverse semigroups, [6].

1. Introduction

A groupoid S on which the following is true

$$(\forall a, b, c \in S) \quad ab \cdot c = cb \cdot a,$$

is called an *Abel-Grassmann's groupoid* (*AG-groupoid*), [9]. It is easy to verify that in every AG-groupoid medial law $ab \cdot cd = ac \cdot bd$ holds. Thus, AG-groupoids belongs to the wider class of entropic groupoids.

We denote the set of all idempotents of S by $E(S)$.

Abel-Grassmann's groupoid S satisfying

$$(\forall a, b, c \in S) \quad ab \cdot c = b \cdot ca$$

is called an *AG^*-groupoid*.

Abel-Grassmann's groupoid S satisfying

$$(\forall a, b, c \in S) \quad a \cdot bc = b \cdot ac$$

is an AG^{**}-groupoid. It is obvious that in AG^{**}-groupoid for $a, b, c, d \in S$ it follows that

$$ab \cdot cd = c(ab \cdot d) = c(db \cdot a) = db \cdot ca.$$

If AG-groupoid S has the left identity e, then

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$$a \cdot bc = ea \cdot bc = eb \cdot ac = b \cdot ac,$$

so S is an AG^{**}-groupoid.

An AG-groupoid S is called *inverse* AG-groupoid if for every $a \in S$ there exists $a' \in S$ such that $a = (aa')a$ and $a' = (a'a)a'$. Then a' is an inverse element of a, and by $V(a)$ we shall mean the set of all inverses of a. It is easy to prove that if $a' \in V(a)$, $b' \in V(b)$, then $a'b' \in V(ab)$ and that aa' or $a'a$ are not necessary idempotents.

Example 1. Let AG-groupoid S be given by the following table:

	1	2	3	4
1	2	3	1	4
2	4	1	3	2
3	3	2	4	1
4	1	4	2	3

Then $23 \cdot 2 = 2,\ 32 \cdot 3 = 3,\ 14 \cdot 1 = 1,\ 41 \cdot 4 = 4$, whence S is an inverse AG-groupoid, and $E(S) = \emptyset$.

Remark 1. Let S be an inverse AG-groupoid, $a \in S,\ a' \in V(a)$ and $aa' = a'a$. Then

$$(aa')^2 = aa' \cdot aa' = (aa' \cdot a')a = (a'a \cdot a')a = a'a = aa',$$

imply that $aa' \in E(S)$.

AG-groupoids are not associative in general, however, there is close relation between them and semigroups as well as with commutative structures.

The relationship of AG-groupoids with semigroups (especially with commutative group) will be illustrated by the following two examples.

Example 2. Let $(S, \cdot)$ be an AG-groupoid, $a \in S$ be the fixed element. Then we define the so-called *sandwich operation* on S as follows,

$$(\forall x, y \in S) \quad x \circ y = xa \cdot y.$$

It is easy to verify that $x \circ y = y \circ x$ for all $x, y \in S$. In other words $(S, \circ)$ is a commutative groupoid. If S is an AG^*-groupoid and $x, y, z \in S$ then

$$(x \circ y) \circ z = ((xa \cdot y)a)z = za \cdot (xa \cdot y)$$

and

$$x \circ (y \circ z) = xa \cdot (y \circ z) = xa \cdot (ya \cdot z) = za \cdot (ya \cdot x) = za \cdot (xa \cdot y),$$

whence $(x \circ y) \circ z = x \circ (y \circ z)$. Consequently, $(S, \circ)$ is a commutative semigroup.

Example 3. Let $(S, \cdot)$ be a commutative group with identity e. If we define operation $*$ on S by

$$(\forall a, b \in S) \ a * b = ba^{-1},$$

then $(S, *)$ is an AG-groupoid. Clearly, $e * e = e$. Moreover,

$$e * b = be^{-1} = be = b,$$

so that e is the left identity on $(S, *)$. It follows that $(S, *)$ is an AG^{**}-groupoid. Since $a * a = aa^{-1} = e$ and $(a * a) * a = a$, we conclude that $(S, *)$ is an inverse AG^{**}-groupoid and for every $a \in S$, $a \in V(a)$.

2. Kernel normal system

We first discuss some properties of inverse AG^{**}-groupoids.

Remark 2. Let S be an AG^{**}-groupoid and $e, f \in E(S)$. Then

$$ef = ee \cdot ff = ef \cdot ef = ff \cdot ee = fe,$$

imply that $E(S)$ is a semilattice.

Remark 3. If S is an inverse AG^{**}-groupoid and $x, y \in V(a)$, then

$$xa = x(ay \cdot a) = ay \cdot xa = ax \cdot ya = y(ax \cdot a) = ya,$$

and

$$x = xa \cdot x = ya \cdot x = xa \cdot y = ya \cdot y = y.$$

It follows that $|V(a)| = 1$.

It follows further that the inverse of $a \in S$ is unique. We shall denote it by a^{-1}. Since in the inverse AG-groupoid S for element a the products aa^{-1} and $a^{-1}a$ are not necessarily idempotents, we need to establish the following lemma.

Lemma 1. *Let S be an inverse AG^{**}-groupoid, $a \in S$. Then*

$$aa^{-1}, a^{-1}a \in E(S) \longleftrightarrow aa^{-1} = a^{-1}a.$$

Proof. Let $aa^{-1}, a^{-1}a \in E(S)$. Then

$$aa^{-1} \cdot a^{-1}a = a^{-1}(aa^{-1} \cdot a) = a^{-1}a,$$
$$a^{-1}a \cdot aa^{-1} = a(a^{-1}a \cdot a^{-1}) = aa^{-1}.$$

It then follows that $aa^{-1} \in E(S)$ and since $E(S)$ is a semilattice it further means that $aa^{-1} = a^{-1}a$. The converse follows easily by Remark 1. □

Example 4. Let AG-groupoid S be given by the following table:

	1	2	3	4
1	2	2	4	4
2	2	2	2	2
3	1	2	3	4
4	1	2	1	2

Then S is an inverse AG^{**}-groupoid, $E(S) = \{2, 3\}$ is a semilattice, elements 1 and 4 are mutually inverse and $1 \cdot 4 \neq 4 \cdot 1$.

The groupoid $(S, *)$ in Example 3 is an inverse AG^{*}-groupoid, $a = a^{-1}$ for each $a \in S$ and $a * a = e$.

Remark 4. From now on we shall denote by S the inverse AG^{**}-groupoid in which $aa^{-1} = a^{-1}a \in E(S)$.

Lemma 2. *If ρ is a congruence relation on S, then S/ρ is an AG^{**}-groupoid, and for every $a, b \in S$ it holds that $a\rho b$ if and only if $a^{-1}\rho b^{-1}$.*

The following definitions are introduced in [7].
Let K be a subset of S, then:

- K is *full* if $E(S) \subseteq K$;
- K is *self-conjugate* if $a^{-1}(Ka) \subseteq K$ for every $a \in K$;
- K is *inverse closed* if from $a \in K$ it follows $a^{-1} \in K$;
- K is *normal* it is full, self-conjugate and inverse closed.

Let ρ be a congruence on S. The restriction $\rho|_{E(S)}$ is the *trace* of ρ and will be denoted by $tr\rho$. The set $ker\rho = \{a \in S \mid (\exists e \in E(S))\ a\rho e\}$ is the *kernel* of ρ.

Lemma 3. *Let ρ be a congruence relation on S, then $ker\rho$ is a normal subgroupoid of S.*

Definition 1. Let K be a normal subgroupoid of S and τ a congruence on $E(S)$ such that

$$ea \in K,\ e\tau a^{-1}a \ \longrightarrow\ a \in K$$

for every $a \in S$ and $e \in E(S)$. Then the pair (K, τ) is a *congruence pair* for S. In such a case, we can define the relation $\rho_{(K,\tau)}$ on S by

$$a\rho_{(K,\tau)}b \ \longleftrightarrow\ a^{-1}a\tau b^{-1}b,\ ab^{-1}, ba^{-1} \in K. \tag{1}$$

Theorem 1. [7] *If* (K, τ) *is a congruence pair for* S*, then* $\rho_{(K,\tau)}$ *is the unique congruence on* S *for which* $ker\rho_{(K,\tau)} = K$ *and* $tr\rho_{(K,\tau)} = \tau$*. Conversely, if* ρ *is a congruence on* S*, then* $(ker\rho, tr\rho)$ *is a congruence pair for* S *and* $\rho_{(ker\rho,tr\rho)} = \rho$*.*

Let ρ be a congruence on S. We can then consider the collection of all ρ-classes containing idempotents:

$$\mathcal{K}(\rho) = \{e\rho \,|\, e \in E(S)\}. \tag{2}$$

Such collections of subsets of S can be characterized in the following fashion also.

Definition 2. Let $\mathcal{K}$ be a family of pairwise disjoint inverse subgroupoids of S satisfying:

(a) $E(S) \subseteq \bigcup_{L\in\mathcal{K}} L$,

(b) for each $a \in S$ and $L \in \mathcal{K}$ there exists $M \in \mathcal{K}$ such that $a^{-1}(La) \subseteq M$,

(c) for each $a, b \in S$ if $a, ab, bb^{-1} \in L, L \in \mathcal{K}$, then $b \in L$.

Then $\mathcal{K}$ is a *kernel normal system* for S. For such a family $\mathcal{K}$ we define a relation $\xi_{\mathcal{K}}$ on S by

$$a\xi_{\mathcal{K}}b \;\longleftrightarrow\; aa^{-1}, bb^{-1}, ab^{-1}, ba^{-1} \in L$$

for some $L \in \mathcal{K}$.

By using (2) and the above definition, we can obtain the second characterization of congruences on an inverse semigroup as follows.

Lemma 4. *Let* $\mathcal{K}$ *be a kernel normal system on* S*,* $K = \bigcup_{L\in\mathcal{K}} L$ *and* τ *the relation on semilattice* $E(S)$ *defined by*

$$e\tau f \;\longleftrightarrow\; e, f \in L \quad (for\ some\ L \in \mathcal{K}).$$

Then τ *is a congruence and a pair* (K, τ) *is a congruence pair for* S*.*

Proof. Let $\mathcal{K}$ is a kernel normal system on S. It is clear that relation τ is an equivalence relation. Let $e, f \in E(S) \cap L$, $g \in E(S)$. Then by the Definition 2 (b) there exists $M \in \mathcal{K}$ such that

$$eg = e \cdot gg = g \cdot eg = g^{-1} \cdot eg \in g^{-1} \cdot Lg \subseteq M,$$

and

$$fg = f \cdot gg = g \cdot fg = g^{-1} \cdot fg \in g^{-1} \cdot Lg \subseteq M.$$

Now $eg, fg \in M$ implies $eg\tau fg$. Hence, τ is a right congruence.

Similarly, τ is a left congruence and hence the congruence on $E(S)$.

Next we prove that K is a subgroupoid on S. Let $a, b \in K$, then there exist $L, T \in \mathcal{K}$ such that $a \in L,\ b \in T$. Since $a^{-1} \in L,\ b^{-1} \in T$, we have that $a^{-1}a = e \in E(L),\ b^{-1}b = f \in E(T)$. Now

$$be = b \cdot ee = e \cdot be \in e \cdot Te,$$

and by the Definition 2 (b) there exists $M \in \mathcal{K}$ such that $be \in e \cdot Te \subseteq M$. In a similar way, there exists $N \in \mathcal{K}$ such that

$$af = f \cdot af \in f \cdot Lf \subseteq N.$$

Since $e\tau = E(L),\ f\tau = E(T)$ and if $(ef)\tau = E(V)$ for some $V \in \mathcal{K}$, then because τ is congruence on semilattice $E(S)$ we obtain

$$E(L) \cdot E(T) = e\tau \cdot f\tau = (ef)\tau = (fe)\tau = E(V).$$

Now, $a, e \in L,\ b, f \in T, be \in M,\ af \in N,\ ef = fe \in V$ and

$$fe \in Te = T \cdot ee = e \cdot Te \in M,\ ef \in Lf = L \cdot ff = f \cdot Le \subseteq N$$

implies that

$$fe \in V \cap M \neq \emptyset, \quad ef \in V \cap N \neq \emptyset,$$

so that $V = M = N$. Hence $be, af \in V$ implies that

$$ab = (aa^{-1} \cdot a)(bb^{-1} \cdot b) = ea \cdot fb = ef \cdot eb = fe \cdot ab = be \cdot af \in V \cdot V \subseteq V \subseteq K,$$

whence K is an subgroupoid of S.

Since each $L \in \mathcal{K}$ is closed under taking of inverses, so is K, and thus it is an inverse subgroupoid of S. Conditions (a) and (b) of the Definition 2 insure that K is full and self-conjugate. Hence, K is a normal subgroupoid of S.

Now, $ea \in K$ and $e\tau aa^{-1}$, implies that there exists $L \in \mathcal{K}$ such that $ea \in L$ so, there exists $T \in \mathcal{K}$ such that $e, aa^{-1} \in E(T) \subseteq T$ and $e \cdot aa^{-1} \in T$. From $ea \in L$ it follows that $(ea)^{-1} = ea^{-1} \in L$, so that $ea \cdot ea^{-1} = e \cdot aa^{-1} \in L$. Hence, $L \cap T \neq \emptyset$ i.e., $L = T$. Now $e, ea^{-1}, aa^{-1} \in L$, and by (c) of the Definition 2, it follows that $a \in L \subseteq K$. Consequently, (K, τ) is a congruence pair on S. □

Theorem 2. *If $\mathcal{K}$ is a kernel normal system for S, then $\xi_{\mathcal{K}}$ is the unique congruence on S for which $\mathcal{K}(\xi_{\mathcal{K}}) = \mathcal{K}$. Conversely, if ξ is a congruence on S, then $\mathcal{K}(\xi)$ is a kernel normal system for S and $\xi_{\mathcal{K}(\xi)} = \xi$.*

Proof. First we prove that $\rho_{(K,\tau)} = \xi_{\mathcal{K}}$. Let

$$a\rho_{(K,\tau)} \longleftrightarrow aa^{-1}\tau bb^{-1},\ ab^{-1}, ba^{-1} \in K.$$

Hence $aa^{-1}, bb^{-1} \in L$ for some $L \in \mathcal{K}$ and there exists $M, N \in \mathcal{K}$ such that $ab^{-1} \in M,\ ba^{-1} \in N$. Now $(ab^{-1})^{-1} = a^{-1}b \in M,\ (ba^{-1})^{-1} = b^{-1}a \in N$ and $ab^{-1} \cdot a^{-1}b \in M,\ ba^{-1} \cdot b^{-1}a \in N$. Also, $ab^{-1} \cdot a^{-1}b = aa^{-1} \cdot bb^{-1} \in L \cdot L \subseteq L$ and $ba^{-1} \cdot b^{-1}a = bb^{-1} \cdot aa^{-1} \in L \cdot L \subseteq L$ and so $L \cap M \neq \emptyset$ implies that $L = M,\ L \cap N \neq \emptyset$ implies that $L = N$. Hence $aa^{-1},\ bb^{-1},\ ab^{-1},\ ba^{-1} \in L$ if and only if $a\xi_{\mathcal{K}}b$, whence $\rho_{(K,\tau)} \subseteq \xi_{\mathcal{K}}$.

Conversely, let

$$a\xi_{\mathcal{K}}b \longleftrightarrow aa^{-1}, bb^{-1}, ab^{-1}, ba^{-1} \in L$$

for some $L \in \mathcal{K}$. Now, $aa^{-1}, bb^{-1} \in E(L)$, so that $aa^{-1}\tau bb^{-1}$. Since $L \subset K$, we have $ab^{-1}, ba^{-1} \in K$. Thus $a\rho_{(K,\tau)}b$ implies that $\xi_{\mathcal{K}} \subseteq \rho_{(K,\tau)}$. Hence $\xi_{\mathcal{K}} = \rho_{(K,\tau)}$.

Next we prove that $\mathcal{K}(\xi_{\mathcal{K}}) = \mathcal{K}$, where $\mathcal{K}(\xi_{\mathcal{K}}) = \{e\xi_{\mathcal{K}} \mid e \in E(S)\}$. First let $L \in \mathcal{K}$. Then L is an inverse subgroupoid of S. In order to prove that $L \in \mathcal{K}(\xi_{\mathcal{K}})$, it is sufficient to prove that L is a $\xi_{\mathcal{K}}$-class. If $a, b \in L$, then $aa^{-1}, bb^{-1}, ab^{-1}, ba^{-1} \in L$. Thus $a\xi_{\mathcal{K}}b$, so that L is a $\xi_{\mathcal{K}}$-class. Let $a \in L$ and $a\xi_{\mathcal{K}}b$. Then $aa^{-1}, bb^{-1}, ab^{-1}, ba^{-1} \in T$ for some $T \in \mathcal{K}$. However, $aa^{-1} \in L \cap T$, so that $L = T$. From $a\xi_{\mathcal{K}}b$ it follows that $a^{-1}\xi_{\mathcal{K}}b^{-1}$. By stability of $\xi_{\mathcal{K}}$ we have

$$aa^{-1}\xi_{\mathcal{K}}bb^{-1},\ ab^{-1}\xi_{\mathcal{K}}bb^{-1},\ aa^{-1}\xi_{\mathcal{K}}ba^{-1},$$

so there exists $U \in \mathcal{K}$ such that $aa^{-1}, bb^{-1}, ab^{-1}, ba^{-1} \in U$, it follows that $L = U$. Also, $a^{-1} \in L,\ a^{-1}b = (ab^{-1})^{-1}, bb^{-1} \in L$ and Definition 2 (c) imply that $b \in L$. Consequently, L is a $\xi_{\mathcal{K}}$-class and thus $L \in \mathcal{K}_(\xi_{\mathcal{K}})$.

Conversely, let $L \in \mathcal{K}_(\xi_{\mathcal{K}})$. Then L contains an idempotent e, and since by Definition 2 (a) it follows that $E(S) \subseteq K$ then there exists $T \in \mathcal{K}$ such that $e \in T$. Since T is also an $\xi_{\mathcal{K}}$-class thus $L = T$.

Consequently, $L \in \mathcal{K}$, which completes the verification that $\mathcal{K}(\xi_{\mathcal{K}}) = \mathcal{K}$.

Let now ρ be any congruence on S for which $\mathcal{K}(\rho) = \mathcal{K}$. Then by the first part of the proof, we obtain $ker\rho = ker\xi_{\mathcal{K}}$ and $tr\rho = tr\xi_{\mathcal{K}}$, and get $\rho = \xi_{\mathcal{K}}$ by Theorem 1. This establishes the uniqueness of $\xi_{\mathcal{K}}$ and completes the proof of the first part of the theorem.

Let ξ be a congruence on S. It is obvious that $\mathcal{K}(\xi) = \{e\xi \mid e \in E(S)\}$ consists of a family of pairwise disjoint inverse subgroupoids of S whose union contains $E(S)$. Let $a \in S$ and let L be a ξ-class containing an idempotent e. Then for any $b \in L$, $a^{-1} \cdot ba\xi a^{-1} \cdot ea = e \cdot a^{-1}a$ so that $a^{-1} \cdot La \subseteq M$

where M is the ξ-class containing idempotent $e \cdot a^{-1}a$. This verifies (b) of Definition 2. With the same notation, assume $a, ab, bb^{-1} \in L$, where L be a ξ-class containing an idempotent e. Then

$$b = bb^{-1} \cdot b \,\xi\, eb \,\xi\, ab \,\xi\, e$$

so that $b \in L$. This verifies (c) of Definition 2 and completes the proof that $\mathcal{K}(\xi)$ is a kernel normal system.

By $\mathcal{K}(\xi_{\mathcal{K}}) = \mathcal{K}$ we have $\mathcal{K}(\xi_{\mathcal{K}(\xi)}) = \mathcal{K}(\xi)$. Thus $\xi_{\mathcal{K}(\xi)}$ and ξ have the same kernel normal system, so by the uniqueness proved in the first part of the theorem, we obtain $\xi_{\mathcal{K}(\xi)} = \xi$. This completes the proof of the theorem. $\square$

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