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# Pure ideals in ternary semigroups

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**Abstract.** In this paper we introduce the notions of pure ideals, weakly pure ideals in ternary semigroups. We also define purely prime ideals of a ternary semigroup and study some properties of these ideals. The space of purely prime two-sided ideal is topologized.

## 1. Introduction

Cayley and Sylvester along with several other mathematicians, in the 19th century considered ternary algebraic structures and cubic relations. The n-ary structures, which are the generalizations of ternary structures create hopes because of their possible applications in Physics. A few important physical applications have been recorded in [2, 3, 12, 19]. Ternary semigroups exhibit natural examples of ternary algebras.

Banach find some applications in ternary semigroup. He gave an example to show that a ternary semigroup is not necessarily reduce to an ordinary semigroup. Los [13] studied some properties of ternary semigroup and proved that every ternary semigroup can be embedded in a semigroup. Sioson at [18] introduced the ideal theory in ternary semigroups. He also introduced the notion of regular ternary semigroups and characterized them by using the properties of quasi-ideals. In [16], Santiago developed the theory of ternary semigroups and semiheaps. He studied regular and completely regular ternary semigroups. Dixit and Dewan studied quasi-ideals and bi-ideals in ternary semigroups at [5, 6]. Ternary regular semigroups are studied in [8] and [17]. The nice characterization of regularity by ideals is given in [8].

M. Shabir and A. Khan at [14] studied prime ideals and prime one sided ideals in semigroups. Ahsan and Takahashi at [1] have brought forwarded

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the concept of pure and purely prime ideals in semigroups.

M. Shabir and S. Bashir at [15] launched prime ideals in ternary semigroups. At cite8 and [17] ternary and $n$-ary semigroups are given along with an immaculate characterization of regularity by their ideals. At [7] applications of ideals to the divisibility theory in ternary and $n$-ary semigroups is presented.

In this paper we start the study of pure ideals, weakly pure ideals and purely prime ideals in ternary semigroups. We characterize ternary semigroups by the properties of pure and weakly pure ideals.

## 2. Preliminaries

A non-empty set $T$ with a ternary operation ( ) is called a *ternary semigroup* if it satisfies the following associative law:

$$((x_1x_2x_3)x_4x_5) = (x_1(x_2x_3x_4)x_5) = (x_1x_2(x_3x_4x_5))$$

for all $x_i \in T$, $1 \leqslant i \leqslant 5$.

To avoid complexity we denote $(x_1x_2x_3)$ as $x_1x_2x_3$ and take the operation ( ) as multiplication. It is evident that each ordinary semigroup $(T,*)$ induces a ternary semigroup $(T,(\,))$ by defining $(abc) = (a*b)*c$. Whereas in [13] it has been demonstrated that every ternary semigroup does not enjoy the status of an ordinary semigroup. A ternary semigroup $T$ is said to be a *ternary semigroup with zero* if there exists an element $0 \in T$ such that $0ab = a0b = ab0 = 0$ for all $a, b \in T$. Then 0 is called the *zero element* of $T$. If $A, B, C$ are non-empty subsets of a ternary semigroup $T$ then their product $ABC$ is defined as

$$ABC = \{abc : a \in A, b \in B \;\; and \; c \in C\}.$$

A non-empty subset $S$ of a ternary semigroup $T$ is called a *ternary subsemigroup* of $T$ if $SSS = S^3 \subseteq S$. A non-empty subset $A$ of a ternary semigroup $T$ is called a *left* (*right, lateral*) *ideal* of $T$ if $TTA \subseteq A$ ($ATT \subseteq A$, $TAT \subseteq A$). If $A$ is a left, right and lateral ideal of $T$, then it is called an *ideal* of $T$ and if $A$ is a left and right ideal of $T$, then it is called *two-sided ideal* of $T$. Lateral ideals are also known as *middle ideals*. It is clear that every left, right and lateral ideal is a ternary subsemigroup of $T$. An ideal $A$ of a ternary semigroup $T$ is called *idempotent* if $A^3 = AAA = A$. A ternary semigroup $T$ is called *semisimple* if each ideal of $T$ is idempotent.

An element $x \in T$ is *regular* if there exists an element $a \in T$ such that $x = xax$, that is $x \in xTx$. A ternary semigroup $T$ is *regular* if each element of $T$ is regular.

The intersection of all the left ideals of $T$ containing $X \subseteq T$ is the smallest left ideal of $T$ containing $X$. It is denoted by $\langle X\rangle_l$ and called the *left ideal generated by* $X$. Clearly $\langle X\rangle_l = X \cup XTT$.

Similarly,

$$\begin{aligned}\langle X\rangle_r &= X \cup TTX\\ \langle X\rangle_m &= X \cup TXT \cup TTXTT\\ \langle X\rangle_t &= X \cup TTX \cup XTT \cup TTXTT\\ \langle X\rangle &= X \cup TTX \cup XTT \cup TXT \cup TTXTT\end{aligned}$$

are the right, lateral, two-sided, and ideal of $T$ generated by $X$, respectively.

It is well known that if $A$, $B$ and $C$ are two-sided ideals of $T$, then $(ABC) = \{abc : a \in A, b \in B, c \in C\}$ is a two-sided ideal of $T$. The intersection of any family of (two-sided) ideals of a ternary semigroup $T$ is either empty or a (two-sided) ideal of $T$. Union of any family of (two-sided) ideals of a ternary semigroup $T$ is a (two-sided) ideal of $T$.

## 3. Pure ideals

In [1], Ahsan and Takahashi studied pure ideals in semigroups. In this section we define pure ideals in ternary semigroups.

**Definition 3.1.** A two-sided ideal $I$ of a ternary semigroup $T$ is called *right* (*left*) *pure* if for each $x \in I$ there exist $y, z \in I$ such that $xyz = x$ ($yzx = x$).

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Similarly we define one-sided right (left) pure ideals.

The following example shows that right pure ideals need not be left pure.

**Example 3.2.** Let $T = \{0, a, b, c, 1\}$. Define the ternary operation ( ) on $T$ as $(abc) = a * (b * c)$ where the binary operation $*$ is defined as

| $*$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | 0 | $a$ | $a$ |
| $b$ | 0 | 0 | $b$ | $b$ | $b$ |
| $c$ | 0 | 0 | $b$ | $c$ | $c$ |
| 1 | 0 | $a$ | $b$ | $c$ | 1 |

Then $(T, (\ ))$ is a ternary semigroup and the ideal $I_1 = \{0, a\}$ is neither right pure nor left pure; the ideal $I_2 = \{0, b\}$ is both right and left pure; the ideal $I_3 = \{0, a, b, c\}$ is right pure but not left pure.

**Proposition 3.3.** *Each right pure right ideal of a ternary semigroup $T$ is contained in a right pure two-sided ideal of $T$.*

*Proof.* Let $A$ be a right pure right ideal of $T$. Then $A \cup TTA$ is a two-sided ideal of $T$ generated by $A$. Let $x \in A \cup TTA$. Suppose $x \in A$, since $A$ is right pure right ideal of $T$, therefore there exist $y, z \in A$ such that $x = xyz$. If $x \in TTA$, then $x = t_1t_2a$ for some $t_1, t_2 \in T$ and $a \in A$. Again, since $A$ is right pure so there exist $b, c \in A$ such that $a = abc$. Hence $x = t_1t_2a = t_1t_2(abc) = (t_1t_2a)bc = xbc$. This shows that $A \cup TTA$ is a right pure two-sided ideal containing the right pure right ideal $A$. □

**Proposition 3.4.** *A two-sided ideal $I$ of a ternary semigroup $T$ is right pure if and only if $J \cap I = JII$ for all right ideals $J$ of $T$.*

*Proof.* Suppose $I$ is a right pure two-side ideal of $T$. For every right ideal $J$ of $T$, $JII \subseteq J \cap I$ always. Let $x \in J \cap I$. Since $I$ is a right pure two-sided ideal, so there exist $y, z \in I$ such that $xyz = x$. Thus $x = xyz \in JII$. Hence $J \cap I \subseteq JII$. Thus $J \cap I = JII$.

Conversely, assume that $J \cap I = JII$ for every right ideal $J$ of $T$. We show that $I$ is a right pure two-sided ideal. Let $x$ be any element of $I$ and $J = x \cup xTT$ be the right ideal of $T$ generated by $x$. Then by hypothesis

$$(x \cup xTT) \cap I = (x \cup xTT)II = xII \cup (xTT)II \subseteq xII \cup xII = xII.$$

Since $x \in (x \cup xTT) \cap I$, so $x \in xII$. Hence there exist $y, z \in I$ such that $x = xyz$. Thus $I$ is right pure. □

Similarly we can show that, *an ideal $I$ of a ternary semigroup $T$ is right pure if and only if $J \cap I = JII$ for all right ideals $J$ of $T$.*

**Definition 3.5.** A ternary semigroup $T$ is said to be *right weakly regular* if for each $x \in T$, $x \in (xTT)^3$.

Every regular ternary semigroup is right weakly regular but the converse is not true.

**Theorem 3.6.** *For a ternary semigroup $T$, the following assertions are equivalent:*

(a) *$T$ is right weakly regular.*
(b) *Every right ideal of $T$ is idempotent, that is $J^3 = J$ for every right ideal $J$ of $T$.*
(c) *$J \cap I = JII$ for every right ideal $J$ and two-sided ideal $I$ of $T$.*
(d) *$J \cap I = JII$ for every right ideal $J$ and for every ideal $I$ of $T$.*

*Proof.* $(a) \Rightarrow (b)$ Let $J$ be a right ideal of $T$, then $J^3 \subseteq JTT \subseteq J$. Let $x \in J$. Then $x \in (xTT)^3 \subseteq J^3$. Thus $J \subseteq J^3$. Hence $J = J^3$.

$(b) \Rightarrow (a)$ Suppose that every right ideal of $T$ is idempotent. Let $x \in T$. Then $J = x \cup xTT$ is the right ideal of $T$, so idempotent, that is

$$\begin{aligned} x \cup xTT &= (x \cup xTT)(x \cup xTT)(x \cup xTT) \\ &= xxx \cup xxxTT \cup xxTTx \cup xxTTxTT \cup xTTxx \cup xTTxxTT \cup \\ &\quad \cup xTTxTTx \cup xTTxTTxTT. \end{aligned}$$

Simple calculations shows that $x \in (xTT)^3$. Hence $T$ is right weakly regular.

$(a) \Rightarrow (c)$ Suppose $T$ is right weakly regular ternary semigroup and $J$ a right ideal and $I$ a two-side ideal of $T$. Then $JII \subseteq J \cap I$ always. Let $x \in J \cap I$. Since $T$ is right weakly regular, so $x \in (xTT)^3$. Thus $x = (xs_1t_1)(xs_2t_2)(xs_3t_3)$ for some $s_1, t_1, s_2, t_2, s_3, t_3 \in T$. Hence $x \in JII$, which shows that $J \cap I \subseteq JII$. Hence $J \cap I = JII$.

$(c) \Rightarrow (d)$ Obvious.

$(d) \Rightarrow (a)$ Let $x \in T$ and $J = x \cup xTT$ be the right ideal of $T$ generated by $x$, $I = x \cup xTT \cup TTx \cup TxT \cup TTxTT$ be the ideal of $T$ generated by $x$. Then, by hypothesis, $(x \cup xTT) \cap (x \cup xTT \cup TTx \cup TxT \cup TTxTT) = (x \cup xTT)(x \cup xTT \cup TTx \cup TxT \cup TTxTT)(x \cup xTT \cup TTx \cup TxT \cup TTxTT) = (xxx \cup xxxTT \cup xxTTx \cup xxTxT \cup xxTTxTT \cup xTTxx \cup xTTxxTT \cup xTTxTTx \cup xTTxTxT \cup xTTxTTxTT \cup xTxTx \cup xTxTxTT \cup xTxTTxT)$.

Simple calculations shows that $x \in (xTT)^3$. Hence $T$ is right weakly regular ternary semigroup. □

**Theorem 3.7.** *For a ternary semigroup $T$, the following assertions are equivalent:*
(1) *$T$ is right weakly regular.*
(2) *Every two-sided ideal $I$ of $T$ is right pure.*
(3) *Every ideal $I$ of $T$ is right pure.*

*Proof.* The proof follows from Theorem 3.6 and Proposition 3.4. □

**Proposition 3.8.** *Let $T$ be a ternary semigroup with 0. Then*
(1) *$\{0\}$ is a right pure ideal of $T$.*
(2) *Set theoretic union of any number of right pure two-sided ideals*

*(ideals) of $T$ is a right pure two-sided ideal (ideal) of $T$.*

(3) *Any finite intersection of right pure two-sided ideals (ideals) of $T$ is a right pure two-sided ideal (ideal) of $T$.*

*Proof.* (1) Obvious.
(2) Let $\{I_k\}_{k\in K}$ be a family of right pure two-sided ideals of $T$. Then $\underset{k\in K}{\cup} I_k$ is a two-sided ideal of $T$. Suppose $x \in \underset{k\in K}{U} I_k$. Then there exists some $k \in K$ such that $x \in I_k$. Since $I_k$ is a right pure two-sided ideal of $T$, so there exist $y, z \in I_k$ such that $x = xyz$. It follows that $y, z \in \underset{k\in K}{U} I_k$ such that $x = xyz$. Hence $\underset{k\in K}{U} I_k$ is a right pure two-sided ideal of $T$.

(3) Let $I_1$, $I_2$ be right pure two-sided ideals of $T$ and $x \in I_1 \cap I_2$. Then $x \in I_1$and $x \in I_2$. Since $I_1$and $I_2$ are right pure two-sided ideals of $T$, so there exist $y_1, z_1 \in I_1$ and $y_2, z_2 \in I_2$ such that $x = xy_1z_1$ and $x = xy_2z_2$. Thus we have $x = xy_1z_1 = (xy_2z_2)y_1z_1 = ((xy_1z_1)y_2z_2)y_1z_1 = x(y_1z_1y_2)(z_2y_1z_1)$, where $y_1z_1y_2$ and $z_2y_1z_1 \in I_1 \cap I_2$. Thus $I_1 \cap I_2$ is a right pure ideal of $T$. □

Similarly we can prove the case of ideals.

**Proposition 3.9.** *Let $I$ be any two-sided ideal of a ternary semigroup $T$ with zero $0$. Then $I$ contains a largest right pure two-sided ideal. (We call it the pure part of $I$ and denote by $\mathcal{S}(I)$).*

*Proof.* Let $\mathcal{S}(I)$ be the union of all right pure two-sided ideals contained in $I$. Such ideals exist because $\{0\}$ is a right pure ideal contained in each two-side ideal. By the above Proposition $\mathcal{S}(I)$ is a right pure two-sided ideal. It is indeed the largest right pure two-sided ideal contained in $I$. □

Similarly we can show that if $I$ is an ideal of $T$ then $I$ contains a largest right pure ideal.

**Proposition 3.10.** *Let $I$, $K$ be two-sided ideals of $T$ and $\{I_k\}_{k\in K}$ be the family of two-sided ideals of a ternary semigroup $T$ with zero $0$. Then*

(1) $\mathcal{S}(I \cap K) = \mathcal{S}(I) \cap \mathcal{S}(K)$.

(2) $\underset{k\in K}{\cup} \mathcal{S}(I_k) \subseteq \mathcal{S}(\underset{k\in K}{\cup} I_k)$.

*Proof.* (1) Since $\mathcal{S}(I) \subseteq I$, $\mathcal{S}(K) \subseteq K$, thus $\mathcal{S}(I) \cap \mathcal{S}(K) \subseteq I \cap K$. But $\mathcal{S}(I) \cap \mathcal{S}(K)$ is right pure by Proposition 3.8, so $\mathcal{S}(I) \cap \mathcal{S}(K) \subseteq \mathcal{S}(I \cap K)$. On the other hand $\mathcal{S}(I \cap K) \subseteq I \cap K \subseteq I$ and $\mathcal{S}(I \cap K)$ is pure, so $\mathcal{S}(I \cap K) \subseteq \mathcal{S}(I)$. Similarly, $\mathcal{S}(I \cap K) \subseteq \mathcal{S}(K)$. Thus $\mathcal{S}(I \cap K) \subseteq \mathcal{S}(I) \cap \mathcal{S}(K)$.

Hence, $\mathcal{S}(I \cap K) = \mathcal{S}(I) \cap \mathcal{S}(K)$.
(2) Since $\mathcal{S}(I_k) \subseteq I_k$ so $\bigcup_{k\in K} \mathcal{S}(I_k) \subseteq \bigcup_{k\in K} I_k$. As $\mathcal{S}(I_k)$ is right pure, so $\bigcup_{k\in K} \mathcal{S}(I_k)$ is right pure. Thus we have $\bigcup_{k\in K} \mathcal{S}(I_k) \subseteq \mathcal{S}(\bigcup_{k\in K} I_k)$. □

**Definition 3.11.** Let $I$ be a right pure two-sided ideal of $T$, then $I$ is called *purely maximal* if $I$ is maximal in the lattice of proper right pure two-sided ideals of $T$.

A proper right pure two-sided ideal $I$ of $T$ is called *purely prime* if $I_1TI_2 \subseteq I$ implies $I_1 \subseteq I$ or $I_2 \subseteq I$ for any right pure two-sided ideals $I_1$ and $I_2$ of $T$. Equivalently $I_1 \cap I_2 \subseteq I$ implies $I_1 \subseteq I$ or $I_2 \subseteq I$ (Because $I_1TI_2 \subseteq I_1 \cap I_2$ and $I_1 \cap I_2 = I_1I_2I_2 \subseteq I_1TI_2$. Thus $I_1TI_2 = I_1 \cap I_2$).

**Proposition 3.12.** *Any purely maximal two-sided ideal is purely prime.*

*Proof.* Suppose $I$ is purely maximal two-sided ideal of $T$ and $I_1, I_2$ are right pure two-sided ideals of $T$ such that $I_1 \cap I_2 \subseteq I$. Suppose $I_1 \nsubseteq I$. Then $I_1 \cup I$ is a right pure ideal such that $I \subsetneq I_1 \cup I$. Since $I$ is purely maximal, so $I_1 \cup I = T$. Thus

$$I_2 = I_2 \cap T = I_2 \cap (I_1 \cup I) = (I_2 \cap I_1) \cup (I_2 \cap I) \subseteq I \cup I = I.$$

Hence $I$ is purely prime. □

**Proposition 3.13.** *The pure part of any maximal two-sided ideal of a ternary semigroup with zero is purely prime.*

*Proof.* Let $M$ be a maximal two-sided ideal of $T$ and $\mathcal{S}(M)$ be its pure part. Suppose $I_1 \cap I_2 \subseteq \mathcal{S}(M)$ where $I_1$, $I_2$ are right pure two-sided ideals of $T$. If $I_1 \subseteq M$ then $I_1 \subseteq \mathcal{S}(M)$. If $I_1 \nsubseteq \mathcal{S}(M)$ then $I_1 \nsubseteq M$. Thus $I_1 \cup M = T$ because $M$ is maximal. Hence we have

$$I_2 = I_2 \cap T = I_2 \cap (I_1 \cup M) = (I_2 \cap I_1) \cup (I_2 \cap M) \subseteq \mathcal{S}(M) \cup M \subseteq M \cup M = M.$$

But $\mathcal{S}(M)$ is the largest right pure two-sided ideal contained in $M$. Thus $I_2 \subseteq \mathcal{S}(M)$. Hence $\mathcal{S}(M)$ is purely prime. □

**Proposition 3.14.** *Let $I$ be a right pure two-sided ideal of $T$ and $a \in T$ such that $a \notin I$, then there exists a purely prime two-sided ideal $J$ of $T$ such that $I \subseteq J$ and $a \notin J$.*

*Proof.* Let

$X = \{J : J \text{ is a right pure two-sided ideal of } T,\ I \subseteq J \text{ and } a \notin J\}$,

then $X \neq \emptyset$ since $I \in X$. $X$ is partially ordered by inclusion. Let $\{J_k\}_{k \in K}$ be any totally ordered subset of $X$. By Proposition 3.8, $\underset{k \in K}{\cup} J_k$ is a right pure two-sided ideal. Since $I \subseteq \underset{k \in K}{\cup} J_k$ and $a \notin \underset{k \in K}{\cup} J_k$, so $\underset{k \in K}{\cup} J_k \in X$. Thus by Zorn's Lemma, $X$ has a maximal element, say, $J$ such that $J$ is pure, $I \subseteq J$ and $a \notin J$. We claim that $J$ is purely prime. Suppose $I_1$ and $I_2$ are right pure two-sided ideals of $T$ such that $I_1 \nsubseteq J$ and $I_2 \nsubseteq J$. Since $I_k (k = 1, 2)$ and $J$ are right pure so $I_k \cup J$ is a right pure two-sided ideal such that $J \subsetneq I_k \cup J$. Thus $a \in I_k \cup J$ $(k = 1, 2)$. As $a \notin J$, so $a \in I_k$ $(k = 1, 2)$. Thus $a \in I_1 \cap I_2$. Hence $I_1 \cap I_2 \nsubseteq J$. This shows that $J$ is purely prime. □

**Proposition 3.15.** *Any proper right pure two sided ideal $I$ of $T$ is the intersection of all the purely prime two-sided ideals of $T$ containing $I$.*

*Proof.* By Proposition 3.14, there exists purely prime two-sided ideals containing $I$ . Let $\{J_k\}_{k \in K}$ be the family of all purely prime two-sided ideals of $T$ which contain $I$. Since $I \subseteq J_k$ for all $k \in K$, so $I \subseteq \underset{k \in K}{\cap} J_k$. To show that $\underset{k \in K}{\cap} J_k \subseteq I$. Let $a \notin I$, then by Proposition 3.14, there exists a purely prime two-sided ideal $J$ such that $I \subseteq J$ and $a \notin J$. It follows that $a \notin \underset{k \in K}{\cap} J_k$. Thus $\underset{k \in K}{\cap} J_k \subseteq I$. Hence $I = \underset{k \in K}{\cap} J_k$. □

## 4. Weakly pure ideals

In this section we generalize the concept of pure two sided ideal and define weakly pure two-sided ideal.

**Definition 4.1.** A two-sided ideal $A$ of a ternary semigroup $T$ is called *left* (resp. *right*) *weakly pure* if $A \cap B = AAB$ (resp. $A \cap B = BAA$) for all two-sided ideals $B$ of $T$.

Every left (right) pure two-sided ideal is left (right) weakly pure.

**Proposition 4.2.** *If $A, B$ are two-sided ideals of a ternary semigroup $T$ with zero* 0*, then*

$$BA^{-1} = \{t \in T : xyt \in B \ for \ all \ x, y \in A\}$$

*and*

$$A_{-1}B = \{t \in T : txy \in B \text{ for all } x, y \in A\}$$

*are two-sided ideals of $T$.*

*Proof.* $BA^{-1} \neq \emptyset$ because $0 \in BA^{-1}$. Let $s, r \in T$ and $t \in BA^{-1}$. Then for all $x, y \in A$, $(xy(srt)) = (x(ysr)t) = xzt \in B$ because $z = ysr \in A$. Hence $srt \in BA^{-1}$. Also, $(xy(tsr)) = (xyt)sr \in BTT \subseteq B$, because $xyt \in B$. Thus $tsr \in BA^{-1}$. Hence $BA^{-1}$ is a two-sided ideal of $T$.

Now, let $s, r \in T$ and $t \in A_{-1}B$. Then $((srt)xy) = sr(txy) = srb \in TTB \subseteq B$ for all $x, y \in A$, because $b = txy \in B$. Hence $srt \in A_{-1}B$.

Also, $(tsr)xy = t(srx)y = tx_1y \in B$ because $x_1 = srx \in A$. Thus $tsr \in A_{-1}B$. Hence $A_{-1}B$ is a two-sided ideal of $T$. □

**Proposition 4.3.** *For a two-sided ideal $A$ of a ternary semigroup $T$, the following assertions are equivalent.*

(1) *$A$ is left (right) weakly pure.*

(2) $(BA^{-1}) \cap A = B \cap A$ $(A_{-1}B \cap A = A \cap B)$ *for all ideals $B$ of $T$.*

*Proof.* (1) ⇒ (2) Suppose $A$ is left weakly pure. Since $BA^{-1}$ is a two sided ideal, we have $(BA^{-1}) \cap A = AA(BA^{-1})$.

Now we show that $AA(BA^{-1}) \subseteq B$. Let $atx \in AA(BA^{-1})$, where $a, t \in A$, $x \in BA^{-1}$. Then $atx \in B$ (by the definition of $BA^{-1}$). Hence $AA(BA^{-1}) \subseteq B$. Also $AA(BA^{-1}) \subseteq ATT \subseteq A$ and $(BA^{-1}) \cap A = AA(BA^{-1}) \subseteq A \cap B$. Thus $(BA^{-1}) \cap A \subseteq B \cap A$.

Let $b \in B \cap A$, then $xyb \in B$ for all $x, y \in A$. Hence $b \in BA^{-1}$. Thus $B \cap A \subseteq (BA^{-1}) \cap A$. Therefore $(BA^{-1}) \cap A = B \cap A$.

(2) ⇒ (1) Assume that $A, B$ are two-sided ideals of a ternary semigroup $T$ and $(BA^{-1}) \cap A = B \cap A$. We show that $A$ is left weakly pure. First we show that $B \subseteq (AAB)A^{-1}$. Let $b \in B$, then for each $x, y \in A$, we have $xyb \in AAB$. Thus $b \in (AAB)A^{-1}$. Hence $B \in (AAB)A^{-1}$. This shows $B \subseteq (AAB)A^{-1}$. Thus $A \cap B \subseteq (AAB)A^{-1} \cap A = AAB \cap A \subseteq AAB$ by hypothesis. But $AAB \subseteq A \cap B$ always. Hence $A \cap B = AAB$. Thus $A$ is left weakly pure. □

**Proposition 4.4.** *For a ternary semigroup $T$ the following assertions are equivalent.*

(1) *Each two-sided ideal of $T$ is left weakly pure.*

(2) *Each two-sided ideal of $T$ is idempotent.*

(3) *Each two-sided ideal of $T$ is right weakly pure.*

*Proof.* (1) $\Rightarrow$ (2) Suppose each two-sided ideal of $T$ is left weakly pure. Let $X$ be a two-sided ideal of $T$, then for each two-sided ideal $Y$ of $T$ we have $X \cap Y = XXY$. In particular $X = X \cap X = XXX$. Hence each two-sided ideal of $T$ is idempotent.

(2) $\Rightarrow$ (1) Suppose each two-sided ideal of $T$ is idempotent. Let $X$ be a two-sided ideal of $T$, then for any two-sided ideal $Y$ of $T$ we always have $XXY \subseteq X \cap Y$. On the other hand,

$$X \cap Y = (X \cap Y)(X \cap Y)(X \cap Y) \subseteq XXY.$$

Hence we have $X \cap Y = XXY$. Thus $X$ is left weakly pure.

(2) $\Rightarrow$ (3) Similarly as (2) $\Rightarrow$ (1).

(3) $\Rightarrow$ (2) Suppose each two-sided ideal of $T$ is right weakly pure. Let $X$ be any two-sided ideal of $T$. Then $X$ is right weakly pure. Hence for each two-sided ideal $Y$ of $T$, we have $X \cap Y = YXX$. In particular $X \cap X = XXX$. Hence each two-sided ideal of $T$ is idempotent. □

**Example 4.5.** Any set $T$ with the ternary operation $(xyz) = x$ if $x = y = z$, and $(xyz) = 0$ otherwise, where 0 is a fixed element of $T$, is a ternary semigroup in which every subset containing 0 is its two-sided ideal. Every two-sided ideal of this semigroup is its right (left) pure ideal.

If $|T| = 1$ or 2, then every two-sided ideal of $T$ is purely prime. But if $|T| \geq 3$, then the ideal $\{0\}$ is not purely prime. Because if $a, b \in T - \{0\}$, then $I = \{0, a\}$ and $J = \{0, b\}$ are right pure ideals of $T$ such that $I \cap J = \{0\}$ but neither $I \nsubseteq \{0\}$ nor $J \nsubseteq \{0\}$.

## 5. Pure spectrum of a ternary semigroup

In this section $T$ is a ternary semigroup with zero such that $T^3 = T$.

Let $\mathcal{P}(T)$ be the set of all right pure ideals of $T$ and $\mathbf{P}(T)$ be the set of all proper purely prime ideals of $T$. Define for each $I \in \mathcal{P}(T)$,

$$\mathcal{B}_I = \{J \in \mathbf{P}(T) : I \nsubseteq J\}, \qquad \Im(T) = \{\mathcal{B}_I : I \in \mathcal{P}(T)\}.$$

**Theorem 5.1.** *$\Im(T)$ forms a topology on $\boldsymbol{P}(T)$.*

*Proof.* As $\{0\}$ is a right pure ideal of $T$, so $\mathcal{B}_{\{0\}} = \{J \in \mathbf{P}(T) : \{0\} \nsubseteq J\} = \emptyset$, because 0 belongs to every right pure ideal. Since $T$ is a right pure ideal of $T$, $\mathcal{B}_T = \{J \in \mathbf{P}(T) : T \nsubseteq J\} = \mathbf{P}(T)$ because $\mathbf{P}(T)$ is the set of all proper purely prime ideals of $T$.

Let $\{\mathcal{B}_{I_\alpha} : \alpha \in \Lambda\} \subseteq \Im(T)$, then

$$\bigcup_{\alpha\in\Lambda} \mathcal{B}_{I_\alpha} = \{J \in \mathbf{P}(T) : I_\alpha \nsubseteq J \text{ for some } \alpha \in \Lambda\} = \{J \in \mathbf{P}(T) : \cup I_\alpha \nsubseteq J\} = \mathcal{B}_{\cup I_\alpha}.$$

To prove that $\mathcal{B}_{I_1} \cap \mathcal{B}_{I_2} \in \Im(T)$ for any $\mathcal{B}_{I_1}, \mathcal{B}_{I_2} \in \Im(T)$ we consider $J \in \mathcal{B}_{I_1} \cap \mathcal{B}_{I_2}$. Then $J \in \mathbf{P}(T)$, $I_1 \nsubseteq J$ and $I_2 \nsubseteq J$.

Suppose that $I_1 \cap I_2 \subseteq J$. Since $J$ is a purely prime ideal, therefore either $I_1 \subseteq J$ or $I_2 \subseteq J$, which is a contradiction, hence $I_1 \cap I_2 \nsubseteq J$, which implies $J \in \mathcal{B}_{I_1 \cap I_2}$. Thus $\mathcal{B}_{I_1} \cap \mathcal{B}_{I_2} \subseteq \mathcal{B}_{I_1 \cap I_2}$.

On the other hand, if $J \in \mathcal{B}_{I_1 \cap I_2}$, then

$$I_1 \cap I_2 \nsubseteq J \Rightarrow I_1 \nsubseteq J \text{ and } I_2 \nsubseteq J \Rightarrow J \in \mathcal{B}_{I_1} \text{ and } J \in \mathcal{B}_{I_2} \Rightarrow J \in \mathcal{B}_{I_1} \cap \mathcal{B}_{I_2}.$$

Hence $\mathcal{B}_{I_1 \cap I_2} \subseteq \mathcal{B}_{I_1} \cap \mathcal{B}_{I_2}$. Consequently, $\mathcal{B}_{I_1 \cap I_2} = \mathcal{B}_{I_1} \cap \mathcal{B}_{I_2}$, which implies $\mathcal{B}_{I_1} \cap \mathcal{B}_{I_2} \in \Im(T)$.

Thus $\Im(T)$ is a topology on $\mathbf{P}(T)$. □

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