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Construction for subdirectly irreducible sloops of cardinality $\mathrm{n}2^{\mathrm{m}}$

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Abstract. Guelzow [8] and similarly Armanious [1] [2] gave generalized doubling constructions to construct nilpotent subdirectly irreducible SQS-skeins and sloops. In [5] the authors have given recursive construction theorems as $n \to 2n$ for subdirectly irreducible sloops and SQS-skeins, these constructions supplies us with a subdirectly irreducible sloop of cardinality $2n$ satisfying that the cardinality of the congruence class of its monolith is equal to 2. In this article, we give a construction for subdirectly irreducible sloops of cardinality $n2^m$ having a monolith with a congruence class of cardinality 2^m for each integer $m \geqslant 2$. This construction supplies us with the fact that each sloop is isomorphic to the homomorphic image of the constructed subdirectly irreducible sloop over its monolith.

1. Introduction

A *Steiner triple system* is a pair $(L; B)$ where L is a finite set and B is a collection of 3-subsets called blocks of L such that every 2-subset of L is contained in exactly one block of B (cf. [7]). Let $\mathbf{STS}(n)$ denote a Steiner triple system (briefly a triple system) of cardinality n. It is well known that an $\mathbf{STS}(n)$ exists iff $n \equiv 1$ or $3 \pmod 6$ (cf. [7] and [9]).

There is one to one correspondence between **STSs** and sloops (Steiner loops) (see [7] and [8]). A *sloop* $L = (L; \bullet, 1)$ is a groupoid with a neutral element 1 satisfying the identities:

$$x \bullet y = y \bullet x,$$
$$1 \bullet x = x,$$
$$x \bullet (x \bullet y) = y.$$

A sloop L is called *Boolean sloop* if the binary operation satisfies in

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addition the associative law. Each Boolean sloop is a group that is also called a Boolean group.

Let $\mathbf{SL}(n)$ denote a sloop of cardinality n. Then $\mathbf{SL}(n)$ exists iff $n \equiv 2$ or $4 \pmod 6$ (cf. [7], [10]). If $\mathbf{SL}(n)$ is Boolean, then $n = 2^m$ for $m \geqslant 1$. Notice that for any a and $b \in L$ the equation $a \bullet x = b$ has the unique solution $x = a \bullet (a \bullet x) = a \bullet b$; i.e., L is a quasigroup [6].

A subsloop N is called a *normal subsloop* of L if and only if :

$$x \bullet (y \bullet N) = (x \bullet y) \bullet N \quad \text{for all} \quad x, y \in L.$$

Equivalently, a subsloop N of L is normal if and only if $N = [1]\theta$ for a congruence θ on L (cf. [7], [10]).

In fact, there is an isomorphism between the lattice of normal subsloops and the congruence lattice of the sloop [10]. Quackenbush has also proved that the congruences of the sloops are permutable, regular and uniform. Moreover, he has shown that for any finite $\mathbf{SL}(n)$, a subsloop N of cardinality $\frac{1}{2}n$ is normal.

Guelzow [8] and Armanious ([1], [2]) gave generalized doubling constructions for nilpotent subdirectly irreducible **SQS** -skeins and sloops of cardinality $2n$. In [5] the authors gave recursive construction theorems as $n \to 2n$ for subdiredtly irreducible sloops. All these constructions supplies us with subdirectly irreducible sloops having a monolith θ satisfying $|[x]\theta| = 2$ (the minimal possible order of a proper normal subsloop). Also in these constructions, the authors begin with a subdirectly irreducible $\mathbf{SL}(n)$ to construct a subdirectly irreducible $\mathbf{SL}(2n)$ satisfying that the cardinality of the congruence class of its monolith is equal 2. Armanious [3] has given another construction of a subdirectly irreducible $\mathbf{SL}(2n)$. He begins with a finite simple $\mathbf{SL}(n)$ to costruct a subdirectly irreducible $\mathbf{SL}(2n)$ having a monolith θ with $|[x]\theta| = n$ (the maximal possible order of a proper normal subsloop).

In this article, we begin with an arbitrary $\mathbf{SL}(n)$ for each possible value $n \geqslant 4$ to construct a subdirectly irreducible $\mathbf{SL}(n2^m)$ for each integer $m \geqslant 2$. This construction enables us to construct a subdirectly irreducible sloop having a monolith θ satisfying that the congruence class containing the identity is a Boolean $\mathbf{SL}(2^m)$. Moreover, its homomorphic image modulo θ is isomorphic to L.

In view of this result, we may construct several distinct examples of subdirectly irreducible sloops that cannot be able to consrtuct by the well known constructions (cf. [1], [2], [3], [5], [8]).

2. Construction of subdirectly irreducible sloops of cardinality $n2^m$

Let $L = (L; *, 1)$ be an $\mathbf{SL}(n)$ and $B = (B; \bullet, 1)$ be a Boolean $\mathbf{SL}(2^m)$, where $L = \{1, x_1, x_2, \ldots, x_{n-1}\}$ and $B = \{1, a_1, a_2, \ldots, a_{2m-1}\}$. In this section we extend the sloop L to a subdireclty irreducible sloop $L \times_\alpha B$ of cardinality $n2^m$ having L as a homomorphic image.

We divide the set of elements of the direct product $L \times B$ into two subsets $\{1, x_1\} \times B$ and $\{x_2, \ldots, x_{n-1}\} \times B$. Consider the cyclic permutation $\alpha = (a_1 a_2 \ldots a_{2m-1})$ on the set $\{1, a_1, a_2, \ldots, a_{2m-1}\}$ and the characteristic function χ from the direct product $L \times B$ to B defined as follows

$$\chi((x,a),(y,b)) = \begin{cases} a \bullet \alpha^{-1}(a) & \text{for } x = 1, y = x_1, \\ b \bullet \alpha^{-1}(b) & \text{for } x = x_1, y = 1, \\ c \bullet \alpha(c) & \text{for } x = x_1 = y \text{ and } a \bullet b = c, \\ 1 & \text{otherwise.} \end{cases}$$

The last term means that $\chi((x,a),(y,b)) = 1$ when $x = y = 1$, $(x,a) \notin \{1, x_1\} \times B$ or $(y,b) \notin \{1, x_1\} \times B$.

Lemma 1. *The characteristic function χ has the following properties:*

(i) $\chi((x,a),(1,1)) = 1$;

(ii) $\chi((x,a),(x,a)) = 1$;

(iii) $\chi((x,a),(y,b)) = \chi((y,b),(x,a))$;

(iv) $\chi((x,a),(x * y, a \bullet b \bullet \chi((x,a),(y,b)))) = \chi((x,a),(y,b))$.

Proof. To prove (i), let $x = x_1$. Then $\chi((x_1,a),(1,1)) = 1 \bullet \alpha^{-1}(1) = 1$. Otherwise if $x \neq x_1$, then $\chi((x,a),(1,1)) = 1$.

Also in (ii), if $x = x_1$, then $\chi((x_1,a),(x_1,a)) = a \bullet a \bullet \alpha(a \bullet a) = 1$. Otherwise, if $x \neq x_1$, then $\chi((x,a),(x,a)) = 1$.

According to the definition of χ, we may deduce that $\chi((x,a),(y,b)) = \chi((y,b),(x,a))$ i.e., (iii) is also valid.

To prove the fourth property we consider four cases:

(1) If $x = x_1$ and $y = 1$, then

$$\begin{aligned} \chi((x_1,a),(x_1 * 1, a \bullet b \bullet \chi((x_1,a),(1,b)))) &= \chi((x_1,a),(x_1, a \bullet \alpha^{-1}(b)) \\ &= a \bullet a \bullet \alpha^{-1}(b)) \bullet \alpha(a \bullet a \bullet \alpha^{-1}(b)) \\ &= b \bullet \alpha^{-1}(b) = \chi((x_1,a),(1,b)). \end{aligned}$$

(2) If $x = 1$ and $y = x_1$, then

$$\begin{aligned} \chi((1,a),(1 * x_1, a \bullet b \bullet \chi((1,a),(x_1,b)))) &= \chi((1,a),(x_1, b \bullet \alpha^{-1}(a)) \\ &= a \bullet \alpha^{-1}(a) = \chi((1,a),(x_1,b)). \end{aligned}$$

(3) If $x = y = x_1$, then

$$\begin{aligned}\chi((x_1,a),(x_1*x_1,a\bullet b\bullet\chi((x_1,a),(x_1,b)))) &= \chi((x_1,a),(1,a\bullet b\bullet c\bullet\alpha(c))\\ &= \chi((x_1,a),(1,\alpha(c)) = c\bullet\alpha(c)\\ &= \chi((x_1,a),(x_1,b)) = 1.\end{aligned}$$

(4) Otherwise, when $x = y = 1$ or when (x,a) or $(y,b) \notin \{1,x_1\}\times B$, we have $\chi((x,a),(y,b)) = \chi(x,a),(x*y,a\bullet b\bullet\chi((x,a),(y,b)))) = 1$, because $\{x, x*y\} \nsubseteq \{1,x_1\}$. This completes the proof of the lemma. □

Lemma 2. *Let $L = (L;*,1)$ be an arbitrary $\mathbf{SL}(n)$, and $B = (B;\bullet,1)$ be a Boolean $\mathbf{SL}(2^m)$ for $m \geqslant 2$. Also let $\circ$ be a binary operation on the set $L\times B$ defined by:*

$$(x,a)\circ(y,b) := (x*y, a\bullet b\bullet\chi((x,a),(y,b))).$$

Then $L\times_a B = (L\times B;\circ,(1,1))$ is an $\mathbf{SL}(n2^m)$ for each possible number $n \geqslant 4$.

Proof. Let $L = \{1,x_1,x_2,\dots,x_{n-1}\}$ and $B = \{1,a_1,a_2,\dots,a_{2m-1}\}$. We note that the operation $\circ$ is the same operation of the direct product $L\times B$ for all elements $(x,a),(y,b)$ of the set $\{x_2,x_3,\dots,x_{n-1}\}\times B$. The difference occurs only if $x,y\in\{1,x_1\}$.

For all $(x,a),(y,b)\in L\times B$, we have:

(1) According to Lemma 1 (i)

$$(x,a)\circ(1,1) = (x*1, a\bullet 1\bullet\chi((x,a),(1,1))) = (x,a).$$

(2) By using Lemma 1 (ii)

$$(x,a)\circ(x,a) = (x*x, a\bullet a\bullet\chi((x,a),(x,a))) = (1,1).$$

(3) Using Lemma 1 (iii) we obtain:

$$\begin{aligned}(x,a)\circ(y,b) &= (x*y, a\bullet b\bullet\chi((x,a),(y,b)))\\ &= (y*x, b\bullet a\bullet\chi((y,b),(x,a)))\\ &= (y,b)\circ(x,a).\end{aligned}$$

(4) Lemma 1 (iv) gives:

$$\begin{aligned}&(x,a)\circ((x,a)\circ(y,b)) = (x,a)\circ(x*y, a\bullet b\bullet\chi((x,a),(y,b)))\\ &= (y, a\bullet a\bullet b\bullet\chi((x,a),(y,b))\bullet\chi((x,a),(x*y,a\bullet b\bullet\chi((x,a),(y,b)))))\\ &= (y,b).\end{aligned}$$

(1), (2), (3) and (4) imply that $L\times_\alpha B = (L\times B;\circ,(1,1))$ is a sloop. □

We note that $((x,a_i))\circ(x_1,a_j)\circ(x_1,a_k)\neq(x,a_i)\circ((x_1,a_j)\circ(x_1,a_k))$, for any $x\notin\{1,x_1\}$ and $a_j\neq a_k$, i.e., the operation $\circ$ is not associative even if the operation $*$ is associative.

In the next theorem we prove that the constructed $L\times_\alpha B$ is a subdirectly irreducible sloop having a monolith θ_1 satisfying that $|[(1,1)]\,\theta_1|=2^m$.

Theorem 3. *The constructed sloop* $L\times_\alpha B=(L\times B;\circ,(1,1))$ *is a subdirectly irreducible sloop.*

Proof. The projection $\Pi:(x,a)\to x$ from $L\times B$ into L is an onto homomorphism and the congruence Ker $\Pi:=\theta_1$ on $L\times_\alpha B$ is given by

$$\theta_1=\cup_{i=0}^{n-1}\left\{(x_i,1),(x_i,a_1),\ldots,(x_i,a_{2m-1})\right\}^2,$$

where $x_0=1$; so one can directly see that

$$[(1,1)]\theta_1=\left\{(1,1),(1,a_1),\ldots,(1,a_{2m-1})\right\}.$$

Now $\mathbf{Con}(L)\cong\ \mathbf{Con}((L\times_\alpha B)/\theta_1)\cong[\theta_1:1]$. Our proof will now be complete if we show that θ_1 is the unique atom of $\mathbf{Con}(L\times_\alpha B)$.

First, assume that θ_1 is not an atom of $\mathbf{Con}(L\times_\alpha B)$. Then we can find an atom γ such that $\gamma\subset\theta_1$ and $|[(1,1)]\,\gamma|=r<|[(1,1)]\,\theta_1|=2^m$. In this case we get a contradiction by proving that $[(1,1)]\,\gamma$ is not a normal subsloop of $L\times_\alpha B$.

Suppose that $[(1,1)]\gamma=\left\{(1,1),(1,a_{s_1}),(1,a_{s_2}),\ldots,(1,a_{s_{r-1}})\right\}$. We will prove that there are two elements $(x,a),(y,b)\in L\times B$ such that:

$$((x,a)\circ(y,b))\circ[(1,1)\gamma]\neq(x,a)\circ((y,b)\circ[(1,1)]\gamma).$$

If $\left\{a_{s_1},a_{s_2},\ldots,a_{s_{r-1}}\right\}$ is an increasing subsequence of $\left\{a_1,a_2,\ldots,a_{2m-1}\right\}$ and if $\alpha(a_{s_i})=a_{s_{i+1}}$ for all $i=1,2,\ldots,r-1$, then $\alpha(a_{s_{r-1}})=a_{s_r}\notin\left\{a_{s_1},a_{s_2},\ldots,a_{s_{r-1}}\right\}$. If $\left\{a_{s_1},a_{s_2},\ldots,a_{s_{r-1}}\right\}$ is increasing and not successive subsequence of $\left\{a_1,a_2,\ldots,a_{2m-1}\right\}$ then there exists an element $a_j\in\left\{a_{s_1},a_{s_2},\ldots,a_{s_{r-1}}\right\}$ such that $\alpha(a_j)=a_{j+1}\notin\left\{a_{s_1},a_{s_2},\ldots,a_{s_{r-1}}\right\}$. For both cases, we can always find an element $(1,a_k)\in[(1,1)]\,\gamma$ such that $(1,\alpha(a_k))\notin[(1,1)]\,\gamma$ ($a_k=a_{s_{r-1}}$ for the first case, and $a_k=a_j$ for the second case).

Consider the two elements (x_1,a_1) and (x_2,a_2) with $x_1\neq x_2\neq 1$, and assume that $((x_2,a_2)\circ(x_1,a_1))\circ[(1,1)]\,\gamma=(x_2,a_2)\circ((x_1,a_1)\circ[(1,1)]\,\gamma)$, then for the element $(1,a_k)$ (determined above) there exists an element $(1,a_{s_1})\in[(1,1)]\,\gamma$ such that

$$((x_2,a_2)\circ(x_1,a_1))\circ(1,a_k)=(x_2,a_2)\circ((x_1,a_1)\circ(1,a_{s_1})).$$

In this case $((x_2,a_2)\circ(x_1,a_1))\circ(1,a_k)=(x_2*x_1,a_2\bullet a_1)\circ(1,a_k)=(x_2*x_1,a_2\bullet a_1\bullet a_k)$ and $(x_2,a_2)\circ((x_1,a_1)\circ(1,a_{s_1}))=(x_2,a_2)\circ(x_1,a_1\bullet\alpha^{-1}(a_{s_1}))=(x_2*x_1,a_2\bullet a_1\bullet\alpha^{-1}(a_{s_1}))$ we obtain $a_k=\alpha^{-1}(a_{s_1})$, which implies $\alpha(a_k)=a_{s_t}$. This contradicts the assumption that $(1,\alpha(a_k))\notin[(1,1)]\gamma$. Hence, we may say that there is no atom γ of $\mathbf{Con}(L\times_\alpha B)$ satisfying $\gamma\subset\theta_1$. Therefore, θ_1 is an atom of the lattice $\mathbf{Con}(L\times_\alpha B)$.

Secondly, θ_1 is the unique atom of $\mathbf{Con}(L\times_\alpha B)$. Indeed, if δ is another atom of $\mathbf{Con}(L\times_\alpha B)$, then $\theta_1\cap\delta=0$. Hence, one can easily see that there is only one element $(x,a_1)\in[(x,a_1)]\delta$ with the first component x (note that $[(x,a_i)]\,\theta_1=\{(x,1),(x,a_1),\dots,(x,a_i),\dots,(x,a_{2m-1})\}$). For this reason we may say that the class $[(1,1)]\delta$ has at most one pair (x_1,a_i) with first component x_1. So we have two possibilities: either

(*i*) $[(1,1)]\delta$ contains only one pair (x_1,a_i) with first component x_1, or

(*ii*) $[(1,1)]\delta$ has no pairs with first component x_1.

For the first case, we choose two elements $(x,a)\&(x_1,a_s)\in L\times B$ such that $1\neq x\neq x_1$, and $a_s\neq a_i$ then

$$((x,a)\circ(x_1,a_s))\circ(x_1,a_i)=(x*x_1,a\bullet a_s)\circ(x_1,a_i)=(x,a\bullet a_s\bullet a_i).$$

Also,

$$(x,a)\circ((x_1,a_s)\circ(x_1,a_i))=(x,a)\circ(1,\alpha(a_s\bullet a_i))=(x,a\bullet\alpha(a_s\bullet a_i)).$$

Since the class $((x,a)\circ((x_1,a_s))\circ[(1,1)]\delta$ contains at most one element with a first component x, it follows that if $((x,a)\circ(x_1,a_s))\circ[(1,1)]\delta=(x,a)\circ((x_1,a_s)\circ[(1,1)]\delta)$, then $\alpha(a_s\bullet a_i)=a_s\bullet a_i$ hence $a_s\bullet a_i=1$, which contradicts the choice that $a_s\neq a_i$. This implies that $[(1,1)]\delta$ is not normal.

For the second case $[(1,1)]\delta$ has no pairs with first component x_1. Let $(x,a),(x,b)\in[(1,1)]\delta$ such that $1\neq x\neq x_1$, and $a\neq b$ Then

$$((x_1,c)\circ(x,a))\circ(x,b)=(x_1*x,c\bullet a)\circ(x,b)=(x_1,c\bullet a\bullet b).$$

Also,

$$(x_1,c)\circ((x,a)\circ(x,b))=(x_1,c)\circ(1,a\bullet b)=(x_1,c\bullet\alpha^{-1}(a\bullet b)).$$

By using the fact that the class $((x_1,c)\circ(x,a))\circ[(1,1)]\delta$ contains only one element with the first component x_1, we may say that if

$$((x_1,c)\circ(x,a))\circ[(1,1)]\delta=(x_1,c)\circ((x,a)\circ[(1,1)]\delta),$$

then $\alpha^{-1}(a \bullet b) = a \bullet b$, hence $a \bullet b = 1$, which contradicts that $a \neq b$. Thus $[(1,1)]\delta$ is not a normal subsloop of $L \times_\alpha B$. This mean that there is no another atom δ, and θ_1 is the unique atom of $\mathbf{Con}(L \times_\alpha B)$. Therefore, $L \times_\alpha B$ is a subdirectly irreducible sloop. □

Note that in the constructed sloop $L \times_\alpha B$, we may choose B a Boolean $\mathbf{SL}(2^m)$ for each $m \geqslant 2$. Therefore, as a consequence of the proof of Theorem 3, the following holds.

Corollary 4. *Let B be a Boolean $\mathbf{SL}(2^m)$ for an integer $m \geqslant 2$. Then the congruence class $[(1,1)]\theta_1$ of the monolith θ_1 of the constucted subdirectly irreducible sloop $L \times_\alpha B$ is a Boolean $\mathbf{SL}(2^m)$.*

Also, Theorem 3 enable us to construct a subdirectly irreducible sloop $L \times_\alpha B$ having a monolith θ_1 satisfying that $(L \times_\alpha B)/\ \theta_1 \cong L$. Then we have the following result.

Corollary 5. *Every sloop L is isomorphic to the homomorphic image of the subdirectly irreducible sloop $L \times_\alpha B$ over its monolith, for each Boolean sloop B.*

In view of these results, we may construct several distinct examples of subdirectly irreducible sloops.

The smallest non-trivial application of our construction is of cardinality 16. Indeed, if we choose two $\mathbf{SL}(4)$s, $L = (\{1, x_1, x_2, x_3\}; *, 1)$ and $B = (\{1, a, b, c\}; \bullet, 1)$, then the constructed sloop $L \times_\alpha B$ is a subdirectly irreducible $\mathbf{SL}(16)$ having 3 normal sub-$\mathbf{SL}(8)$s:

$$\mathbf{S}_1 = \{(1,1), (1,a), (1,b), (1,c), (x_1,1), (x_1,a), (x_1,b), (x_1,c)\}\,,$$

$$\mathbf{S}_2 = \{(1,1), (1,a), (1,b), (1,c), (x_2,1), (x_2,a), (x_2,b), (x_2,c)\} \text{ and}$$

$$\mathbf{S}_3 = \{(1,1), (1,a), (1,b), (1,c), (x_3,1), (x_3,a), (x_3,b), (x_3,c)\}\,.$$

The constructed $\mathbf{SL}(16)$ corresponds to an $\mathbf{STS}(15)$ having 3 sub-$\mathbf{STS}(7)$s.

In the classification of all subdirectly irreducible $\mathbf{SL}(32)$ given in [5] there are two classes having a monolith θ_1 satisfying $|[(1,1)]\theta_1| = 4$ and 8. The well-known constructions for subdirectly irreducible sloops given in [1], [2], [3], [5], [8] dose not enable us to construct examples for these classes.

In the following example we apply our construction to describe subdirectly irreducible $\mathbf{SL}(32)$ having a monolith θ_1 satisfying $|[(1,1)]\theta_1| = 4$ (or 8).

Example. Let L be the Boolean $\mathbf{SL}(8)$ (or $\mathbf{SL}(4)$), B be the Boolean $\mathbf{SL}(4)$ (or $\mathbf{SL}(8)$) and α be the cyclic permutation on the non-unit elements of B. By apply our construction $L \times_\alpha B$, we get a subdirectly irreducible $\mathbf{SL}(32)$ having a monolith θ_1 satisfying $(L \times_\alpha B)/\theta_1 \cong L \cong \mathbf{SL}(8)$ (or $\mathbf{SL}(4)$) in which its monolith θ_1 satisfying $|[(1,1)]\theta_1| = 4$ (or 8).

This example of an $\mathbf{SL}(32)$ corresponds to a subdirectly irreducible $\mathbf{SL}(32)$ having exactly 7 normal sub-$\mathbf{SL}(16)$s. (or 3 normal sub-$\mathbf{SL}(16)$s).

Similarly, we can use our construction to give an example for a subdirectly irreducible $\mathbf{SL}(n\mathbf{2}^m)$ having a monolith θ_1 satisfying $|[(1,1)]\theta_1| = 2^m$ for each possible $n \geqslant 4$ and each integer $m \geqslant 2$.

References

[1] **M. H. Armanious**, *Construction of nilpotent sloops of class n*, Discrete Math. **171** (1997), 17 – 25.

[2] **M. H. Armanious**, *Nilpotent SQS-skeins with nilpotent derived sloops*, Ars Combin. **56** (2000), 193 – 200.

[3] **M. H. Armanious**, *On subdirectly irreducible Steiner loops of cardinality* $2n$, Beitrage zur Algebra and Geometrie **43** (2002), 325 – 331.

[4] **M. H. Armanious and E. M. Elzayat**, *Extending sloops of cardinality* 16 *to SQS-skeins with all possible congruence lattices*, Quasigroups and Related Systems **12** (2004), 1 – 12.

[5] **M. H. Armanious and E. M. Elzayat**, *Subdirectly irreducible sloops and SQS-skeins*, Quasigroups and Reelated Systems **15** (2007), 233 – 250.

[6] **O. Chein, H. O. Pflugfelder, and J. D. H. Smith**, *Quasigroups and loops*, Theory and Applications, Sigma Series in Pure Math., Heldermann Verlag, Berlin, 1990.

[7] **B. Ganter and H. Werner**, *Co-ordinatizing Steiner systems*, Ann. Discrete Math. **7** (1980), 3 – 24.

[8] **A. J. Guelzow**, *The structure of nilpotent Steiner quadruple systems*, J. Comb. Designs **1** (1993), 301 – 321.

[9] **C. C. Lindner and A. Rosa**, *Steiner quadruple system*, Discrete Math. **21** (1978), 147 – 181.

[10] **R. W. Quackenbush**, *Varieties of Steiner loops and Steiner quasigroups*, Canada J. Math. **28** (1976), 11087 – 11098.

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