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Generalized quadratic quasigroup equations with three variables

Aleksandar Krapež

Abstract. F. M. Sokhats'kyi recently posed the problem of classification of (un)cancellable generalized quadratic quasigroup equations. Refining relevant results of S. Krstić, A. Krapež and D. Živković solved this problem by reducing it to the classification of connected (3–connected) cubic graphs. They also started systematic investigation by solving all equations with two variables. Here we consider equations with exactly three variables. There are 330 of them and they split into five classes of parastrophic equivalence. We give solutions to five representative equations, one from each class.

1. Introduction

This paper is a sequel of [7] by A. Krapež and D. Živković. Although we define the most important notions and state essential results of [7], it is assumed that the reader is thoroughly familiar with it.

In [7] authors consider the correspondence between generalized quadratic quasigroup functional equations and connected cubic graphs as established by S. Krstić in his PhD thesis [8]. It is proved that the important notion of parastrophic equivalence of quadratic equations corresponds to the isomorphism of graphs obtained from given equations. The set of nine normal equations with two variables is divided into two classes of parastrophic equivalence corresponding to two nonisomorphic graphs.

Much more is known about the special case of parastrophically uncancellable equations. Various authors obtained instances of number u_n of classes of parastrophically nonequivalent uncancellable equations with small number n of variables. The results are: $u_2 = 1$ (Krapež and Živković [7]),

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$u_3 = 1$ (Sokhats'kyi [10] after Duplák [4]), $u_4 = 2$ (Sokhats'kyi [10]), $u_5 = 4$ (Koval' [5]) and $u_6 = 14$ (Krapež, Simić and Tošić [6]).

In this paper we prove that there are five classes of cancellable and uncancellable equations with three variables, we give their Krstić graphs and solve five parastrophically nonequivalent representative equations.

2. Quasigroups and functional equations

Let us recall necessary definitions and results of [7]. For bare essentials on quasigroups see [7]. More can be found in standard references V. D. Belousov [2], O. Chein, H. O. Pflugfelder and J. D. H. Smith [3] and H. O. Pflugfelder [9]. We just state that the language of quasigroups contains six binary operations: multiplication ($\cdot$), left ($\backslash$) and right ($/$) division and their respective dual operations: $*$ (dual of $\cdot$), $\backslash\!\backslash$ (dual of $\backslash$) and $/\!\!/$ (dual of $/$). These six operations are known as parastrophes of $\cdot$ (and of each other) and the connection between them is: $xy = z$ iff $x\backslash z = y$ iff $z/y = x$ iff $y * x = z$ iff $z\backslash\!\backslash x = y$ iff $y/\!\!/z = x$.

When we use prefix notation for operations and a quasigroup operation is A, we define: $A(x_1, x_2) = x_3$ iff $A^{(1)}(x_1, x_2) = x_3$ iff $A^{(12)}(x_2, x_1) = x_3$ iff $A^{(13)}(x_3, x_2) = x_1$ iff $A^{(23)}(x_1, x_3) = x_2$ iff $A^{(123)}(x_2, x_3) = x_1$ iff $A^{(132)}(x_3, x_1) = x_2$. In general, $A(x_1, x_2) = x_3$ iff $A^{\sigma}(x_{\sigma(1)}, x_{\sigma(2)}) = x_{\sigma(3)}$ for $\sigma \in S_3$.

We assume that all operations are quasigroups. Further:

Definition 2.1. Functional equation $s = t$ is *quadratic* if every object variable appears exactly twice in $s = t$.

Definition 2.2. Functional equation $s = t$ is *generalized* if every functional variable F (including all parastrophes of F) appears only once in $s = t$.

We also need the following:

Definition 2.3. Let $Eq[F_1, \dots, F_n]$ be a generalized quadratic functional equation on quasigroups. We write $F_i \sim F_j$ $(1 \leqslant i, j \leqslant n)$ and say that F_i and F_j are *necessarily isostrophic* if in every solution $Q_1, \dots, Q_n$ of Eq the operations Q_i and Q_j are isostrophic.

A functional variable F_i is *loop, group, abelian* if Q_i is isostrophic to a loop, group, abelian group respectively.

Definition 2.4. A $\sim$–class with one or two elements is called *small*, otherwise it is *big*.

Definition 2.5. Two equations Eq and Eq' are *parastrophically equivalent* ($Eq\,\mathrm{PE}\,Eq'$) if one of them can be obtained from the other by applying a finite number of the following steps:

1. Renaming object and/or functional variables.
2. Replacing $s = t$ by $t = s$.
3. Replacing equation $A(t_1, t_2) = t_3$ by one of the following equations: $A^\sigma(t_{\sigma(1)}, t_{\sigma(2)}) = t_{\sigma(3)}$ for some $\sigma \in S_3$.
4. Replacing a subterm $A(t_1, t_2)$ of s or t by $A^{(12)}(t_2, t_1)$.
5. Replacing a subterm $A(x, t_2)$ by a new variable y and simultaneously replacing all other occurrences of x by either $A^{(13)}(y, t_2)$ or $A^{(123)}(t_2, y)$.
6. Replacing a subterm $A(t_1, x)$ by a new variable y and simultaneously replacing all other occurrences of x by either $A^{(23)}(t_1, y)$ or $A^{(132)}(y, t_1)$.

If we use notation $Eq[\ldots, A, \ldots]$, we denote by $Eq'[\ldots, A^\sigma, \ldots]$ the equation obtained by one of the steps $(3) - (7)$ above, always preserving the order of other functional variables. Using this convention we get:

Theorem 2.6 (Krstić [8]) **.** *If equations $Eq[F_1, \ldots, F_n]$ and $Eq'[G_1, \ldots, G_n]$ are parastrophically equivalent and $Q_1, \ldots, Q_n$ and $R_1, \ldots, R_n$ are solutions of respectively Eq, Eq' on a set S, then the operations Q_i and R_i* $(1 \leqslant i \leqslant n)$ *are mutually isostrophic.*

3. Graphs and functional equations

Following S. Krstić [8] we represent functional equations by graphs. These 'graphs' may have loops and multiple edges between two vertices and are technically known as *multigraphs*.

We define graphs as relation systems $(V, E; I)$ with $I \subseteq V \times E$. It is assumed that the sets V of vertices and E of edges are disjoint and that for every edge e there are at most two vertices incident to e. A *loop* is an edge with a unique vertex incident to it. A loop in a graph should not be confused with a loop as a quasigroup with an identity.

A graph is *cubic* if for every vertex v there are exactly three edges to which v is incident, provided that if edge is a loop it is counted twice.

Definition 3.1. Two vertices v_1, v_2 of a graph G are 3*–connected* (and we write $v_1 \equiv v_2$) if there are three disjoint paths in G from v_1 to v_2. A graph G is 3*–connected* if all vertices of G are 3–connected.

In graph theory, 3–connectedness, as defined above, is usually called 3–edge–connectedness, but we shortened it to 3–connectedness. A graph is 3–connected iff removal of any two edges does not disconnect it. Obviously, a cubic graph G is 3–connected iff the relation $\equiv$ is a full relation on V.

Definition 3.2. A $\equiv$–class with one or two elements is called *small*, otherwise it is *big*.

Based on the theory of S. Krstić [8], two constructioins are presented in [7] - the one which produces the graph $\mathrm{K}(Eq)$ for a given generalized quadratic functional equation Eq and the other, which gives an equation $\mathrm{QE}(G)$ for a given finite connected cubic graph G.

We have:

Theorem 3.3 (Krapež and Živković [7] after Krstić [8]). *Generalized quadratic quasigroup functional equations Eq and Eq' are parastrophically equivalent iff their Krstić graphs $\mathrm{K}(Eq)$ and $\mathrm{K}(Eq')$ are isomorphic.*

The following theorem is also important.

Theorem 3.4 (Krstić [8]). *Let $Eq[F_1, \ldots, F_n]$ be a generalized quadratic functional equation. Then $F_i \sim F_j$ in Eq iff $F_i \equiv F_j$ in $\mathrm{K}(Eq)$. Moreover:*

Every F_i is a loop functional variable.

A symbol F_i is a group functional variable iff $F_i/\equiv$ is big iff K_4 is homeomorphically embeddable in $\mathrm{K}(Eq)$ within $F_i/\equiv$.

A symbol F_i is an abelian functional variable iff the subgraph of $\mathrm{K}(Eq)$ defined by $F_i/\equiv$ is not planar iff $K_{3,3}$ is homeomorphically embeddable in $\mathrm{K}(Eq)$ within $F_i/\equiv$.

4. Equations with three variables

In the paper [7] A. Krapež and D. Živković defined sequences (E_n), (e_n) and $(\pi_n)(n \geq 1)$, where E_n is the number of generalized quadratic quasigroup functional equations with n variables, e_n is the number of normal equations among them and π_n is the number of classes of parastrophically equivalent equations with n variables. By the Theorem 5.9 of [7] π_n is also the number of nonisomorphic cubic graphs with $2(n-1)$ vertices. We have $E_3 = 3780$ and $e_3 = 330$. It is announced that $\pi_3 = 5$. We give the proof of this fact now but also a new proof that $\pi_2 = 2$.

By the Lemma 5.2 of [7], equations with 2, 3 variables have Krstić graphs which are connected, cubic and have 2, 4 vertices and 3, 6 edges respectively.

Figure 1. Graphs with two vertices

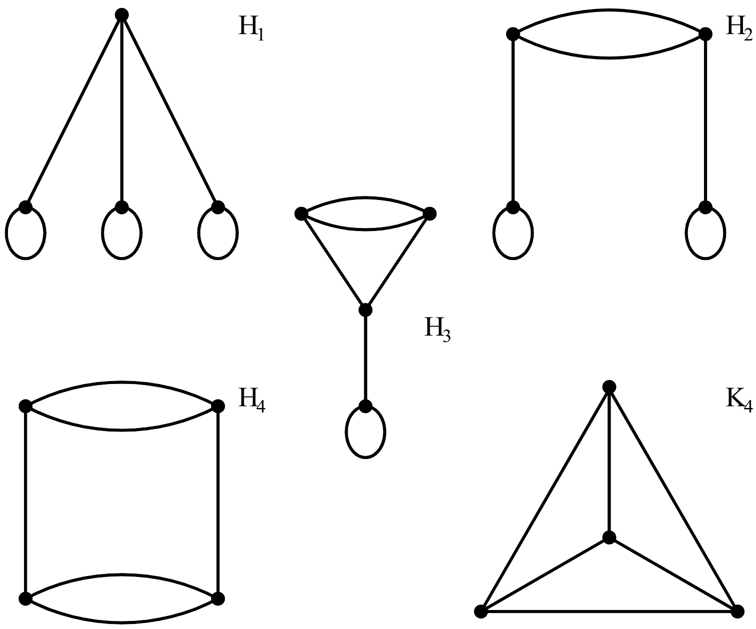

Figure 2. Graphs with four vertices

Theorem 4.1. *Every connected cubic graph with two vertices is isomorphic to either the dumbbell graph H_0 or to the dipole graph D_3 (Figure 1). Every connected cubic graph with four vertices is isomorphic to either one of: H_1, H_2, H_3, H_4, K_4 (Figure 2). Consequently, $\pi_2 = 2$ and $\pi_3 = 5$.*

Proof. Let G be a connected cubic graph with either two or four vertices. There are four possibilities:

(1) G has a loop,
(2) G has no loops but has a triple edge,
(3) G has no loops or triple edges but has a double edge,
(4) G has no loops or multiple edges.

(1) G has a loop. Then there is a vertex, say 1, with the loop. Since G is cubic, there is another edge in 1 connecting it to a new vertex 2. There are three possibilities:

(11) the vertex 2 has a loop,

(12) 2 has no loop but has a double edge,

(13) 2 has no loops or double edges.

(11) The vertex 2 has a loop. Since G is cubic and connected, no further extension is possible. Therefore G is isomorphic to the dumbbell graph H_0.

(12) The vertex 2 has no loop but has a double edge. Let the vertex 2 connects to the vertex 3 by the double edge. The single remaining edge at 3 has to connect it to the new vertex 4. All vertices except 4 now have three edges. Therefore 4 has to connect to itself by the loop. The graph G is isomorphic to H_2.

(13) The vertex 2 has no loops or double edges. Therefore 2 has to connect to two more vertices 3 and 4 by single edges. There are two possibilities:

(131) there is a loop in 3,

(132) there is no loop in 3.

(131) There is a loop in 3. There must be a loop in 4 as well and G is isomorphic to H_1.

(132) There is no loop in 3. Then 3 and 4 must be connected by the double edge. The graph G is isomorphic to H_3.

(2) G has no loops but has a triple edge. Then two vertices 1 and 2 are triply connected and no further extension is possible. The graph G is isomorphic to the dipole graph D_3.

(3) G has no loops or triple edges but has a double edge. Assume that the vertex 1 has a double edge to the vertex 2 and consequently a single edge to another vrtex 3. There are two possibilities:

(31) there is an edge connecting vertices 2 and 3,

(32) there is no such edge.

(31) There is an edge connecting vertices 2 and 3. The edge must be a single one since 2 is connected to 1 by the double edge. Then 3 must be connected to the only remaining vertex 4 by the single edge. But then the vertex 4 must have a loop which contradicts assumption (3).

(32) There is no edge connecting 2 and 3. Since no loops are alowed, 2 must be singly and 3 doubly connected to 4. The graph G is isomorphic to H_4.

(4) G has no loops or multiple edges. Therefore 1 is singly connected to 2, 3 and 4. Since no loops or multiple edges are alowed, 2 must connect to both 3 and 4. Also, the 3 and 4 are connected and the graph G is isomorphic to the graph K_4. $\square$

We prove four usefull lemmas. They generalize Lemmas 8.1–8.4 from [7].

Lemma 4.2. *Let a, b and e be elements and σ a permutation of a set S. A general solution to the equation*

$$\sigma F(a,b) = e \tag{1}$$

on a set S is given by:

$$F(x,y) = \alpha L(\lambda x, \varrho y)$$

where:

- *L is an arbitrary loop on S with the identity e,*
- *α, λ and ϱ are arbitrary permutations of S such that: $\alpha = \sigma^{-1}$, $\lambda a = e$ and $\varrho b = e$.*

Proof. It is trivial to check that the above formulas always give a solution to the equation (1). Next, we prove that every solution to the equation (1) is of the form given in the statement of the Lemma.

Let F be a particular quasigroup on S which satisfies (1). Define $\alpha = \sigma^{-1}, \lambda x = \sigma F(x,b), \varrho x = \sigma F(a,x)$ and $L(x,y) = \sigma F(\lambda^{-1}x, \varrho^{-1}y)$. We see that λ and ϱ are permutations of S such that $\lambda a = \varrho b = e$ and $F(x,y) = \alpha L(\lambda x, \varrho y)$. The operation L is a quasigroup as an isotope of the quasigroup F. Moreover, it is a loop, as follows from: $L(e,x) = \sigma F(\lambda^{-1}e, \varrho^{-1}x) = \sigma F(a, \varrho^{-1}x) = \varrho\varrho^{-1}x = x$ and $L(x,e) = \sigma F(\lambda^{-1}x, \varrho^{-1}e) = \sigma F(\lambda^{-1}x, b) = \lambda\lambda^{-1}x = x$. □

Lemma 4.3. *Let b be an element and γ, σ and τ permutations of a set S. A general solution to the equation*

$$\sigma F(\gamma x, b) = \tau x \tag{2}$$

on a set S is given by:

$$F(x,y) = \alpha L(\lambda x, \varrho y)$$

where:

- *L is an arbitrary loop on S with the identity e,*
- *α, λ and ϱ are arbitrary permutations of S such that: $\alpha = \sigma^{-1}$, $\lambda\gamma = \tau$ and $\varrho b = e$.*

Proof. It is easy to check that the above formulas always give a solution to the equation (2).

Assume that a quasigroup F is a solution of (2). We are proving that F must be of the form indicated in the statement of the Theorem.

Take $a \in S$ and define $e = \tau a, \alpha = \sigma^{-1}, \lambda x = \sigma F(x, b), \varrho x = \sigma F(\gamma a, x)$. Operations α, λ and ϱ are permutations such that $\lambda\gamma x = \sigma F(\gamma x, b) = \tau x$ and $\varrho b = \sigma F(\gamma a, b) = \tau a = e$.

Define a quasigroup L by $L(u, v) = \alpha^{-1} F(\lambda^{-1} u, \varrho^{-1} v)$. We have $L(e, x) = \alpha^{-1} F(\lambda^{-1} e, \varrho^{-1} x) = \sigma F(\gamma\tau^{-1} e, \varrho^{-1} x) = \sigma F(\gamma a, \varrho^{-1} x) = \varrho\varrho^{-1} x = x$ and $L(x, e) = \alpha^{-1} F(\lambda^{-1} x, \varrho^{-1} e) = \sigma F(\lambda^{-1} x, b) = \lambda\lambda^{-1} x = x$ proving that L is a loop with the identity e. □

By duality we have:

Lemma 4.4. *Let a be an element and δ, σ and τ permutations of a set S. A general solution to the equation*

$$\sigma F(a, \delta x) = \tau x \tag{3}$$

on a set S is given by:

$$F(x, y) = \alpha L(\lambda x, \varrho y)$$

where:

- *L is an arbitrary loop on S with the identity e,*
- *α, λ and ϱ are arbitrary permutations of S such that: $\alpha = \sigma^{-1}$, $\lambda a = e$ and $\varrho\delta = \tau$.*

Lemma 4.5. *Let e be an element and γ, δ and σ permutations of a set S. A general solution to the equation*

$$\sigma F(\gamma x, \delta x) = e \tag{4}$$

on a set S is given by:

$$F(x, y) = \alpha L^{(23)}(\lambda x, \varrho y)$$

where:

- *L is an arbitrary loop on S with the identity e,*
- *α, λ and ϱ are arbitrary permutations of S such that: $\sigma\alpha e = e$, $\lambda\gamma = \sigma$ and $\varrho\delta = \sigma$.*

Proof. Since L is a loop, we have $L(x, e) = x$ i.e., $L^{-2}(x, x) = e$. Therefore $\sigma F(\gamma x, \delta x) = \sigma\alpha L^{-2}(\lambda\gamma x, \varrho\delta x) = \sigma\alpha L^{-2}(\sigma x, \sigma x) = \sigma\alpha e = e$ so F satisfies (4).

Assume that a quasigroup F is a particular solution of (4). Define $\alpha x = F(\gamma\sigma^{-1} e, \delta\sigma^{-1} x)$. The function α is a permutation and $\sigma\alpha e = \sigma F(\gamma\sigma^{-1} e, \delta\sigma^{-1} e) = e$.

Define also $\lambda = \sigma\gamma^{-1}$ and $\varrho = \sigma\delta^{-1}$. It follows that $\lambda\gamma = \sigma$ and $\varrho\delta = \sigma$.

If a quasigroup L is defined by $L(u, v) = \varrho F^{-2}(\lambda^{-1} u, \alpha v)$ then $F(x, y) = \alpha L^{-2}(\lambda x, \varrho y)$, $L(e, x) = \varrho F^{-2}(\lambda^{-1} e, \alpha x) = \varrho F^{-2}(\lambda^{-1} e, F(\lambda^{-1} e, \varrho^{-1} x)) =$

$\varrho\varrho^{-1}x = x$ and $L(x,e) = \varrho F^{-2}(\lambda^{-1}x, \alpha e) = \varrho F^{-2}(\lambda^{-1}x, F(\lambda^{-1}x, \varrho^{-1}x)) = \varrho\varrho^{-1}x = x$. Therefore L is a loop. □

There are three equations corresponding to the graph H_1:

$$A(B(x,x), C(y,y)) = D(z,z),$$

$$A(x, B(C(y,y), D(z,z))) = x, \qquad A(B(C(x,x), D(y,y)), z) = z.$$

To reduce some space we shall not write appropriate generalized equations, as above, but the corresponding equations in the language with the single operation $\cdot$. So the three equations representing generalized equations which correspond to the graph H_1 are:

$$\boxed{xx.yy = zz} \qquad x(yy.zz) = x \qquad (xx.yy)z = z$$

One of the equations is also boxed, indicating the equation chosen to represent the whole PE–class. The distinguished equation is then written in full form and its solution is given in the following theorem. In this case the representative equation is:

$$A(B(x,x), C(y,y)) = D(z,z) \tag{5}$$

and the corresponding theorem is:

Theorem 4.6. *A general solution of the equation* (5) *on a set* S *is given by:*

$$\begin{cases} A(x,y) = L(\lambda x, \varrho y) \\ B(x,y) = \lambda^{-1}U_2(x,y) \\ C(x,y) = \varrho^{-1}U_3(x,y) \\ D(x,y) = U_4(x,y) \end{cases}$$

where:
- L *is an arbitrary loop on* S *with an identity* e,
- U_i $(2 \leqslant i \leqslant 4)$ *are arbitrary unipotent quasigroups with a common idempotent* e,
- λ *and* ϱ *are arbitrary permutations of* S.

Proof. 1) Let quasigroups A, B, C, D be given by the formulas above. Then $A(B(x,x), C(y,y)) = L(B(x,x), C(y,y)) = L(\lambda\lambda^{-1}U_2(x,x), \varrho\varrho^{-1}U_3(y,y)) = L(e,e) = e = U_4(z,z) = D(z,z)$. Therefore a quadruple of such quasigroups is a solution to (5).

2) Let a quadruple of quasigroups A, B, C, D be a solution to (5). Assume p, q be arbitrary but fixed elements from S. Define $b = B(p,p), c = C(q,q)$ and $e = A(b,c)$. Fixing x and y in the equation yields $D(z,z) = e$. We can easily infer $B(x,x) = b$ and $C(y,y) = c$. From $A(b,c) = e$, by the Lemma 4.2, we find $A(x,y) = \alpha L(\lambda x, \varrho y)$ where L is a loop on S with an identity e and α, λ, ϱ are permutations of S such that $\alpha = \mathrm{Id}$ and $\lambda b = \varrho c = e$. But then $\lambda B(x,x) = \lambda b = e$ and if we define $U_2(x,y) = \lambda B(x,y)$ we get $B(x,y) = \lambda^{-1} U_2(x,y)$. We can define U_3 and U_4 similarly. □

The $\sim$–classes of (5) are all singletons.

The question arises as to why we use unipotent quasigroups to express the solutions to functional equations when the Theorem 3.4 stresses the role of loops, groups and/or Abelian groups. The reason is pure convenience since we could use loops instead of unipotent quasigroups. Namely, by the Lemma 4.5, for every unipotent quasigroup U the quasigroup $U^{(23)}$ with the (unique) idempotent e is a loop with the identity e, and conversely, if L is a loop with the identity e, then the quasigroup $L^{(23)}$ is unipotent and has a unique idempotent e. The alternative general solution to (5) is then given in:

Theorem 4.7. *A general solution of the equation* (5) *on a set S is given by:*

$$\begin{cases} A(x,y) = L(\lambda x, \varrho y) \\ B(x,y) = \lambda^{-1} L_2^{(23)}(x,y) \\ C(x,y) = \varrho^{-1} L_3^{(23)}(x,y) \\ D(x,y) = L_4^{(23)}(x,y) \end{cases}$$

where:

– *L, L_2, L_3 and L_4 are arbitrary loops on S with an identity e,*
– *λ and ϱ are arbitrary permutations of S.*

Further on, we state only one version of the solution, the one using unipotent quasigroups.

There are 19 equations corresponding to the graph H_2:

$\boxed{x.yy = x.zz}$	$xx.y = y.zz$	$xx.y = zz.y$
$x(x.yy) = zz$	$x(yy.x) = zz$	$(x.yy)x = zz$
$(xx.y)y = zz$	$x.y(y.zz) = x$	$x.y(zz.y) = x$
$x.(y.zz)y = x$	$x.(yy.z)z = x$	$xx.(y.zz) = y$
$xx.(yy.z) = z$	$(x.yy).zz = x$	$(xx.y).zz = y$
$x(x.yy).z = z$	$x(yy.x).z = z$	$(x.yy)x.z = z$
	$(xx.y)y.z = z$	

The $\sim$–classes of operations are again singletons. The representative equation is

$$A(x, B(y, y)) = C(x, D(z, z)) \tag{6}$$

and its solution is given in the following theorem.

Theorem 4.8. *A general solution of the equation* (6) *on a set* S *is given by:*

$$\begin{cases} A(x,y) = L_1(\lambda_1 x, \varrho_1 y) \\ B(x,y) = \varrho_1^{-1} U_2(x,y) \\ C(x,y) = L_3(\lambda_3 x, \varrho_3 y) \\ D(x,y) = \varrho_3^{-1} U_4(x,y) \end{cases}$$

where:

- L_1 *and* L_3 *are arbitrary loops on* S *with a common identity* e*,*
- U_2 *and* U_4 *are arbitrary unipotent quasigroups on* S *with a common idempotent* e*,*
- $\lambda_1, \varrho_1, \lambda_3$ *and* ϱ_3 *are arbitrary permutations of* S *such that* $\lambda_1 = \lambda_3$*.*

Proof. 1) Let quasigroups A, B, C, D be given by the formulas above. Then $A(x, B(y, y)) = L_1(\lambda_1 x, \varrho_1 \varrho_1^{-1} U_2(y, y)) = L_1(\lambda_1 x, e) = \lambda_1 x = \lambda_3 x = L_3(\lambda_3 x, e) = L_3(\lambda_3 x, \varrho_3 \varrho_3^{-1} U_4(z, z)) = C(x, D(z, z))$. Therefore the quadruple of such quasigroups is a solution to (6).

2) Let a quadruple of quasigroups A, B, C, D be a solution to (6). Suppose that p, q, r are arbitrary but fixed elements from S and define $b = B(q, q), e = A(p, b), d = D(r, r)$. Fixing x and y we get $C(p, d) = e$.

Define also $\lambda_1 x = A(x, b), \varrho_1 x = A(p, x), \lambda_3 x = C(x, d)$ and $\varrho_3 x = C(p, x)$. The relation $\lambda_1 = \lambda_3$ immediately follows. The equation (6) reduces to the system: $A(x, b) = \lambda_3 x$, $\varrho_1 B(y, y) = e$, $C(x, d) = \lambda_1 x$, $\varrho_3 D(z, z) = e$.

By the Lemma 4.3 we can choose $A(x, y) = L_1(\lambda_1 x, \varrho_1 y)$ for some L_1. It is rather obvious that we have to take $L_1(u, v) = A(\lambda_1^{-1} u, \varrho_1^{-1} v)$. Since λ_1 and ϱ_1 are translations of A, the operation L_1 must be a loop with the identity e. Analogously, $C(x, y) = L_3(\lambda_3 x, \varrho_3 y)$ for a suitable loop L_3 with the identity e.

If we define $U_2(x, y) = \varrho_1 B(x, y)$ we get $U_2(x, x) = \varrho_1 b = e$ and $B(x, y) = \varrho_1^{-1} U_2(x, y)$. Similarly, $D(x, y) = \varrho_3^{-1} U_4(x, y)$ for a unipotent quasigroup U_4 with the idempotent e. $\square$

There are 94 equations corresponding to the graph H_3.

$\boxed{x.xy = y.zz}$	$x.yx = y.zz$	$x.yy = z.xz$
$x.yy = z.zx$	$xx.y = z.yz$	$xx.y = z.zy$
$xy.x = y.zz$	$xy.y = x.zz$	$xx.y = yz.z$
$xx.y = zy.z$	$xy.x = zz.y$	$xy.y = zz.x$
$x(y.xy) = zz$	$x(y.yx) = zz$	$x(y.zz) = xy$
$x(y.zz) = yx$	$x(xy.y) = zz$	$x(yx.y) = zz$
$x(yy.z) = xz$	$x(yy.z) = zx$	$xx.yz = yz$
$xx.yz = zy$	$xy.xy = zz$	$xy.yx = zz$
$xy.zz = xy$	$xy.zz = yx$	$(x.xy)y = zz$
$(x.yx)y = zz$	$(x.yy)z = xz$	$(x.yy)z = zx$
$(xx.y)z = yz$	$(xx.y)z = zy$	$(xy.x)y = zz$
$(xy.y)x = zz$	$x.x(y.zz) = y$	$x.y(x.zz) = y$
$x.y(z.yz) = x$	$x.y(z.zy) = x$	$x.x(yy.z) = z$
$x.y(yz.z) = x$	$x.y(zy.z) = x$	$x.y(zz.x) = y$
$x(xy.zz) = y$	$x(yx.zz) = y$	$x(yy.xz) = z$
$x(yy.zx) = z$	$x(yz.yz) = x$	$x(yz.zy) = x$
$x.(x.yy)z = z$	$x.(y.yz)z = x$	$x.(y.zy)z = x$
$x.(y.zz)x = y$	$x.(yy.x)z = z$	$x.(yy.z)x = z$
$x.(yz.y)z = x$	$x.(yz.z)y = x$	$xx.(y.yz) = z$
$xx.(y.zy) = z$	$xy.(x.zz) = y$	$xy.(y.zz) = x$
$xx.(yz.y) = z$	$xx.(yz.z) = y$	$xy.(zz.x) = y$
$xy.(zz.y) = x$	$(x.xy).zz = y$	$(x.yx).zz = y$
$(x.yy).xz = z$	$(x.yy).zx = z$	$(xx.y).yz = z$
$(xx.y).zy = z$	$(xy.x).zz = y$	$(xy.y).zz = x$
$x(y.xy).z = z$	$x(y.yx).z = z$	$x(y.zz).x = y$
$x(y.zz).y = x$	$x(xy.y).z = z$	$x(yx.y).z = z$
$x(yy.z).x = z$	$x(yy.z).z = x$	$(xx.yz)y = z$
$(xx.yz)z = y$	$(xy.xy)z = z$	$(xy.yx)z = z$
$(xy.zz)x = y$	$(xy.zz)y = x$	$(x.xy)y.z = z$
$(x.yx)y.z = z$	$(x.yy)z.x = z$	$(x.yy)z.z = x$
$(xx.y)z.y = z$	$(xx.y)z.z = y$	$(xy.x)y.z = z$
	$(xy.y)x.z = z$	

In this case we have two $\sim$–classes which are singletons and one class with two elements. The representative equation is

$$A(x, B(x, y)) = C(y, D(z, z)) \tag{7}$$

and its solution is given in the following theorem.

Theorem 4.9. *A general solution of the equation* (7) *on a set* S *is given by:*

$$\begin{cases} A(x,y) = L_1^{(23)}(\lambda_1 x, \varrho_1 y) \\ B(x,y) = \varrho_1^{-1} L_1(\lambda_2 x, \varrho_2 y) \\ C(x,y) = L_3(\lambda_3 x, \varrho_3 y) \\ D(x,y) = \varrho_3^{-1} U(x,y) \end{cases}$$

where:

- L_1 *and* L_3 *are arbitrary loops on* S *with a common identity* e*,*
- U *is an unipotent quasigroup with the idempotent* e*,*
- $\lambda_1, \varrho_1, \lambda_2, \varrho_2, \lambda_3, \varrho_3$ *are arbitrary permutations of* S *such that* $\lambda_1 = \lambda_2$ *and* $\varrho_2 = \lambda_3$.

Proof. 1) Let quasigroups A, B, C, D be given by the formulas above. Then $A(x, B(x,y)) = L_1^{(23)}(\lambda_1 x, \varrho\varrho^{-1} L_1(\lambda_2 x, \varrho_2 y)) = L_1^{(23)}(\lambda_2 x, L_1(\lambda_2 x, \varrho_2 y)) = \varrho_2 y = \lambda_3 y = L_3(\lambda_3 y, e) = L_3(\lambda_3 y, \varrho_3 \varrho_3^{-1} U(z,z)) = C(y, D(z,z))$. Therefore the quadruple of such quasigroups is a solution to (7).

2) Let a quadruple of particular quasigroups A, B, C, D be a solution to (7). Let p, q, r be arbitrary but fixed elements from S. Define $b = B(p,q), d = D(r,r)$ and $e = A(p,b)$. Fixing x and y in the equation yields $C(q,d) = e$. Define $A_2 x = A(p,x), B_1 x = B(x,q), B_2 x = B(p,x), C_1 x = C(x,d)$ and $C_2 x = C(q,x)$ and their various compositions: $\lambda_1 = \lambda_2 = A_2 B_1, \varrho_1 = A_2, \varrho_2 = A_2 B_2, \lambda_3 = C_1, \varrho_3 = C_2$. Equation (7) is equivalent to the system:

$$\begin{cases} A(x, B(x,y)) = \lambda_3 y \\ C(y,d) = \varrho_2 y \\ \varrho_3 D(z,z) = e\ . \end{cases}$$

Moreover, $\varrho_2 = \lambda_3$.

By the Lemma 4.2, there is a unipotent quasigroup U such that $D(x,y) = \varrho_3 U(x,y)$ with a unipotent e. Also, by the Lemma , there is a loop L_3 with the identity e such that $C(x,y) = L_3(\lambda_3 x, \varrho_3 y)$. If we define a quasigroup L_1 by $L_1(u,v) = A^{(23)}(\lambda_1 u, \varrho_1 v)$, then it is a loop with the identity e and $A(x,y) = L_1^{(23)}(\lambda_1 x, \varrho_1 y), B(x,y) = \varrho_1 L_1(\lambda_2 x, \varrho_2 y)$.

The rest of the requirements of the Theorem are satisfied too, which completes the proof. □

There are 114 equations corresponding to the graph H_4.

$x.xy = z.yz$	$x.xy = z.zy$	$x.yx = z.yz$
$x.yx = z.zy$	$\boxed{x.yz = x.yz}$	$x.yz = x.zy$
$xy.x = z.yz$	$xy.x = z.zy$	$xy.y = z.xz$
$xy.y = z.zx$	$xy.z = z.xy$	$xy.z = z.yx$
$xy.x = yz.z$	$xy.x = zy.z$	$xy.y = xz.z$
$xy.y = zx.z$	$xy.z = xy.z$	$xy.z = yx.z$
$x(x.yz) = yz$	$x(x.yz) = zy$	$x(y.yz) = xz$
$x(y.yz) = zx$	$x(y.zy) = xz$	$x(y.zy) = zx$
$x(yz.x) = yz$	$x(yz.x) = zy$	$x(yz.y) = xz$
$x(yz.y) = zx$	$x(yz.z) = xy$	$x(yz.z) = yx$
$(x.xy)z = yz$	$(x.xy)z = zy$	$(x.yx)z = yz$
$(x.yx)z = zy$	$(x.yz)x = yz$	$(x.yz)x = zy$
$(xy.x)z = yz$	$(xy.x)z = zy$	$(xy.y)z = xz$
$(xy.y)z = zx$	$(xy.z)z = xy$	$(xy.z)z = yx$
$x.x(y.yz) = z$	$x.x(y.zy) = z$	$x.y(y.xz) = z$
$x.y(y.zx) = z$	$x.y(z.xz) = y$	$x.y(z.zx) = y$
$x.x(yz.y) = z$	$x.x(yz.z) = y$	$x.y(xz.y) = z$
$x.y(xz.z) = y$	$x.y(zx.y) = z$	$x.y(zx.z) = y$
$x.(y.xy)z = z$	$x.(y.xz)y = z$	$x.(y.yx)z = z$
$x.(y.yz)x = z$	$x.(y.zx)y = z$	$x.(y.zy)x = z$
$x.(xy.y)z = z$	$x.(xy.z)z = y$	$x.(yx.y)z = z$
$x.(yx.z)z = y$	$x.(yz.y)x = z$	$x.(yz.z)x = y$
$xy.(z.xy) = z$	$xy.(z.xz) = y$	$xy.(z.yx) = z$
$xy.(z.yz) = x$	$xy.(z.zx) = y$	$xy.(z.zy) = x$
$xy.(xy.z) = z$	$xy.(xz.z) = y$	$xy.(yx.z) = z$
$xy.(yz.z) = x$	$xy.(zx.z) = y$	$xy.(zy.z) = x$
$(x.xy).yz = z$	$(x.xy).zy = z$	$(x.yx).yz = z$
$(x.yx).zy = z$	$(x.yz).yz = x$	$(x.yz).zy = x$
$(xy.x).yz = z$	$(xy.x).zy = z$	$(xy.y).xz = z$
$(xy.y).zx = z$	$(xy.z).xy = z$	$(xy.z).yx = z$
$x(x.yz).y = z$	$x(x.yz).z = y$	$x(y.yz).x = z$
$x(y.yz).z = x$	$x(y.zy).x = z$	$x(y.zy).z = x$
$x(yz.x).y = z$	$x(yz.x).z = y$	$x(yz.y).x = z$
$x(yz.y).z = x$	$x(yz.z).x = y$	$x(yz.z).y = x$
$(x.xy)z.y = z$	$(x.xy)z.z = y$	$(x.yx)z.y = z$
$(x.yx)z.z = y$	$(x.yz)x.y = z$	$(x.yz)x.z = y$
$(xy.x)z.y = z$	$(xy.x)z.z = y$	$(xy.y)z.x = z$
$(xy.y)z.z = x$	$(xy.z)z.x = y$	$(xy.z)z.y = x$

There are two $\sim$–classes with two elements each. The representative equation is

$$A(x, B(y,z)) = C(x, D(y,z)) \tag{8}$$

and its solution is given in the following theorem.

Theorem 4.10. *A general solution of the equation* (8) *on a set* S *is given by:*

$$\begin{cases} A(x,y) = L_1(\lambda_1 x, \varrho_1 y) \\ B(x,y) = \varrho_1^{-1} L_2(\lambda_2 x, \varrho_2 y) \\ C(x,y) = L_1(\lambda_3 x, \varrho_3 y) \\ D(x,y) = \varrho_3^{-1} L_2(\lambda_4 x, \varrho_4 y) \end{cases}$$

where:

- L_1 *and* L_2 *are arbitrary loops on* S *with a common unit* e*,*
- $\lambda_1, \varrho_1, \lambda_2, \varrho_2, \lambda_3, \varrho_3, \lambda_4, \varrho_4$ *are arbitrary permutations of* S *such that* $\lambda_1 = \lambda_3$*,* $\lambda_2 = \lambda_4$*,* $\varrho_2 = \varrho_4$*.*

Proof. 1) Let quasigroups A, B, C and D be given by the formulas above. Then

$$\begin{aligned} A(x, B(y,z)) &= L_1(\lambda_1 x, \varrho_1 \varrho_1^{-1} L_2(\lambda_2 y, \varrho_2 z)) \\ &= L_1(\lambda_3 x, \varrho_3 \varrho_3^{-1} L_2(\lambda_4 y, \varrho_4 z)) = C(x, D(y,z)) \end{aligned}$$

and the quadruple A, B, C, D is a solution to (8).

2) Let a quadruple A, B, C, D of quasigroups be a solution to (8) and p, q, r arbitrary fixed elements from S. Define $b = B(q,r)$, $e = A(p,b)$ and $d = D(q,r)$. It follows that $C(p,d) = e$.

Define also $\lambda_1 x = A(x,b)$, $\varrho_1 x = A(p,x)$, $\lambda_2 x = \varrho_1 B(x,r)$, $\varrho_2 x = \varrho_1 B(q,x)$, $\lambda_3 x = C(x,d)$, $\varrho_3 x = C(p,x)$, $\lambda_4 x = \varrho_3 D(x,r)$, and $\varrho_4 x = \varrho_3 D(q,x)$. It follows that $\lambda_1 x = A(x,b) = A(x, B(q,r)) = C(x, D(q,r)) = C(x,d) = \lambda_3 x$ and $\lambda_2 y = \varrho_1 B(y,r) = A(p, B(y,r)) = C(p, D(y,r)) = \varrho_3 D(y,r) = \lambda_4 y$. Analogously $\varrho_2 z = \varrho_4 z$.

Let us define quasigroups $L_1(u,v) = A(\lambda_1^{-1} u, \varrho_1^{-1} v)$ and $L_2(u,v) = \varrho_1 B(\lambda_2^{-1} u, \varrho_2^{-1} v)$. It is easy to check that L_1 and L_2 are both loops with a common identity e. Trivially $A(x,y) = L_1(\lambda_1 x, \varrho_1 y)$ and $B(x,y) = \varrho_1^{-1} L_2(\lambda_2 x, \varrho_2 y)$. Also $C(x, \varrho_3^{-1} \lambda_4 y) = C(x, \varrho_3^{-1} \varrho_3 D(y,r)) = C(x, D(y,r))$ $= A(x, B(y,r)) = A(x, \varrho_1^{-1} \varrho_1 B(y,r)) = A(x, \varrho_1^{-1} \lambda_2 y) = L_1(\lambda_1 x, \varrho_1 \varrho_1^{-1} \lambda_2 y)$ $= L_1(\lambda_3 x, \varrho_3 \varrho_3^{-1} \lambda_4 y)$. Consequently $C(x,y) = L_1(\lambda_3 x, \varrho_3 y)$.

Finally, $D(y,z) = \varrho_3^{-1} \varrho_3 D(y,z) = \varrho_3^{-1} C(p, D(y,z)) = \varrho_3^{-1} A(p, B(y,z))$ $= \varrho_3^{-1} \varrho_1 B(y,z) = \varrho_3^{-1} L_2(\lambda_2 y, \varrho_2 z) = \varrho_3^{-1} L_2(\lambda_4 y, \varrho_4 z)$. $\square$

There are 100 equations corresponding to the graph K_4.

$$\begin{array}{ccc}
x.yz = y.xz & x.yz = y.zx & x.yz = z.xy \\
x.yz = z.yx & \boxed{xy.z = x.yz} & xy.z = x.zy \\
xy.z = y.xz & xy.z = y.zx & xy.z = xz.y \\
xy.z = yz.x & xy.z = zx.y & xy.z = zy.x \\
x(y.xz) = yz & x(y.xz) = zy & x(y.zx) = yz \\
x(y.zx) = zy & x(xy.z) = yz & x(xy.z) = zy \\
x(yx.z) = yz & x(yx.z) = zy & xy.xz = yz \\
xy.xz = zy & xy.yz = xz & xy.yz = zx \\
xy.zx = yz & xy.zx = zy & xy.zy = xz \\
xy.zy = zx & (x.yz)y = xz & (x.yz)y = zx \\
(x.yz)z = xy & (x.yz)z = yx & (xy.z)x = yz \\
(xy.z)x = zy & (xy.z)y = xz & (xy.z)y = zx \\
x.y(x.yz) = z & x.y(x.zy) = z & x.y(z.xy) = z \\
x.y(z.yx) = z & x.y(xy.z) = z & x.y(yx.z) = z \\
x.y(yz.x) = z & x.y(zy.x) = z & x(xy.yz) = z \\
x(xy.zy) = z & x(yx.yz) = z & x(yx.zy) = z \\
x(yz.xy) = z & x(yz.xz) = y & x(yz.yx) = z \\
x(yz.zx) = y & x.(x.yz)y = z & x.(x.yz)z = y \\
x.(y.xz)z = y & x.(y.zx)z = y & x.(xy.z)y = z \\
x.(yx.z)y = z & x.(yz.x)y = z & x.(yz.x)z = y \\
xy.(x.yz) = z & xy.(x.zy) = z & xy.(y.xz) = z \\
xy.(y.zx) = z & xy.(xz.y) = z & xy.(yz.x) = z \\
xy.(zx.y) = z & xy.(zy.x) = z & (x.yz).xy = z \\
(x.yz).xz = y & (x.yz).yx = z & (x.yz).zx = y \\
(xy.z).xz = y & (xy.z).yz = x & (xy.z).zx = y \\
(xy.z).zy = x & x(y.xz).y = z & x(y.xz).z = y \\
x(y.zx).y = z & x(y.zx).z = y & x(xy.z).y = z \\
x(xy.z).z = y & x(yx.z).y = z & x(yx.z).z = y \\
(xy.xz)y = z & (xy.xz)z = y & (xy.yz)x = z \\
(xy.yz)z = x & (xy.zx)y = z & (xy.zx)z = y \\
(xy.zy)x = z & (xy.zy)z = x & (x.yz)y.x = z \\
(x.yz)y.z = x & (x.yz)z.x = y & (x.yz)z.y = x \\
(xy.z)x.y = z & (xy.z)x.z = y & (xy.z)y.x = z \\
 & (xy.z)y.z = x &
\end{array}$$

There is just one $\sim$–class with four elements. The representative equation is

$$A(B(x,y),z) = C(x,D(y,z)) \tag{9}$$

and its solution is given in the following theorem.

Theorem 4.11 (Aczél, Belousov, Hosszú [1]). *A general solution of the generalized associativity equation* (9) *on a set S is given by:*

$$\begin{cases} A(x,y) = \lambda_1 x \cdot \varrho_1 y \\ B(x,y) = \lambda_1^{-1}(\lambda_2 x \cdot \varrho_2 y) \\ C(x,y) = \lambda_3 x \cdot \varrho_3 y \\ D(x,y) = \varrho_3^{-1}(\lambda_4 x \cdot \varrho_4 y) \end{cases}$$

where:

– $\cdot$ *is an arbitrary group on S,*
– $\lambda_1, \varrho_1, \lambda_2, \varrho_2, \lambda_3, \varrho_3, \lambda_4, \varrho_4$ *are arbitrary permutations of S such that:* $\lambda_2 = \lambda_3, \quad \varrho_2 = \lambda_4, \quad \varrho_1 = \varrho_4.$

The results are summarized in the Table 1.

PE–class	Graph	Number of $\sim$–classes	Number of equations	Representative equation
1	H_1	4	3	(5)
2	H_2	4	19	(6)
3	H_3	3	94	(7)
4	H_4	2	114	(8)
5	K_4	1	100	(9)

Table 1: Equations with 3 variables – summary

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Mathematical Institute of the Serbian Academy of Sciences and Arts,
Knez Mihailova 36, 11001 Belgrade, Serbia
E-mail: sasa@mi.sanu.ac.rs