Quasigroups and Related Systems **17** (2009), 17 – 28

Rigid and super rigid quasigroups

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Abstract. The paper deals with quasigroups having a trivial group of automorphisms and a trivial group of autotopisms. Examples of such quasigroups and methods of their verification are given.

1. Introduction

Let $Q = \{1, 2, 3, \ldots, n\}$ be a finite set, φ and ψ permutations of Q. The multiplication (composition) of permutations is defined as $\varphi\psi(x) = \varphi(\psi(x))$. Permutations will be written in the form of cycles and cycles will be separated by points, e.g.

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 4 & 6 \end{pmatrix} = (123.45.6.)$$

As it is well known, any permutation φ of the set Q of order n can be decomposed into $r \leqslant n$ cycles of the length $k_1, k_2, \ldots, k_r$ and $k_1+k_2+\ldots+k_r = n$. We denote this fact by

$$Z(\varphi) = [k_1, k_2, \ldots, k_r].$$

Two permutations are conjugate if and only if they have the same number of cycles of each length (Theorem 5.1.3 in [8]). So, for any two permutations φ and ψ we have

$$Z(\varphi) = Z(\psi) \longleftrightarrow \beta\varphi\beta^{-1} = \psi.$$

From the proof of Theorem 5.1.3 and Lemma 5.1.1 in [8] follows a method of determination of β. This method is also used here, so let us recall it.

If $\beta\varphi\beta^{-1} = \psi$ and

$$\varphi = (a_{11}\, a_{12} \ldots a_{1k_1}) \ldots (a_{r1} \ldots a_{rk_r})$$
$$\psi = (b_{11}\, b_{12} \ldots b_{1k_1}) \ldots (b_{r1} \ldots b_{rk_r})$$

2000 Mathematics Subject Classification: 05B15; 20N05
Keywords: quasigroups, automorphism, autotopism.

then, according to [8], β has the form

$$\beta = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k_1} & \dots & a_{r1} & \dots & a_{rk_r} \\ b_{11} & b_{12} & \dots & b_{1k_1} & \dots & b_{r1} & \dots & b_{rk_r} \end{pmatrix}, \qquad (1)$$

where the first row contains all elements of φ, the second – elements of ψ written in the same order as in decompositions into cycles. Replacing in φ the cycle $(a_{11}\, a_{12} \dots a_{1k_1})$ by $(a_{12}\, a_{13} \dots a_{1k_1}\, a_{11})$ we save the permutation φ but we obtain a new β. Similarly for an arbitrary cycle of φ and ψ. One can prove that in this way we obtain all β satisfying the equality $\beta\varphi\beta^{-1} = \psi$.

Definition 1.1. Let $Q(\cdot)$ be a quasigroup. Each permutation φ_i of Q satisfying the identity

$$x \cdot \varphi_i(x) = i, \qquad (2)$$

where $i \in Q$, is called a *track* or a *right middle translation*.

Such permutations were firstly studied by V. D. Belousov [1] in connection with some groups associated with quasigroups. The investigations of such permutations were continued, for example, in [5, 6] and [11].

The above condition says that in a Latin square $n \times n$ associated with a quasigroup $Q(\cdot)$ of order n we select n cells, one in each row, one in each column, containing the same fixed element i. $\varphi_i(x)$ means that to find in the row x the cell containing i we must select the column $\varphi_i(x)$. It is clear that for a quasigroup $Q(\cdot)$ of order n the set $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ uniquely determines its Latin square, and conversely, any Latin square $n \times n$ uniquely determines the set $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$.

Connections between tracks of isotopic quasigroups are described in [5] and [6]. Namely, if $\{\varphi_1, \varphi_2, \dots \varphi_n\}$ are tracks of $Q(\cdot)$, $\{\psi_1, \psi_2, \dots \psi_n\}$ – tracks of $Q(\circ)$ and

$$\gamma(x \circ y) = \alpha(x) \cdot \beta(y),$$

then

$$\varphi_{\gamma(i)} = \beta\psi_i\alpha^{-1}. \qquad (3)$$

So, tracks of isomorphic quasigroups ($\alpha = \beta = \gamma$) are connected by the formula

$$\varphi_{\alpha(i)} = \alpha\psi_i\alpha^{-1}.$$

Thus, for any automorphism α of a quasigroup $Q(\cdot)$ we have

$$\varphi_{\alpha(i)} = \alpha\varphi_i\alpha^{-1} \qquad (4)$$

and

$$Z(\varphi_i) = Z(\varphi_{\alpha(i)}). \tag{5}$$

Definition 1.2. A track φ_k of $Q(\cdot)$ is called *special* if $Z(\varphi_k) \neq Z(\varphi_i)$ for all $i \in Q$, $i \neq k$.

Example 1.3. Consider two isotopic quasigroups:

$\cdot$	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

$\circ$	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

The first has the following tracks: $\varphi_1 = (1.23.)$, $\varphi_2 = (12.3.)$, $\varphi_3 = (13.2.)$, the second: $\psi_1 = (1.2.3.) = \varepsilon$, $\psi_2 = (123.)$, $\psi_3 = (132.)$. The first has no special tracks, the second has one.

The above examples suggest that any unipotent quasigroup has a special track. Indeed, if $x \cdot x = a$ for all $x \in Q$ and some fixed $a \in Q$, then, as it is not difficult to see, $\varphi_a = \varepsilon$ is its special track. Moreover, $\varphi_a = \varepsilon$ if and only if $x \cdot x = a$ for all $x \in Q$.

Lemma 1.4. *If φ_k is a special track of a quasigroup $Q(\cdot)$, then*
(*a*) $\alpha(k) = k$,
(*b*) $\varphi_k\alpha = \alpha\varphi_k$,
(*c*) $\varphi_k(k) = \alpha(\varphi_k(k))$
for any $\alpha \in \operatorname{Aut} Q(\cdot)$.

Proof. Indeed, $Z(\varphi_k) \neq Z(\varphi_i) = Z(\varphi_{\alpha(i)})$ for every $i \neq k$ and $\alpha \in \operatorname{Aut} Q(\cdot)$ implies $\alpha(i) \neq k$ for every $i \neq k$. Hence $\alpha(k) = k$. The second statement is a consequence of (4). (*c*) follows from (*a*) and (*b*). □

As a consequence of (4) and Lemma 1.4 (*a*) we obtain more general result.

Proposition 1.5. *If φ_i, φ_j are special track of a quasigroup $Q(\cdot)$, then*

$$\varphi_i(j) = \alpha(\varphi_i(j))$$

for any $\alpha \in \operatorname{Aut} Q(\cdot)$. □

Example 1.6. The unipotent quasigroup from Example 1.3 has no special tracks. Its prolongation

$\cdot$	1	2	3	4
1	4	2	3	1
2	3	1	4	2
3	2	4	1	3
4	1	3	2	4

obtained by the method proposed by Belousov (see [2] or [7]) also has no special tracks.

Example 1.7. The idempotent quasigroup of order 3 has no special track, but its prolongation obtained by Bruck's method (see [3] or [7]) is an unipotent quasigroup with one special track.

Example 1.8. The cyclic group of order 4 has no special tracks. Its prolongation

$\cdot$	1	2	3	4	5
1	1	2	5	4	3
2	2	3	4	5	1
3	3	4	1	2	5
4	5	1	2	3	4
5	4	5	3	1	2

obtained according to the formula (9) from [7] has three special tracks:

$$\varphi_2 = (12.34.5.)\,, \quad \varphi_4 = (145.23.)\,, \quad \varphi_5 = (13524.)\,.$$

2. Rigid quasigroups

Autotopies of a quasigroup form a group. Isotopic quasigroups have isomorphic groups of autotopies (see for example [2] or [4]) but groups of automorphisms of such quasigroups may not be isomorphic. Below we give examples of such quasigroups.

Example 2.1. Let $Q(\cdot)$ be a quasigroup defined by the following table:

$\cdot$	1	2	3	4
1	1	2	3	4
2	3	1	4	2
3	4	3	2	1
4	2	4	1	3

It is not difficult to see that this quasigroup is isotopic to a cyclic group of order 4 and has the following four tracks:

$$\varphi_1 = (1.2.34.)\,, \quad \varphi_2 = (124.3.)\,, \quad \varphi_3 = (132.4.)\,, \quad \varphi_4 = (1423.)\,.$$

Tracks φ_1 and φ_4 are special. So, according to Lemma 1.4, for any $\alpha \in \operatorname{Aut} Q(\cdot)$ we have

$$\alpha(1) = 1\,, \qquad \alpha(4) = 4\,,$$

which by Proposition 1.5 implies $\alpha(3) = 3$. Hence $\alpha(2) = 2$, i.e., $\alpha = \varepsilon$. This means that this quasigroup has only one (trivial) automorphisms while a cyclic group of order 4 has two automorphisms.

Definition 2.2. A quasigroup having only one automorphism is called *rigid.*

The above examples prove that a quasigroup isotopic to a rigid quasigroup may not be rigid. Quasigroups of order two are rigid.

Proposition 2.3. *No rigid quasigroups of order three.*

Proof. Indeed, if a quasigroup of order 3 has an idempotent e then $\alpha = (e.xy.)$ is its non-trivial automorphism. If it has no idempotents then it is commutative and has an automorphism $\alpha = (123.)$. □

Each finite quasigroup containing at least 5 elements is isotopic to some rigid quasigroup [9]. The same is true for quasigroups defined on countable sets. So, *for every $k > 3$ there exists at least one rigid quasigroup of order k.*

There are no rigid medial quasigroups of finite order $k \geqslant 2$ [12], but on the additive group of integers we can define infinitely many rigid medial quasigroups [13]. A simple example of such quasigroup is the quasigroup $(\mathbb{Z}, \circ)$ with the operation $x \circ y = -x - y + 1$. Finite rigid T-quasigroups are characterized in [12].

Note, by the way, that prolongation does not save this property. Nevertheless in some cases a prolongation of a rigid quasigroup is also a rigid quasigroup. Moreover, a prolongation of a non-rigid quasigroup may be a rigid quasigroup.

Example 2.4. The cyclic group of order 4 is not a rigid quasigroup. Its prolongation from Example 1.8 is rigid. Indeed, it has three special tracks φ_2, φ_4 and φ_5. Thus, according to Lemma 1.4, for any its automorphism α we have $\alpha(2) = 2$, $\alpha(4) = 4$, $\alpha(5) = 5$. Since $\varphi_2(2) = 1$, $\varphi_2(4) = 3$, Proposition 1.5 implies $\alpha(1) = 1$ and $\alpha(3) = 3$. Hence $\alpha = (1.2.3.4.5.)$, which proves that this quasigroup is rigid.

Example 2.5. The loop $Q(\cdot)$ with the multiplication table

$\cdot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	4	5	6	3
3	3	6	2	1	4	5
4	4	5	6	2	3	1
5	5	3	1	6	2	4
6	6	4	5	3	1	2

has the following tracks:

$$\varphi_1 = (1.2.3465.),\quad \varphi_2 = (12.3.4.5.6.),\quad \varphi_3 = (13.2645.),$$
$$\varphi_4 = (14.2356.),\quad \varphi_5 = (15.24.36.),\quad \varphi_6 = (16.2543.)\,.$$

Since

$$Z(\varphi_1) = [1,1,4],\quad Z(\varphi_2) = [1,1,1,1,2],\quad Z(\varphi_3) = [2,4],$$
$$Z(\varphi_4) = [2,4],\quad Z(\varphi_5) = [2,2,2],\quad Z(\varphi_6) = [2,4],$$

tracks φ_1, φ_2, φ_5 are special. So, according to Lemma 1.4, for any automorphism α of this quasigroup should be

$$\alpha(1) = 1,\quad \alpha(2) = 2,\quad \alpha(5) = 5.$$

By Proposition 1.5, we also have $\alpha(3) = \alpha(\varphi_1(5)) = \varphi_1(5) = 3$ and $\alpha(4) = \alpha(\varphi_1(3)) = \varphi_1(3) = 4$. Thus $\alpha = (1.2.3.4.5.6.) = \varepsilon$, which means that this loop is a rigid quasigroup.

In a similar way we can verify that the following four loops are rigid:

$\cdot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	4	5	6	3
3	3	5	1	6	2	4
4	4	6	5	1	3	2
5	5	3	6	2	4	1
6	6	4	2	3	1	5

$\circ$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	4	3	6	5
3	3	5	1	6	2	4
4	4	6	2	5	1	3
5	5	3	6	2	4	1
6	6	4	5	1	3	2

$\ast$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	6	1	4	5
3	3	4	5	2	6	1
4	4	5	2	6	1	3
5	5	6	1	3	2	4
6	6	1	4	5	3	2

$\star$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	5	1	6	4
3	3	1	2	6	4	5
4	4	5	6	2	1	3
5	5	6	4	3	2	1
6	6	4	1	5	3	2

We say that two quasigroups $Q(\cdot)$ and $Q(*)$ are *dual* if

$$x * y = y \cdot x$$

holds for all $x, y \in Q$. Dual quasigroups have the same automorphisms. This means that *a quasigroup $Q(\cdot)$ is rigid if and only if its dual quasigroup $Q(*)$ is rigid.*

3. Super rigid quasigroups

The next interesting class of quasigroups is a class of quasigroups having only one (trivial) autotopism. Quasigroups with this property are called *super rigid.* Clearly, a super rigid quasigroup has only one automorphism. Hence a super rigid quasigroup is rigid. So, there are no super rigid quasigroups of order 2 and 3.

We remind some definitions and basic facts from [5] and [6].

Definition 3.1. By a *spin* of quasigroup $Q(\cdot)$ we mean the permutation

$$\varphi_{ij} = \varphi_i \varphi_j^{-1},$$

where φ_i, φ_j are tracks of $Q(\cdot)$. The spin φ_{ii} is called *trivial.*

The set Φ_Q of all non-trivial spins of a quasigroup $Q(\cdot)$ is called a *halo.* It can be divided into n disjoint parts $\Phi_1, \Phi_2, \ldots, \Phi_n$, where

$$\Phi_i = \{\varphi_{i1}, \varphi_{i2}, \ldots, \varphi_{i(i-1)}, \varphi_{i(i+1)}, \ldots, \varphi_{in}\}.$$

Let $\Phi = \{\sigma_1, \sigma_2, \ldots, \sigma_k\} \subseteq S_Q$ be a collection of permutations of the set Q. According to [6], the set

$$Sp\,(\Phi) = [Z(\sigma_1), Z(\sigma_2), \ldots, Z(\sigma_k)]$$

is called the *spectrum* of Φ. The spectrum of all spins of $Q(\cdot)$ is called the *spin-spectrum.*

Example 3.2. The quasigroup considered in the Example 2.1 has the following proper spins:

$$\begin{array}{lll}\varphi_{12} = (1342.), & \varphi_{13} = (1243.), & \varphi_{14} = (14.23.),\\ \varphi_{21} = (1243.), & \varphi_{23} = (14.23.), & \varphi_{24} = (1243.),\\ \varphi_{31} = (1342.), & \varphi_{32} = (14.23.), & \varphi_{34} = (1243.),\\ \varphi_{41} = (14.23.), & \varphi_{42} = (1243.), & \varphi_{43} = (1342.).\end{array}$$

Thus $Sp\,(\Phi_1) = [[4],[4],[2,2]]$, $Sp\,(\Phi_2) = [[4],[2,2],[4]] = Sp\,(\Phi_3)$, $Sp\,(\Phi_4) = [[2,2],[4],[4]]$. In the abbreviated form it will be written as $Sp\,(\Phi_i) = 2A + B$, where $A = [4]$, $B = [2,2]$.

Finite isotopic quasigroups have the same spin-spectrum ([6], Theorem 2.5). Moreover, spins of isotopic quasigroups are pairwise conjugated. Namely, if quasigroups $Q(\cdot)$ and $Q(\circ)$ are isotopic and

$$\gamma(x \circ y) = \alpha(x) \cdot \beta(y),$$

then spins φ_{ij} of $Q(\cdot)$ and ψ_{ij} of $Q(\circ)$ are connected by the equality

$$\varphi_{\gamma(i)\gamma(j)} = \beta\psi_{ij}\beta^{-1}.$$

Hence, identifying $Q(\cdot)$ and $Q(\circ)$, we obtain

$$\varphi_{\gamma(i)\gamma(j)} = \beta\varphi_{ij}\beta^{-1}. \tag{6}$$

This means that for any fixed $i \in Q$ and an arbitrary permutation γ of Q, we have

$$Sp\,(\Phi_i) = Sp\,(\Phi_{\gamma(i)}).$$

If $Sp\,(\Phi_i) \neq Sp\,(\Phi_k)$ for all $k \in Q$, $k \neq i$, then we say that the part Φ_i is *special.*

It is not difficult to see that the following lemma is valid.

Lemma 3.3. *If Φ_i is a special part of Φ_Q, then $\gamma(i) = i$ for any autotopism (α, β, γ) of $Q(\cdot)$.* □

Proposition 3.4. *Dual quasigroups have the same spin-spectrum and their special parts have the same numbers.*

Proof. Let $Q(\cdot)$ and $Q(\circ)$ be dual quasigroups. If φ_i is a track of $Q(\cdot)$, then

$$\varphi_i(x) \circ x = x \cdot \varphi_i(x) = i$$

for every $x \in Q$. From this, replacing x by $\varphi_i^{-1}(x)$, we obtain $x \circ \varphi_i^{-1}(x) = i$, which means that $\psi_i = \varphi_i^{-1}$ is a track of $Q(\cdot)$. So, spins ψ_{ij} of $Q(\circ)$ have the form

$$\psi_{ij} = \psi_i \psi_j^{-1} = \varphi_i^{-1} \varphi_j = (\varphi_j^{-1} \varphi_i)^{-1}.$$

Since for any conjugate permutations σ_1, σ_2 of the same set Q we have $Z(\sigma_1) = Z(\sigma_2)$ (cf. [8]), for any permutations α, β, from $\alpha\beta = \beta^{-1}(\beta\alpha)\beta$ it follows $Z(\alpha\beta) = Z(\beta\alpha)$. Thus

$$Z(\psi_{ij}) = Z((\varphi_j^{-1}\varphi_i)^{-1}) = Z(\varphi_j^{-1}\varphi_i) = Z(\varphi_i\varphi_j^{-1}) = Z(\varphi_{ij}),$$

for $i, j = 1, 2, \ldots, n$. Consequently $Sp\,(\Psi_i) = Sp\,(\Phi_i)$ for all $i = 1, 2, \ldots, n$. □

Proposition 3.5. *A quasigroup $Q(\cdot)$ is super rigid if and only if its dual quasigroup $Q(\circ)$ is super rigid.*

Proof. Let $Q(\cdot)$ be a super rigid quasigroup. If (α, β, γ) is an autotopism of a dual quasigroup $Q(\circ)$, then (β, α, γ) is an autotopism of $Q(\cdot)$. Hence $\alpha = \beta = \gamma = \varepsilon$. □

Now we give examples of super rigid quasigroups.

Example 3.6. Consider the following quasigroup:

$\cdot$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	1	7	6	4	5	3
3	3	6	1	2	7	4	5
4	4	5	2	1	3	7	6
5	5	7	4	3	6	2	1
6	6	3	5	7	2	1	4
7	7	4	6	5	1	3	2

This quasigroup has seven tracks:

$$\varphi_1 = (1.2.3.4.57.6.), \quad \varphi_2 = (12.34.56.7.), \quad \varphi_3 = (13.276.45.),$$
$$\varphi_4 = (14.25367.), \quad \varphi_5 = (15.26374.), \quad \varphi_6 = (16.2473.5.),$$
$$\varphi_7 = (17.235.46.).$$

After the calculation of all spins we can see that each spin can be decomposed into cycles in one of the following ways:

$$A = [7], \qquad B = [3,4], \qquad C = [2,2,3], \qquad D = [2,5].$$

Moreover,

$$\begin{aligned}
Sp\,(\Phi_1) &= A \phantom{{}+B} + 2C + 3D,\\
Sp\,(\Phi_2) &= A + B + 2C + 2D,\\
Sp\,(\Phi_3) &= \phantom{2A+{}} 2B + 2C + 2D,\\
Sp\,(\Phi_4) &= 2A \phantom{{}+B} + C + 3D,\\
Sp\,(\Phi_5) &= 2A + B + C + 2D,\\
Sp\,(\Phi_6) &= 2C + 4D,\\
Sp\,(\Phi_7) &= 2C + 4D.
\end{aligned}$$

Since parts $\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5$ are special, from Lemma 3.3 it follows that for any autotopism (α, β, γ) of this quasigroup we have $\gamma = (1.2.3.4.5.67)$ or $\gamma = \varepsilon$.

Below we prove that the first case is impossible. For this we consider two spins

$$\varphi_{15} = (1736245.) \quad \text{and} \quad \varphi_{52} = (16.153.47.).$$

According to (6), we have

$$\varphi_{15} = \beta\varphi_{15}\beta^{-1} \quad \text{and} \quad \varphi_{52} = \beta\varphi_{52}\beta^{-1}.$$

In view of Theorem 5.1.3 from [8] any β satisfying the first equality has the form

$$\beta = \varphi_{15}^{i}, \qquad i = 1, 2, 3, \ldots, 7.$$

The second equality is satisfied by $\beta = \varphi_{52}^{j}$. So, $\varphi_{15}^{i} = \varphi_{52}^{j}$ for some i, j. Since $\varphi_{52}^{j}(1) = 6$ or $\varphi_{52}^{j}(1) = 1$, we have $\varphi_{15}^{i}(1) = 6$ or $\varphi_{15}^{i}(1) = 1$. The first case holds for $i = 3$, the second – for $i = 7$. The case $i = 3$ is impossible because $\varphi_{15}^{3}(6) = 5 \neq \varphi_{52}^{j}(6)$. So, $i = 7$ and $\beta = \varphi_{15}^{7} = \varepsilon$. Thus

$$\gamma(x \cdot y) = \alpha(x) \cdot y,$$

which implies $\gamma(x) = \gamma(x \cdot 1) = \alpha(x)$ for every $x \in Q$. Consequently,

$$\gamma(6) = \gamma(3 \cdot 2) = \alpha(3) \cdot 2 = \gamma(3) \cdot 2 = 3 \cdot 2 = 6.$$

Hence $\gamma = \alpha = \varepsilon$. This proves that this quasigroup is super rigid.

It is the smallest super rigid quasigroup. To prove this fact first we select all rigid quasigroups of order $k < 7$, next we prove that these quasigroups are not super rigid.

Example 3.7. Consider the quasigroup:

$\cdot$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	3	1	8	6	7	5	9	4
3	3	1	2	9	7	5	6	4	8
4	4	5	6	7	9	8	1	3	2
5	5	6	4	2	1	9	8	7	3
6	6	4	5	3	8	1	9	2	7
7	7	8	9	5	3	2	4	6	1
8	8	9	7	1	4	3	2	5	6
9	9	7	8	6	2	4	3	1	5

Using the same method as in Example 3.6 we can see that $Sp\,(\Phi_3) = Sp\,(\Phi_4)$ and $Sp\,(\Phi_i) \neq Sp\,(\Phi_j)$ for all $i \neq j, \quad i \neq 3,4$. This means that $\Phi_1, \Phi_2, \Phi_5, \Phi_6, \Phi_7, \Phi_8$ and Φ_9 are special. Thus, by Lemma 3.3, for any autotopism (α, β, γ) of this quasigroup should be $\gamma = (1.2.34.5.6.7.8.9.)$ or $\gamma = \varepsilon$.

We prove that $\gamma = \varepsilon$. For this consider two spins

$$\varphi_{68} = (197286345.) \quad \text{and} \quad \varphi_{13} = (123.46.59.78.).$$

Then, similarly as in the previous example, $\varphi_{68} = \beta\varphi_{68}\beta^{-1}$ and $\varphi_{13} = \beta\varphi_{13}\beta^{-1}$ imply

$$\beta = \varphi_{68}^i = \varphi_{13}^j$$

for some $i = 1, 2, 3, \ldots, 9$ and $j = 1, 2, \ldots, 6$. Since $\varphi_{13}^j(4) = 4$ or $\varphi_{13}^j(4) = 6$, also $\varphi_{68}^i(4) = 4$ or $\varphi_{68}^i(4) = 6$. Thus $i = 9$ or $i = 7$. For $i = 7$ we have $\varphi_{68}^7(3) = 8$. But $\varphi_{13}^j(3) \neq 8$ for every j. So, this case is impossible. Therefore $i = 9$. Consequently $\beta = \varphi_{68}^9 = \varepsilon$, i.e., $\gamma(x \cdot y) = \alpha(x) \cdot y$, which implies $\gamma(x) = \gamma(x \cdot 1) = \alpha(x)$ for every $x \in Q$. Now, using the above we obtain

$$\gamma(3) = \gamma(2 \cdot 2) = \alpha(2) \cdot 2 = \gamma(2) \cdot 2 = 2 \cdot 2 = 3.$$

Hence $\gamma = \alpha = \varepsilon$. This means that this quasigroup has no nontrivial autotopisms. So, it is super rigid.

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Received March 28, 2009

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