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New identities in universal Osborn loops

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Abstract. A question associated with the 2005 open problem of Michael Kinyon (Is every Osborn loop universal?), is answered. Two nice identities that characterize universal (left and right universal) Osborn loops are established. Numerous new identities are established for universal (left and right universal) Osborn loops like CC-loops, VD-loops and universal weak inverse property loops. Particularly, Moufang loops are discovered to obey the new identity $[y(x^{-1}u) \cdot u^{-1}](xu) = [y(xu) \cdot u^{-1}](x^{-1}u)$ surprisingly. For the first time, new loop properties that are weaker forms of well known loop properties like inverse property, power associativity and diassociativity are introduced and studied in universal (left and right universal) Osborn loops. Some of them are found to be necessary and sufficient conditions for a universal Osborn to be 3 power associative. For instance, four of them are found to be new necessary and sufficient conditions for a CC-loop to be power associative. A conjugacy closed loop is shown to be diassociative if and only if it is power associative and has a weak form of diassociativity.

1. Introduction

The isotopic invariance of varieties of quasigroups and loops described by one or more equivalent identities, especially those that fall in the class of Bol-Moufang type loops have been of interest to researchers in loop theory in the recent past. These types of identities were first named by Fenyves [18] and [17] in the 1960s and later on in this 21^{st} century by Phillips and Vojtěchovský [32], [33] and [25]. Among such are Etta Falconer [15] and [16] which investigated isotopy invariants in quasigroups. Loops such as Bol loops, Moufang loops, central loops and extra loops are the most popular loops of Bol-Moufang type whose isotopic invariance have been considered. For an overview of the theory of loops, readers may check [30, 8, 10, 12, 19, 34].

Consider $(G,\cdot)$ and $(H,\circ)$ been two distinct groupoids (quasigroups, loops).

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Let A, B and C be three bijective mappings, that map G onto H. The triple $\alpha = (A, B, C)$ is called an *isotopism* of $(G, \cdot)$ onto $(H, \circ)$ if and only if

$$xA \circ yB = (x \cdot y)C \quad \forall\ x, y \in G.$$

So, $(H, \circ)$ is called a groupoid (quasigroup, loop) *isotope* of $(G, \cdot)$.

If $C = I$ is the identity map on G so that $H = G$, then the triple $\alpha = (A, B, I)$ is called a *principal isotopism* of $(G, \cdot)$ onto $(G, \circ)$ and $(G, \circ)$ is called a *principal isotope* of $(G, \cdot)$. Eventually, the equation of relationship now becomes

$$x \cdot y = xA \circ yB \quad \forall\ x, y \in G$$

which is easier to work with. But if $A = R_g$ and $B = L_f$ where $R_x : G \to G$, the *right translation* is defined by $yR_x = y \cdot x$ and $L_x : G \to G$, the *left translation* is defined by $yL_x = x \cdot y$ for all $x, y \in G$, for some $f, g \in G$, the relationship now becomes

$$x \cdot y = xR_g \circ yL_f \quad \forall\ x, y \in G$$

or

$$x \circ y = xR_g^{-1} \cdot yL_f^{-1} \quad \forall\ x, y \in G.$$

With this new form, the triple $\alpha = (R_g, L_f, I)$ is called an *f, g-principal isotopism* of $(G, \cdot)$ onto $(G, \circ)$, f and g are called *translation elements* of G or at times written in the pair form (g, f), while $(G, \circ)$ is called an *f, g-principal isotope* of $(G, \cdot)$.

The last form of α above gives rise to an important result in the study of loop isotopes of loops.

Theorem 1.1. [8] *Let $(G, \cdot)$ and $(H, \circ)$ be two distinct isotopic loops. For some $f, g \in G$, there exists an f, g-principal isotope $(G, *)$ of $(G, \cdot)$ such that $(H, \circ) \cong (G, *)$.* □

With this result, to investigate the isotopic invariance of an isomorphic invariant property in loops, one simply needs only to check if the property in consideration is true in all f, g-principal isotopes of the loop. A property is *isotopic invariant* if whenever it holds in the domain loop i.e., $(G, \cdot)$ then it must hold in the co-domain loop i.e., $(H, \circ)$ which is an isotope of the formal. In such a situation, the property in consideration is said to be a *universal property* hence the loop is called a *universal loop* relative to the property in consideration as often used by Nagy and Strambach [28] in their algebraic and geometric study of the universality of some types of loops. For instance, if every isotope of a "certain" loop is a "certain" loop, then the formal is called a *universal "certain" loop*. So, we can now restate Theorem 1.1 as:

Theorem 1.2. *Let $(G,\cdot)$ be a "certain" loop where "certain" is an isomorphic invariant property. $(G,\cdot)$ is a universal "certain" loop if and only if every f,g-principal isotope $(G,*)$ of $(G,\cdot)$ has the "certain" loop property.* □

From the earlier discussions, if $(H,\circ)=(G,\cdot)$ then the triple $\alpha=(A,B,C)$ is called an *autotopism* where $A,B,C\in \mathrm{Sym}(G,\cdot)$, the set of all bijections on $(G,\cdot)$ called the *symmetric group* of $(G,\cdot)$. Such triples form a group $\mathrm{Aut}(G,\cdot)$ called the *autotopism group* of $(G,\cdot)$.

Bol-Moufang type of quasigroups (loops) are not the only quasigroups (loops) that are isomorphic invariant and whose universality have been considered. Some others are weak inverse property loops (WIPLs) and cross inverse property loops (CIPLs). The universality of WIPLs and CIPLs have been addressed by Osborn [29] and Artzy [1] respectively. In 1970, Basarab [3] later continued the work of Osborn of 1961 on universal WIPLs by studying isotopes of WIPLs that are also WIPLs after he had studied a class of WIPLs ([2]) in 1967. Osborn [29], while investigating the universality of WIPLs discovered that a universal WIPL $(G,\cdot)$ satisfies the identity

$$yx\cdot(zE_y\cdot y)=(y\cdot xz)\cdot y\quad \forall\ x,y,z\in G \tag{1.1}$$

where $E_y=L_yL_{y^\lambda}=R_{y^\rho}^{-1}R_y^{-1}=L_yR_yL_y^{-1}R_y^{-1}$ and y^λ and y^ρ are respectively the left and right inverse elements of y.

Eight years after Osborn's [29] 1960 work on WIPL, in 1968, Huthnance Jr. [20] studied the theory of generalized Moufang loops. He named a loop satysfying (1.1) a *generalized Moufang loop* and later on in the same thesis, he called them *M-loops*. On the other hand, he called a *universal WIPL* an *Osborn loop* and this same definition was adopted by Chiboka [11]. Basarab dubbed a loop $(G,\cdot)$ satisfying the identity:

$$x(yz\cdot x)=(x\cdot yE_x)\cdot zx\quad \forall\ x,y,z\in G \tag{1.2}$$

an *Osborn loop* where $E_x=R_xR_{x^\rho}=(L_xL_{x^\lambda})^{-1}=R_xL_xR_x^{-1}L_x^{-1}$.

It will look confusing if both Basarab's and Huthnance's definitions of an Osborn loop are both adopted because an Osborn loop of Basarab is not necessarily a universal WIPL (Osborn loop of Huthnance). So in this work, Huthnance's definition of an Osborn loop will be dropped while we shall stick to that of Basarab which was actually adopted by Kinyon [21] and the open problem we intend to solve is relative to Basarab's definition of an Osborn loop and not that of Huthnance. Huthnance [20] was able to deduce some properties of E_x relative to (1.1). $E_x = E_{x^\lambda} = E_{x^\rho}$. So, since $E_x = R_x R_{x^\rho}$, then $E_x = E_{x^\lambda} = R_{x^\lambda} R_x$ and $E_x = (L_{x^\rho} L_x)^{-1}$. So, we now have two identities equivalent to identities (1.1) and (1.2) defining an Osborn loop.

$$\mathrm{OS}_0 \ : \ x(yz \cdot x) = x(yx^\lambda \cdot x) \cdot zx \tag{1.3}$$

$$\mathrm{OS}_1 \ : \ x(yz \cdot x) = x(yx \cdot x^\rho) \cdot zx \tag{1.4}$$

Although Basarab [4] and [7] considered universal Osborn loops but the universality of Osborn loops was raised as an open problem by Kinyon in 2005 at a conference tagged *Milehigh Conference on Loops, Quasigroups and Non-associative Systems* held at the University of Denver, where he presented a talk titled *A survey of Osborn loops*. The present authors have been able to find a counter example to prove that not every Osborn loop is universal (in a different paper, submitted for publication) thereby putting the open problem to rest. Kinyon [21] further raised the question concerning the problem by asking if there exists a 'nice' identity that characterizes a universal Osborn loop.

In this study, a question associated with the 2005 open problem of Kinyon (Is every Osborn loop universal?), is answered. Two nice identities that characterize universal (left and right universal) Osborn loops are established. Numerous new identities are established for universal Osborn loops like CC-loops, VD-loops and universal weak inverse property loops. Particularly, Moufang loops are discovered to obey the new identity $[y(x^{-1}u) \cdot u^{-1}](xu) = [y(xu) \cdot u^{-1}](x^{-1}u)$ surprisingly. For the first time, new loop properties that are weaker forms of well known loop properties like inverse property, power associativity and diassociativity are introduced and studied in universal (left and right universal) Osborn loops. Some of them are found to be necessary and sufficient conditions for a universal Osborn to be 3 power associative. For instance, four of them are found to be new necessary and sufficient conditions for a CC-loop to be power associative. A conjugacy closed loop is shown to be diassociative if and only if it is power associative and has a weak form of diassociativity.

2. Preliminaries

Let G be a non-empty set. Define a binary operation $(\cdot)$ on G. If each of the equations:

$$a \cdot x = b \qquad \text{and} \qquad y \cdot a = b$$

has unique solutions in G for x and y respectively, then $(G, \cdot)$ is called a *quasigroup*.

If there exists a unique element $e \in G$ called the *identity element* such that for all $x \in G$, $x \cdot e = e \cdot x = x$, $(G, \cdot)$ is called a *loop*. We write xy instead of $x \cdot y$, and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied. For instance, $x \cdot yz$ stands for $x(yz)$.

It can now be seen that a groupoid $(G, \cdot)$ is a quasigroup if it's left and right translation mappings are bijections or permutations. Since the left and right translation mappings of a loop are bijective, then the inverse mappings L_x^{-1} and R_x^{-1} exist. Let

$$x \backslash y = yL_x^{-1} = y\mathbb{L}_x \qquad \text{and} \qquad x/y = xR_y^{-1} = x\mathbb{R}_y$$

and note that $x \backslash y = z \longleftrightarrow x \cdot z = y$ and $x/y = z \longleftrightarrow z \cdot y = x$. Hence, $(G, \backslash)$ and $(G, /)$ are also quasigroups. Using the operations $(\backslash)$ and $(/)$, the definition of a loop can be stated as follows.

Definition 2.3. A *loop* $(G, \cdot, /, \backslash, e)$ is a set G together with three binary operations $(\cdot)$, $(/)$, $(\backslash)$ and one nullary operation e such that

(i) $x \cdot (x \backslash y) = y$, $(y/x) \cdot x = y$ for all $x, y \in G$,

(ii) $x \backslash (x \cdot y) = y$, $(y \cdot x)/x = y$ for all $x, y \in G$ and

(iii) $x \backslash x = y/y$ or $e \cdot x = x$ for all $x, y \in G$.

We also stipulate that $(/)$ and $(\backslash)$ have higher priority than $(\cdot)$ among factors to be multiplied. For instance, $x \cdot y/z$ and $x \cdot y \backslash z$ stand for $x(y/z)$ and $x \cdot (y \backslash z)$ respectively.

In a loop $(G, \cdot)$ with identity element e, the *left inverse element* of $x \in G$ is the element $xJ_\lambda = x^\lambda \in G$ such that $x^\lambda \cdot x = e$ while the *right inverse element* of $x \in G$ is the element $xJ_\rho = x^\rho \in G$ such that $x \cdot x^\rho = e$.

Definition 2.4. A loop $(Q, \cdot)$ is called

- a 3 *power associative property loop (3-PAPL)* if $xx \cdot x = x \cdot xx$,

- a *left self inverse property loop* (LSIPL) if $x^\lambda \cdot xx = x$,
- a *right self inverse property loop* (RSIPL) if $xx \cdot x^\rho = x$,
- a *self automorphic inverse property loop* (SFAIPL) if $(xx)^\rho = x^\rho x^\rho$,
- a *self weak inverse property loop* (SWIPL) if $x \cdot (xx)^\rho = x^\rho$,
- a *left^1 bi-self inverse property loop* (L^1BSIPL) if $x^\lambda(xx \cdot x) = xx$,
- a *left^2 bi-self inverse property loop* (L^2BSIPL) if $x^\lambda(x \cdot xx) = xx$

for all $x \in Q$.

Definition 2.5. Let $w_1(q_1, q_2, \cdots, q_n)$ and $w_2(q_1, q_2, \cdots, q_n)$ be words in terms of variables $q_1, q_2, \cdots, q_n$ of the loop Q with equal lengths $N(N \in \mathbb{N}, N > 1)$ such that the variables $q_1, q_2, \cdots, q_n$ appear in them in equal number of times. Q is called a $\mathrm{N}^{m_1, m_2, \cdots, m_n}_{w_1(r_1, r_2, \cdots, r_n) = w_2(r_1, r_2, \cdots, r_n)}$ loop if it satisfies the identity

$$w_1(q_1, q_2, \cdots, q_n) = w_2(q_1, q_2, \cdots, q_n),$$

where $m_1, m_2, m_3, \cdots, m_n \in \mathbb{N}$ represent the number of times the variables $q_1, q_2, \cdots, q_n \in Q$ respectively appear in the word w_1 or w_2 such that the mappings $q_1 \mapsto r_1, q_1 \mapsto r_1, \cdots, q_n \mapsto r_n$ are assumed, $r_1, r_2, \cdots r_n \in \mathbb{N}$.

In this study, we shall concentrate on when $N = 4$.

The identities describing the most popularly known varieties of Osborn loops are given below.

Definition 2.6. A loop $(Q, \cdot)$ is called:

- a *VD-loop* if $(\cdot)_x = (\cdot)^{L_x^{-1}R_x}$ and ${}_x(\cdot) = (\cdot)^{R_x^{-1}L_x}$ i.e., $R_x^{-1}L_x \in PS_\lambda(Q, \cdot)$ with companion $c = x$ and $L_x^{-1}R_x \in PS_\rho(Q, \cdot)$ with companion $c = x$ for all $x \in Q$, where $PS_\lambda(Q, \cdot)$ and $PS_\rho(Q, \cdot)$ are respectively the left and right pseudo-automorphism groups of Q,
- a *Moufang loop* if the identity $(xy) \cdot (zx) = (x \cdot yz)x$ holds in Q,
- a conjugacy closed loop (*CC-loop*) if the identities $x \cdot yz = (xy)/x \cdot xz$ and $zy \cdot x = zx \cdot x\backslash(yx)$ hold in Q,
- a *universal WIPL* if the identity $x(yx)^\rho = y^\rho$ or $(xy)^\lambda x = y^\lambda$ holds in Q and all its isotopes.

All these four varieties of Osborn loops are universal. CC-loops, and VD-loops are *G-loops*. i.e., are isomorphic to all their loop isotopes.

Definition 2.7. Let $\alpha = (A, B, C)$ be an isotopism of $(G, \cdot)$ onto $(H, \circ)$. For $C = B$ it is called a *left isotopism*; for $C = A$ – a *right isotopism*; for $B = C = I$ – a *left principal isotopism*; for $A = C = I$ – a *right principal isotopism*.

A loop is a *left (right) universal "certain" loop* if and only if all its left (right) isotopes are "certain" loops.

Theorem 2.8. *Let $(G,\cdot)$ and $(H,\circ)$ be two distinct left (right) isotopic loops with the former having an identity element e. For some g $(f)\in G$, there exists an e,g (f,e)-principal isotope $(G,*)$ of $(G,\cdot)$ such that $(H,\circ)\cong(G,*)$.*

Proof. The proof of this is similar to that of Theorem III.2.1 of [30]. □

Theorem 2.9. *Let $(G,\cdot)$ be a "certain" loop where "certain" is an isomorphic invariant property. $(G,\cdot)$ is a left (right) universal "certain" loop if and only if every left (right) principal isotope $(G,*)$ of $(G,\cdot)$ has the "certain" loop property.*

Proof. Use Theorem 2.8. □

3. Main results

Theorem 3.1. *A loop $(Q,\cdot,\backslash,/)$ is a universal Osborn loop if and only if it satisfies the identity*

$$OS_0':\begin{cases} x\, u\backslash\{(yz)/v\cdot[u\backslash(xv)]\}=\\ (x\cdot u\backslash\{[y(u\backslash([(uv)/(u\backslash(xv))]v))]/v\cdot[u\backslash(xv)]\})/v\cdot u\backslash[((uz)/v)(u\backslash(xv))]\end{cases}$$

or

$$OS_1':\begin{cases} x\cdot x\cdot u\backslash\{(yz)/v\cdot[u\backslash(xv)]\}=\\ \{x\cdot u\backslash\{[y(u\backslash(xv))]/v\cdot[x\backslash(uv)]\}\}/v\cdot u\backslash[((uz)/v)(u\backslash(xv))].\end{cases}$$

Proof. Let $\mathcal{Q}=(Q,\cdot,\backslash,/)$ be an Osborn loop with any arbitrary principal isotope $\mathfrak{Q}=(Q,\vartriangle,\nwarrow,\nearrow)$ such that

$$x\vartriangle y=xR_v^{-1}\cdot yL_u^{-1}=(x/v)\cdot(u\backslash y)\quad\forall\ u,v\in Q.$$

If $\mathcal{Q}$ is a universal Osborn loop then, $\mathfrak{Q}$ is an Osborn loop. Then OS_0 implies

$$x\vartriangle[(y\vartriangle z)\vartriangle x]=\{x\vartriangle[(y\vartriangle x^{\lambda'})\vartriangle x]\}\vartriangle(z\vartriangle x)\tag{3.1}$$

where $x^{\lambda'}=xJ_{\lambda'}$ is the left inverse of x in $\mathfrak{Q}$. The identity element of the loop $\mathfrak{Q}$ is uv. So, $x\vartriangle y=xR_v^{-1}\cdot yL_u^{-1}$ implies $y^{\lambda'}\vartriangle y=y^{\lambda'}R_v^{-1}\cdot yL_u^{-1}=uv$, whence we obtain $y^{\lambda'}R_v^{-1}R_{yL_u^{-1}}=uv$, consequently $yJ_{\lambda'}=(uv)R^{-1}_{yL_u^{-1}}R_v=$

$(uv)R^{-1}_{(u\backslash y)}R_v = [(uv)/(u\backslash y)]v$. Thus, using the fact that $x \vartriangle y = (x/v)\cdot(u\backslash y)$, $\mathfrak{Q}$ is an Osborn loop if and only if

$$(x/v)\cdot u\backslash\{[(y/v)\cdot(u\backslash z)]/v\cdot(u\backslash x)\} = \\ ((x/v)\cdot u\backslash\{[(y/v)(u\backslash([(uv)/(u\backslash x)]v))]/v\cdot(u\backslash x)\})/v\cdot u\backslash[(z/v)(u\backslash x)].$$

Do the following replacements:

$$x' = x/v \to x = x'v, \quad z' = u\backslash z \to z = uz', \quad y' = y/v \to y = y'v$$

we have

$$x'\cdot u\backslash\{(y'z')/v\cdot[u\backslash(x'v)]\} = \\ (x'\cdot u\backslash\{[y'(u\backslash([(uv)/(u\backslash(x'v))]v))]/v\cdot[u\backslash(x'v)]\})/v\cdot u\backslash[((uz')/v)(u\backslash(x'v))].$$

This is precisely $0S_0'$ by replacing x', y' and z' by x, y and z respectively.

Conversely, let $\mathcal{Q} = (Q,\cdot,\backslash,/)$ be an Osborn loop satisfying OS_0'. Doing the reverse process of the proof of the necessary part, it will be observed that equation (3.1) is true for any arbitrary u,v-principal isotope $\mathfrak{Q} = (Q,\vartriangle,\nwarrow,\nearrow)$ of $\mathcal{Q}$. So, every f,g-principal isotope $\mathfrak{Q}$ of $\mathcal{Q}$ is an Osborn loop. Following Theorem 1.2, $\mathcal{Q}$ is a universal Osborn loop if and only if $\mathfrak{Q}$ is an Osborn loop.

The proof for the second identity is similar. □

Lemma 3.2. *A loop Q with the multiplication group $\mathrm{Mult}(Q)$ is a universal Osborn loop if and only if $\big(\alpha(x,u,v),\beta(x,u,v),\gamma(x,u,v)\big) \in \mathrm{Aut}(Q)$ or $\Big(R_{[u\backslash(xv)]}\mathbb{R}_vR_{[x\backslash(uv)]}\mathbb{R}_{[u\backslash(xv)]}R_v\gamma(x,u,v)\mathbb{R}_v,\beta(x,u,v),\gamma(x,u,v)\Big) \in \mathrm{Aut}(Q)$ for all $x,u,v \in Q$, where $\alpha(x,u,v) = R_{(u\backslash([(uv)/(u\backslash(xv))]v))}\mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_uL_x\mathbb{R}_v$, $\beta(x,u,v) = L_u\mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_u$ and $\gamma(x,u,v) = \mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_uL_x$ are elements of $\mathrm{Mult}(Q)$.*

Proof. This is obtained from identity OS_0' or OS_1' of Theorem 3.1. □

Theorem 3.3. *If Q is a universal Osborn loop with the multiplication group $\mathrm{Mult}(Q)$, then $\big(\gamma(x,u,v)\mathbb{R}_{(u\backslash[(u/v)(u\backslash(xv))])},\beta(x,u,v),\gamma(x,u,v)\big) \in \mathrm{Aut}(Q)$ for all elements $x,u,v \in Q$, where $\beta(x,u,v) = L_u\mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_u$ and $\gamma(x,u,v) = \mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_uL_x$ are from $\mathrm{Mult}(Q)$.*

Proof. Theorem 3.1 will be employed. Let $z = e$ in identity OS_0', then

$$x\cdot u\backslash\{y/v\cdot[u\backslash(xv)]\} = \\ (x\cdot u\backslash\{[y(u\backslash([(uv)/(u\backslash(xv))]v))]/v\cdot[u\backslash(xv)]\})/v\cdot u\backslash[(u/v)(u\backslash(xv))].$$

So, identity OS_0' can now be written as

$$x \cdot u\backslash\{(yz)/v \cdot [u\backslash(xv)]\} = \Big\{\{x \cdot u\backslash[y/v \cdot (u\backslash(xv))]\}/\{u\backslash[(u/v)(u\backslash(xv))]\}\Big\} \cdot u\backslash[((uz)/v)(u\backslash(xv))].$$

Thus $\Big(\gamma(x,u,v)\mathbb{R}_{(u\backslash[(u/v)(u\backslash(xv))])}, \beta(x,u,v), \gamma(x,u,v)\Big) \in \mathrm{Aut}(Q)$. $\square$

Lemma 3.4. *In a universal Osborn loop $(Q,\cdot,\backslash,/)$ the following identities are satisfied:*

$$\begin{aligned}
OSI_{01}: &\quad y\{u\backslash([(uv)/(u\backslash(xv))]v)\} = \{(y[u\backslash(xv)])/v \cdot [x\backslash(uv)]\}/[u\backslash(xv)] \cdot v,\\
OSI_{01.1}: &\quad \begin{cases} \{(uz)/v \cdot u\backslash(\{(yv)(u\backslash([(uv)/z]v))\}/v \cdot z)\}/v \cdot (u\backslash[(u/v)z]) \\ = (uz)/v \cdot u\backslash(yz), \end{cases}\\
OSI_{01.2}: &\quad \begin{cases} (uz)/v \cdot u\backslash\{(yv \cdot z)/v \cdot [((uz)/v)\backslash(uv)]\} \\ = [(uz)/v \cdot u\backslash(yz)]/\{u\backslash[(u/v)z]\} \cdot v, \end{cases}\\
OSI_{01.1.1}: &\quad \{u\backslash(\{(uy \cdot u)(u\backslash(uu \cdot u))\}/u)\}/u \cdot u^\rho = y,\\
&\quad uu \cdot u\backslash(uu \cdot u) = (u \cdot uu)u,\\
OSI_{01.2.1}: &\quad v^\lambda \cdot u\backslash\{(yv \cdot u^\rho)/v \cdot [v^\lambda\backslash(uv)]\} = [v^\lambda \cdot u\backslash(yu^\rho)]/\{u\backslash[(u/v)u^\rho]\} \cdot v,\\
OSI_{01.2.2}: &\quad v^\lambda(y \cdot v^\lambda\backslash v) = (v^\lambda y)/v^\lambda \cdot v,\\
&\quad v^\lambda \cdot (v \cdot v^\lambda\backslash v) = v^{\lambda^2} \cdot v = (v^\lambda \cdot vv)v\\
&\quad v(v^\rho \cdot v\backslash v^\rho) = v^\lambda \cdot v^\rho.
\end{aligned}$$

Proof. To prove these identities, we shall make use of the three autotopisms in Lemma 3.2 and Theorem 3.3. In a quasigroup, any two components of an autotopism uniquely determine the third. So equating the first components of the three autotopisms, it is easy to see that

$$\begin{aligned}\alpha(x,u,v) &= \gamma(x,u,v)\mathbb{R}_{(u\backslash[(u/v)(u\backslash(xv))])}\\ &= R_{[u\backslash(xv)]}\mathbb{R}_v R_{[x\backslash(uv)]}\mathbb{R}_{[u\backslash(xv)]}R_v\gamma(x,u,v)\mathbb{R}_v.\end{aligned}$$

The establishment of the identities OSI_{01}, $\mathrm{OSI}_{01.1}$ and $\mathrm{OSI}_{01.2}$ follows by using the bijections appropriately to map an arbitrary element $y \in Q$ as follows:

OSI_{01} : $\alpha(x,u,v) = R_{[u\backslash(xv)]}\mathbb{R}_v R_{[x\backslash(uv)]}\mathbb{R}_{[u\backslash(xv)]}R_v\gamma(x,u,v)\mathbb{R}_v$ implies that $R_{(u\backslash([(uv)/(u\backslash(xv))]v))}\mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x\mathbb{R}_v = R_{(u\backslash([(uv)/(u\backslash(xv))]v))}\gamma(x,u,v)\mathbb{R}_v = R_{[u\backslash(xv)]}\mathbb{R}_v R_{[x\backslash(uv)]}\mathbb{R}_{[u\backslash(xv)]}R_v\gamma(x,u,v)\mathbb{R}_v$ which gives $R_{(u\backslash([(uv)/(u\backslash(xv))]v))} = R_{[u\backslash(xv)]}\mathbb{R}_v R_{[x\backslash(uv)]}\mathbb{R}_{[u\backslash(xv)]}R_v$. So, for any $y \in Q$, $yR_{(u\backslash([(uv)/(u\backslash(xv))]v))} =$

$yR_{[u\backslash(xv)]}\mathbb{R}_vR_{[x\backslash(uv)]}\mathbb{R}_{[u\backslash(xv)]}R_v$ implies

$$y\{u\backslash([(uv)/(u\backslash(xv))]v)\} = \{(y[u\backslash(xv)])/v\cdot[x\backslash(uv)]\}/[u\backslash(xv)]\cdot v.$$

$\mathrm{OSI}_{01.1}$: Let $\alpha(x,u,v) = \gamma(x,u,v)\mathbb{R}_{(u\backslash[(u/v)(u\backslash(xv))])}$. Then for all $y\in Q$

$$\begin{aligned} y\alpha(x,u,v) &= yR_{(u\backslash([(uv)/(u\backslash(xv))]v))}\mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_uL_x\mathbb{R}_v \\ &= y\gamma(x,u,v)\mathbb{R}_{(u\backslash[(u/v)(u\backslash(xv))])} = y\mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_uL_x\mathbb{R}_{(u\backslash[(u/v)(u\backslash(xv))])}.\end{aligned}$$

Consequently,

$$\begin{aligned} &\{x\cdot u\backslash(\{y(u\backslash([(uv)/(u\backslash(xv))]v))\}/v\cdot[u\backslash(xv)])\}/v \\ &\qquad = \{x\cdot u\backslash(y/v\cdot[u\backslash(xv)])\}/(u\backslash[(u/v)(u\backslash(xv))]).\end{aligned}$$

Now, replacing y/v by y and multiplying both sides by $(u\backslash[(u/v)(u\backslash(xv))])$ we get

$$\begin{aligned} &\{x\cdot u\backslash(\{(yv)(u\backslash([(uv)/(u\backslash(xv))]v))\}/v\cdot[u\backslash(xv)])\}/v\cdot(u\backslash[(u/v)(u\backslash(xv))]) \\ &\qquad = x\cdot u\backslash(y\cdot[u\backslash(xv)]),\end{aligned}$$

whence, for $z = u\backslash(xv)$, i.e., $x = (uz)/v$, we deduce

$$\{(uz)/v\cdot u\backslash(\{(yv)(u\backslash([(uv)/z]v))\}/v\cdot z)\}/v\cdot(u\backslash[(u/v)z]) = (uz)/v\cdot u\backslash(yz).$$

$\mathrm{OSI}_{01.2}$: Let

$$R_{[u\backslash(xv)]}\mathbb{R}_vR_{[x\backslash(uv)]}\mathbb{R}_{[u\backslash(xv)]}R_v\gamma(x,u,v)\mathbb{R}_v = \gamma(x,u,v)\mathbb{R}_{(u\backslash[(u/v)(u\backslash(xv))])}.$$

Then for all $y\in Q$, we obtain $yR_{[u\backslash(xv)]}\mathbb{R}_vR_{[x\backslash(uv)]}\mathbb{R}_{[u\backslash(xv)]}R_v\gamma(x,u,v)\mathbb{R}_v = y\gamma(x,u,v)\mathbb{R}_{(u\backslash[(u/v)(u\backslash(xv))])}$. Whence, as a simple consequence we obtain that

$$\begin{aligned} &(\{\big[(y[u\backslash(xv)])/v\cdot[x\backslash(uv)]\big]/[u\backslash(xv)]\cdot v\}\gamma(x,u,v))/v = \\ &\qquad \big(y\gamma(x,u,v)\big)/(u\backslash[(u/v)(u\backslash(xv))]),\end{aligned}$$

which implies

$$\begin{aligned} &x\cdot u\backslash(\{\big[(y[u\backslash(xv)])/v\cdot[x\backslash(uv)]\big]/[u\backslash(xv)]\cdot v\}/v\cdot[u\backslash(xv)]) \\ &\qquad = [x\cdot u\backslash\big(y/v\cdot[u\backslash(xv)]\big)]/(u\backslash[(u/v)(u\backslash(xv))])\cdot v.\end{aligned}$$

Since $z = u\backslash(xv)$ means that $x = (uz)/v$, from the above we have

$$(uz)/v\cdot u\backslash\big[(yz)/v\cdot[(uz)/v\backslash(uv)]\big] = [(uz)/v\cdot u\backslash\big(y/v\cdot z\big)]/(u\backslash[(u/v)z])\cdot v.$$

Now, replacing y by yv we get

$$(uz)/v\cdot u\backslash\big[(yv\cdot z)/v\cdot[(uz)/v\backslash(uv)]\big] = [(uz)/v\cdot u\backslash\big(yz\big)]/(u\backslash[(u/v)z])\cdot v.$$

$\mathrm{OSI}_{01.1.1}$: For $u = v$ the identity $\mathrm{OSI}_{01.1}$ has the form

$$\{(uz)/u \cdot u\backslash(\{(yu)(u\backslash([(uu)/z]u))\}/u \cdot z)\}/u \cdot (u\backslash z) = (uz)/u \cdot u\backslash(yz),$$

whence replacing z by uz we get

$$\{(u \cdot uz)/u \cdot u\backslash(\{(yu)(u\backslash([(uu)/(uz)]u))\}/u \cdot uz)\}/u \cdot z = (u \cdot uz)/u \cdot u\backslash(y \cdot uz),$$

which for $z = u^\rho$ gives $\{u\backslash(\{(yu)(u\backslash(uu \cdot u))\}/u)\}/u \cdot u^\rho = u\backslash y$. Now, replacing y by uy, we obtain $\{u\backslash(\{(uy \cdot u)(u\backslash(uu \cdot u))\}/u)\}/u \cdot u^\rho = y$.
$\mathrm{OSI}_{01.2.1}$: Putting $z = u^\rho$ in $\mathrm{OSI}_{01.2}$ we get

$$v^\lambda \cdot u\backslash\big[(yv \cdot u^\rho)/v \cdot [v^\lambda\backslash(uv)]\big] = [v^\lambda \cdot u\backslash\big(yu^\rho\big)]/(u\backslash[(u/v)u^\rho]) \cdot v.$$

$\mathrm{OSI}_{01.2.2}$: Putting $u = e$ in $\mathrm{OSI}_{01.2.1}$ we get $v^\lambda(y \cdot v^\lambda\backslash v) = (v^\lambda y)/v^\lambda \cdot v$.

By putting $y = e$ in $\mathrm{OSI}_{01.1.1}$, we have $uu \cdot u\backslash(uu \cdot u) = (u \cdot uu)u$. Also, substituting $y = v$ into $\mathrm{OSI}_{01.2.2}$ and using the fact that $x^{\lambda^2} = x^\lambda \cdot xx$ we get $v^\lambda \cdot (v \cdot v^\lambda\backslash v) = v^{\lambda^2} \cdot v = (v^\lambda \cdot vv)v$ and $v(v^\rho \cdot v\backslash v^\rho) = v^\lambda \cdot v^\rho$. □

Lemma 3.5. *A universal Osborn loop is a 3-PAPL if and only if it is a* $\not{4}^1_{11\cdot 11=(1\cdot 11)1}$ *and a* $\not{4}^1_{11\cdot 11=(11\cdot 1)1}$ *loop.*

Proof. In Lemma 3.4, it was shown that $uu \cdot u\backslash(uu \cdot u) = (u \cdot uu)u$ in a universal Osborn loop. The necessary and sufficient parts are easy to prove using this identity. □

Lemma 3.6. *In a universal Osborn loop Q, the following are equivalent.*

1. *Q is a 3-PAPL.*
2. *Q is a* $\not{4}^1_{11\cdot 11=(1\cdot 11)1}$ *loop and a* $\not{4}^1_{11\cdot 11=(11\cdot 1)1}$ *loop.*
3. *Q is a LSIPL.*
4. *Q satisfies the identity* $v[v^\lambda \cdot (v \cdot v^\lambda\backslash v)] = v^\lambda\backslash v \cdot v$.
5. *Q is a* $\not{4}^{1,3}_{12\cdot 22=(1\cdot 22)2}$ *loop.*
6. *Q is a* $\not{4}^1_{11\cdot 11=(1\cdot 11)1}$ *loop.*

Proof. By using the identities $uu \cdot u\backslash(uu \cdot u) = (u \cdot uu)u$ and $v^\lambda \cdot (v \cdot v^\lambda\backslash v) = (v^\lambda \cdot vv)v$ and applying Lemmas 3.13, 3.16, 3.17. □

Corollary 3.7. *In a universal Osborn loop, the* $\not{4}^1_{11\cdot 11=(1\cdot 11)1}$ *and* $\not{4}^1_{11\cdot 11=(11\cdot 1)1}$ *loop properties are equivalent.*

Proof. This follows from Lemma 3.6. □

Corollary 3.8. *A universal Osborn loop that is a LSIPL or RSIPL or 3-PAPL or* $\measuredangle^{1,3}_{12\cdot 22=(1\cdot 22)2}$ *or* $\measuredangle^{1}_{11\cdot 11=(1\cdot 11)1}$ *loop is a* L^2*BSIPL and a* L^1*BSIPL.*

Proof. It is a consequence of Corollary 3.21, Lemma 3.14 and Lemma 3.6. $\square$

Theorem 3.9. *A loop* $(Q,\cdot,\backslash,/)$ *is a left universal Osborn loop if and only if it satisfies the identity*

$$OS_0^\lambda:\ x\cdot[(y\cdot zv)/v\cdot(xv)]=(x\cdot\{[y([v/(xv)]v)]/v\cdot(xv)\})/v\cdot[z\cdot xv]\quad or$$
$$OS_1^\lambda:\ x\cdot[(y\cdot zv)/v\cdot(xv)]=\{x\cdot[(y\cdot xv)/v\cdot(x\backslash v)]\}/v\cdot[z(xv)].$$

Proof. The procedure of the proof of this theorem is similar to the procedure used to prove Theorem 3.1 by just using the arbitrary left principal isotope $\mathfrak{Q}=(Q,\vartriangle,\nwarrow,\nearrow)$ such that $x\vartriangle y=xR_v^{-1}\cdot y=(x/v)\cdot y\ \forall\ v\in Q$. $\square$

Lemma 3.10. *A loop Q with the multiplication group* $\mathrm{Mult}(Q)$ *is a left universal Osborn loop if and only if the triple* $\big(\alpha(x,v),\beta(x,v),\gamma(x,v)\big)\in\mathrm{Aut}(Q)$ *or* $\big(R_{[xv]}\mathbb{R}_vR_{[x\backslash v]}\mathbb{R}_{[xv]}R_v\gamma(x,v)\mathbb{R}_v,\beta(x,v),\gamma(x,v)\big)\in\mathrm{Aut}(Q)$ *for all* $x,v\in Q$, *where* $\alpha(x,v)=R_{([v/(xv)]v)}\mathbb{R}_vR_{[xv]}L_x\mathbb{R}_v$, $\beta(x,v)=\mathbb{R}_vR_{[xv]}$ *and* $\gamma(x,v)=\mathbb{R}_vR_{[xv]}L_x$ *are elements of* $\mathrm{Mult}(Q)$.

Proof. From OS_0^λ or OS_1^λ of Theorem 3.9. $\square$

Theorem 3.11. *If a loop Q with the multiplication group* $\mathrm{Mult}(Q)$ *is a left universal Osborn loop, then* $\big(\gamma(x,v)\mathbb{R}_{[v^\lambda\cdot xv]},\beta(x,v),\gamma(x,v)\big)\in\mathrm{Aut}(Q)$ *for all* $x,v\in Q$, *where* $\beta(x,v)=\mathbb{R}_vR_{(xv)}$ *and* $\gamma(x,v)=\mathbb{R}_vR_{(xv)}L_x$ *are elements of* $\mathrm{Mult}(Q)$.

Proof. This follows from OS_0^λ or OS_1^λ of Theorem 3.9. $\square$

Lemma 3.12. *Let* $(Q,\cdot,\backslash,/)$ *be a left universal Osborn loop. The following identities are satisfied:*

$$\begin{aligned}
OSI_{01}^\lambda:&\quad y\{[v/(xv)]v\}=\{[y(xv)]/v\cdot(x\backslash v)\}/(xv)\cdot v,\\
OSI_{01.2}^\lambda:&\quad z\{(yv\cdot zv)/v\cdot z\backslash v\}=[z(y\cdot zv)]/(v^\lambda\cdot zv)\cdot v,\\
OSI_{01.1}^\lambda:&\quad \{z\cdot\{[(yv)(v/(zv)\cdot v)]/v\cdot zv\}\}/v\cdot v^\lambda(zv)=z\cdot y(vz),\\
OSI_{01.1.1}^\lambda:&\quad \{v^\lambda\{[(yv)(vv)]/v\}\}/v\cdot v^\lambda=v^\lambda y,
\end{aligned}$$

$$\begin{array}{ll} OSI^{\lambda}_{01.1.2}: & \{z\{[v(v/(zv)\cdot v)]z\}\}/v\cdot v^{\lambda}(zv)=z\cdot zv,\\ OSI^{\lambda}_{01.2.1}: & v\{(yv\cdot vv)/v\}=[v(y\cdot vv)]/(v^{\lambda}\cdot vv)\cdot v,\\ OSI^{\lambda}_{01.2.2}: & v[(v\cdot vv)/v]=(v\cdot vv)/(v^{\lambda}\cdot vv)\cdot v,\\ OSI^{\lambda}_{01.2.3}: & v[(vv\cdot vv)/v]=[v(v\cdot vv)]/(v^{\lambda}\cdot vv)\cdot v,\\ OSI^{\lambda}_{01.2.4}: & v^{\lambda}[y\cdot v^{\lambda}\backslash v]=(v^{\lambda}y)/v^{\lambda}\cdot v,\\ & v\cdot vv=v^{\lambda}\backslash v\cdot v,\qquad vv\cdot vv=v^{\lambda}\backslash(v^{\lambda^2}v)\cdot v. \end{array}$$

Proof. To prove these identities, we shall make use of the three autotopisms in Lemma 3.10 and Theorem 3.11. In a quasigroup, any two components of an autotopism uniquely determine the third. So equating the first components of the three autotopisms, it is easy to see that

$$\alpha(x,v)=\gamma(x,v)\mathbb{R}_{[v^{\lambda}\cdot xv]}=R_{[xv]}\mathbb{R}_vR_{[x\backslash v]}\mathbb{R}_{[xv]}R_v\gamma(x,v)\mathbb{R}_v.$$

This implies

$$R_{([v/(xv)]v)}\mathbb{R}_vR_{[xv]}L_x\mathbb{R}_v=R_{([v/(xv)]v)}\gamma(x,v)\mathbb{R}_v=R_{[xv]}\mathbb{R}_vR_{[x\backslash v]}\mathbb{R}_{[xv]}R_v\gamma(x,v)\mathbb{R}_v,$$

which gives $R_{([v/(xv)]v)}=R_{[xv]}\mathbb{R}_vR_{[x\backslash v]}\mathbb{R}_{[xv]}R_v$. So, for any $y\in Q$, $yR_{([v/(xv)]v)}=yR_{[xv]}\mathbb{R}_vR_{[x\backslash v]}\mathbb{R}_{[xv]}R_v$ implies $\mathrm{OSI}^{\lambda}_{01}$.

Let $\alpha(x,v)=\gamma(x,v)\mathbb{R}_{[v^{\lambda}\cdot xv]}$. Then for all $y\in Q$,

$$y\alpha(x,v)=yR_{([v/(xv)]v)}\mathbb{R}_vR_{[xv]}L_x\mathbb{R}_v=y\gamma(x,v)\mathbb{R}_{[v^{\lambda}\cdot xv]}=y\mathbb{R}_vR_{(xv)}L_x\mathbb{R}_{[v^{\lambda}\cdot xv]}.$$

Consequently, $\{x\cdot(\{y([v/(xv)]v)\}/v\cdot xv)\}/v=\{x\cdot(y/v\cdot xv)\}/[v^{\lambda}\cdot xv]$.

Replacing y/v by y and multiplying both sides by $[v^{\lambda}\cdot xv]$ we obtain $\mathrm{OSI}^{\lambda}_{01.1}$.

Now, consider $R_{[xv]}\mathbb{R}_vR_{[x\backslash v]}\mathbb{R}_{[xv]}R_v\gamma(x,v)\mathbb{R}_v=\gamma(x,v)\mathbb{R}_{[v^{\lambda}\cdot xv]}$. Then for all $y\in Q$, we have $yR_{[xv]}\mathbb{R}_vR_{[x\backslash v]}\mathbb{R}_{[xv]}R_v\gamma(x,v)\mathbb{R}_v=y\gamma(x,v)\mathbb{R}_{[v^{\lambda}\cdot xv]}$, i.e.,

$$(\{\big[[y(xv)]/v\cdot(x\backslash v)\big]/(xv)\cdot v\}\gamma(x,v))/v=\big(y\gamma(x,v)\big)/[v^{\lambda}\cdot xv]$$

which gives $x\{[y(xv)]/v\cdot(x\backslash v)\}=(x\cdot[y/v\cdot(xv)])/[v^{\lambda}\cdot xv]\cdot v$, whence, replacing y by yv, we obtain $\mathrm{OSI}^{\lambda}_{01.2}$.

$\mathrm{OSI}^{\lambda}_{01.1.1}$ and $\mathrm{OSI}^{\lambda}_{01.1.2}$ can be deduced from $\mathrm{OSI}^{\lambda}_{01.1}$; $\mathrm{OSI}^{\lambda}_{01.2.1}$ and $\mathrm{OSI}^{\lambda}_{01.2.4}$ are consequence of $\mathrm{OSI}^{\lambda}_{01.2}$ while identities $\mathrm{OSI}^{\lambda}_{01.2.2}$ and $\mathrm{OSI}^{\lambda}_{01.2.3}$ are deduced from $\mathrm{OSI}^{\lambda}_{01.2.1}$.

Putting $z=v^{\lambda}$ in $\mathrm{OSI}^{\lambda}_{01.1}$ we get $\mathrm{OSI}^{\lambda}_{01.1.1}$. Putting $y=e$ in $\mathrm{OSI}^{\lambda}_{01.1}$ we obtain $\mathrm{OSI}^{\lambda}_{01.1.2}$. Putting $z=v$ in $\mathrm{OSI}^{\lambda}_{01.2}$ we get $\mathrm{OSI}^{\lambda}_{01.2.1}$. Replacing in $\mathrm{OSI}^{\lambda}_{01.2.1}$ y by e we get $\mathrm{OSI}^{\lambda}_{01.2.2}$. Similarly, $\mathrm{OSI}^{\lambda}_{01.2.1}$ for $y=v$ gives

$\mathrm{OSI}^{\lambda}_{01.2.3}$; $\mathrm{OSI}^{\lambda}_{01.2}$ for $z = v^{\lambda}$ gives $\mathrm{OSI}^{\lambda}_{01.2.4}$. From $\mathrm{OSI}^{\lambda}_{01.1.1}$ for $y = e$ it follows $\{v^{\lambda}\{[v(vv)]/v\}\}/v \cdot v^{\lambda} = v^{\lambda}$ which implies $v^{\lambda}\{[v(vv)]/v\} = v$, hence, $v(vv) = (v^{\lambda}\backslash v) \cdot v$. Again, $\mathrm{OSI}^{\lambda}_{01.1.1}$ for $y = v$ proves $\{v^{\lambda}\{[(vv)(vv)]/v\}\}/v \cdot v^{\lambda} = e$ which implies $v^{\lambda}\{[(vv)(vv)]/v\} = v^{\lambda^2}v$, hence, $vv \cdot vv = v^{\lambda}\backslash(v^{\lambda^2}v) \cdot v$. □

Lemma 3.13. *A left universal Osborn loop is a LSIPL if and only if it is a* 3 *PAPL.*

Proof. This is proved by using $v \cdot vv = v^{\lambda}\backslash v \cdot v$ in Lemma 3.12. □

Lemma 3.14. *A left universal Osborn loop* $(Q, \cdot, \backslash, /)$ *is a* $\not{4}^{1}_{11\cdot 11=(11\cdot 1)1}$ *loop if and only if it obeys the identity* $v^{\lambda}(vv \cdot v) = v^{\lambda^2}v$.

Proof. This is proved by using $vv \cdot vv = v^{\lambda}\backslash(v^{\lambda^2}v) \cdot v$ in Lemma 3.12. □

Corollary 3.15. *A left universal Osborn loop* $(Q, \cdot, \backslash, /)$ *that is a* $\not{4}^{1}_{11\cdot 11=(11\cdot 1)1}$ *loop is a* L^1*BSIPL if and only if it is a LSIPL. Hence, it is a* L^2*BSIPL.*

Proof. This follows from Lemma 3.14 by using the fact that in an Osborn loop, $x^{\lambda^2} = x \mapsto x^{\lambda} \cdot xx$. □

Lemma 3.16. *A left universal Osborn loop is a LSIPL if and only if it is a* $\not{4}^{1,3}_{12\cdot 22=(1\cdot 22)2}$ *loop.*

Proof. This is proved by using the identity $\mathrm{OSI}^{\lambda}_{01.2.1}$ of Lemma 3.12. □

Lemma 3.17. *A left universal Osborn loop is a LSIPL if and only if it is a* $\not{4}^{1}_{11\cdot 11=(1\cdot 11)1}$ *loop.*

Proof. This is proved by using the identity $\mathrm{OSI}^{\lambda}_{01.2.3}$ of Lemma 3.12. □

Lemma 3.18. *Let* G *be a left universal Osborn loop. The following are equivalent:*

1. G *is a LSIPL and a* $\not{4}^{1,3}_{12\cdot 22=(12\cdot 2)2}$ *loop.*
2. G *is a left alternative property loop.*
3. G *is a Moufang loop.*

Proof. This is proved by using the identity $\mathrm{OSI}^{\lambda}_{01.2.1}$ of Lemma 3.12. □

Lemma 3.19. *In a left universal Osborn loop* $[y(yy \cdot y^{\rho})]y = y \cdot yy$.

Proof. This is proved by using the identity $\mathrm{OSI}^{\lambda}_{01.2}$ of Lemma 3.12. □

Lemma 3.20. *In an Osborn loop, the following properties are equivalent. LSIP, RSIP,* $|J_\lambda| = 2$, $|J_\rho| = 2$ *and* $J_\rho = J_\lambda$.

Proof. This can be proved by using the facts that in an Osborn loop, $J_\rho^2 \,:\, x \mapsto xx \cdot x^\rho$ and $J_\lambda^2 \,:\, x \mapsto x^\lambda \cdot xx$. □

Corollary 3.21. *In a left universal Osborn loop, the following properties are equivalent. LSIP, RSIP,* 3*-PAP,* $J_\rho = J_\lambda$, $4^{1,3}_{12\cdot 22=(1\cdot 22)2}$ *and* $4^{1}_{11\cdot 11=(1\cdot 11)1}$ *properties.*

Proof. Use Lemma 3.20, Lemma 3.13, Lemma 3.16 and Lemma 3.17. □

Corollary 3.22. *For a CC-loop L the following are equivalent:*

1. *L is a power associativity loop.*
2. *L is a 3-PAPL.*
3. *L obeys* $x^\rho = x^\lambda$ *for all* $x \in L$.
4. *L is a LSIPL.*
5. *L is a RSIPL.*
6. *L is a* $4^{1,3}_{12\cdot 22=(1\cdot 22)2}$ *loop.*
7. *L is a* $4^{1}_{11\cdot 11=(1\cdot 11)1}$ *loop.*

Proof. The proof of the equivalence of the first three is shown in Lemma 3.20 of [26] and mentioned in Lemma 1.2 of [31]. The proof of the equivalence of the last four and the first three can be deduced from the last result of Corollary 3.21. □

Corollary 3.23. *A CC-loop is a diassociative loop if and only if it is a power associative loop and a* $4^{1,3}_{12\cdot 22=(12\cdot 2)2}$ *loop.*

Proof. The proof of this follows from Corollary 3.22 and Lemma 3.18. □

Theorem 3.24. *A loop* $(Q, \cdot, \backslash, /)$ *is a right universal Osborn loop if and only if it satisfies the identity*

$$OS_0^\rho :\ (ux) \cdot u\backslash\{yz \cdot x\} = ((ux) \cdot u\backslash\{[y(u\backslash[u/x])] \cdot x\}) \cdot u\backslash[(uz)x] \quad \text{or}$$
$$OS_1^\rho :\ (ux) \cdot u\backslash\{(yz) \cdot x\} = \{(ux) \cdot u\backslash[yx \cdot (ux)\backslash u]\} \cdot u\backslash[(uz)x].$$

Proof. The proof is similar to the proof of Theorem 3.1 but now use the arbitrary right principal isotope $\mathfrak{Q} = (Q, \vartriangle, \nwarrow, \nearrow)$ such that

$$x \vartriangle y = x \cdot yL_u^{-1} = x \cdot (u\backslash y) \quad \forall\ u \in Q.$$ □

Lemma 3.25. *A loop Q with the multiplication group* $\mathrm{Mult(Q)}$ *is a right universal Osborn loop if and only if* $\big(\alpha(x,u),\beta(x,u),\gamma(x,u)\big)\in \mathrm{Aut}(Q)$ *or* $\big(R_{[u\backslash x]}R_{[x\backslash u]}\mathbb{R}_{[u\backslash x]}\gamma(x,u),\beta(x,u),\gamma(x,u)\big)\in \mathrm{Aut}(Q)$ *for all* $x,u\in Q$, *where* $\alpha(x,u)=R_{(u\backslash[u/(u\backslash x)])}R_{[u\backslash x]}\mathbb{L}_uL_x$, $\beta(x,u)=L_uR_{[u\backslash x]}\mathbb{L}_u$ *and* $\gamma(x,u)=R_{[u\backslash x]}\mathbb{L}_uL_x$ *are elements of* $\mathrm{Mult(Q)}$.

Proof. This is obtained by using identity OS_0^ρ or OS_1^ρ of Theorem 3.24. □

Theorem 3.26. *Let Q is a right universal Osborn loop with the multiplication group* $\mathrm{Mult(Q)}$, *then the triple* $\big(\gamma(x,u)\mathbb{R}_{(u\backslash x)},\beta(x,u),\gamma(x,u)\big)\in \mathrm{Aut}(Q)$ *for all* $x,u\in Q$, *where* $\beta(x,u)=L_uR_{[u\backslash x)]}\mathbb{L}_u$ *and* $\gamma(x,u)=R_{[u\backslash x]}\mathbb{L}_uL_x$ *are elements of* $\mathrm{Mult(Q)}$.

Proof. It is a consequence of OS_0^ρ or OS_1^ρ and the method used in the proof of Theorem 3.1. □

Theorem 3.27. *In a right universal Osborn loop* $(Q,\cdot,\backslash,/)$ *the following identities are satisfied:*

$$
\begin{aligned}
OSI^\rho_{01}: &\quad y\{u\backslash(u/x)\}=\{(yx)\cdot[(ux)\backslash u]\}/x,\\
OSI^\rho_{01.2}: &\quad \{(uz)\cdot u\backslash[(yz)[(uz)\backslash u]]\}z=(uz)\cdot u\backslash(yz),\\
OSI^\rho_{01.1}: &\quad \{(uz)\cdot u\backslash(\{y(u\backslash(u/z))\}\cdot z)\}z=(uz)\cdot u\backslash(yz),\\
OSI^\rho_{01.1.1}: &\quad \{(uz)\cdot u\backslash(\{z^\lambda(u\backslash(u/z))\}\cdot z)\}z=(uz)\cdot u^\rho,\\
OSI^\rho_{01.1.2}: &\quad \{(uu)\cdot u\backslash(u^\lambda u^\rho\cdot u)\}u=uu\cdot u^\rho,\\
OSI^\rho_{01.1.3}: &\quad \{(uz)\cdot u\backslash(\{z(u\backslash(u/z))\}\cdot z)\}z=(uz)\cdot u\backslash(zz),\\
OSI^\rho_{01.1.4}: &\quad \{u\backslash(\{u^\rho(u\backslash(u/u^\rho))\}\cdot u^\rho)\}z=u\backslash(u^\rho u^\rho),\\
OSI^\rho_{01.1.5}: &\quad \{(uz)\cdot u\backslash(\{z^\rho(u\backslash(u/z))\}\cdot z)\}z=(uz)\cdot u\backslash(z^\rho z),\\
OSI^\rho_{01.1.6}: &\quad z^\lambda\backslash[\{z^\rho(z^\lambda\backslash(z^\lambda/z))\}\cdot z]\cdot z=z^\lambda\backslash(z^\rho z),\\
OSI^\rho_{01.1.7}: &\quad \{(zz)\cdot z\backslash(z^\rho z^\rho\cdot z)\}z=(zz)\cdot z\backslash(z^\rho z),\\
OSI^\rho_{01.2.1}: &\quad \{(uz)\cdot u\backslash[(uz)\backslash u]\}z=(uz)\cdot u^\rho,\\
OSI^\rho_{01.2.2}: &\quad \{(uu)\cdot u\backslash[(uu)\backslash u]\}u=(uu)\cdot u^\rho,\\
OSI^\rho_{01.2.3}: &\quad \{(uu^\lambda)\cdot u\backslash[(uu^\lambda)\backslash u]\}u^\lambda=(uu^\lambda)\cdot u^\rho,\\
OSI^\rho_{01.2.4}: &\quad \{(uz)\cdot u\backslash[z[(uz)\backslash u]]\}z=(uz)\cdot u\backslash z,\\
OSI^\rho_{01.2.5}: &\quad \{(uu^\lambda)\cdot u\backslash[u^\lambda[(uu^\lambda)\backslash u]]\}u^\lambda=(uu^\lambda)\cdot u\backslash(uu^\lambda),
\end{aligned}
$$

$$\begin{aligned}
OSI^{\rho}_{01.2.6}: &\quad \{(uz)\cdot u\backslash[(zz)[(uz)\backslash u]]\}z = (uz)\cdot u\backslash(zz),\\
OSI^{\rho}_{01.2.7}: &\quad \{(uz)\cdot u\backslash[(z^{\rho}z)[(uz)\backslash u]]\}z = (uz)\cdot u\backslash(z^{\rho}z),\\
OSI^{\rho}_{01.2.8}: &\quad \{(uu)\cdot u\backslash[(u^{\rho}u)[(uu)\backslash u]]\}u = (uu)\cdot u\backslash(u^{\rho}u),\\
OSI^{\rho}_{01.2.9}: &\quad (uu\cdot u^{\rho})u^{\rho} = u\{u\backslash[(uu\cdot u^{\rho})u^{\rho}\cdot u]\cdot u^{\rho}\},\\
OSI^{\rho}_{01.2.10}: &\quad \{(uu^{\lambda})\cdot u\backslash[(uu^{\lambda})[(uu^{\lambda})\backslash u]]\}u^{\lambda} = (uu^{\lambda})\cdot u\backslash(uu^{\lambda}),\\
&\quad u\cdot[u\backslash(u^{\rho}u)]u^{\rho} = u^{\rho}.
\end{aligned}$$

Proof. To prove these identities, we shall make use of the three autotopisms in Lemma 3.25 and Theorem 3.26. Identifying the first components of these three autotopisms, we obtain

$$\alpha(x,u) = \gamma(x,u)\mathbb{R}_{(u\backslash x)} = R_{[u\backslash x]}R_{[x\backslash u]}\mathbb{R}_{[u\backslash x]}\gamma(x,u),$$

which we can deduce that $R_{(u\backslash[u/(u\backslash x)])}R_{[u\backslash x]}\mathbb{L}_u L_x = R_{[u\backslash x]}R_{[x\backslash u]}\mathbb{R}_{[u\backslash x]}\gamma(x,u)$ $= R_{[u\backslash x]}R_{[x\backslash u]}\mathbb{R}_{[u\backslash x]}R_{[u\backslash x]}\mathbb{L}_u L_x$. So, $R_{(u\backslash[u/(u\backslash x)])} = R_{[u\backslash x]}R_{[x\backslash u]}\mathbb{R}_{[u\backslash x]}$. Thus for any $y\in Q$, $yR_{(u\backslash[u/(u\backslash x)])} = yR_{[u\backslash x]}R_{[x\backslash u]}\mathbb{R}_{[u\backslash x]}$ implies $y(u\backslash[u/z]) = \{(yz)[(uz)\backslash u]\}/z$. This for $z = u\backslash x$, i.e., $x = uz$ gives $y(u\backslash[u/(u\backslash x)]) = \{[y(u\backslash x)][x\backslash u]\}/[u\backslash x]$, which is equivalent to OSI^{ρ}_{01}.

Now let $\alpha(x,u) = \gamma(x,u)\mathbb{R}_{(u\backslash x)}$. Then $y\alpha(x,u) = yR_{(u\backslash[u/(u\backslash x)])}R_{[u\backslash x]}\mathbb{L}_u L_x$ $= y\gamma(x,u)\mathbb{R}_{(u\backslash x)} = yR_{[u\backslash x]}\mathbb{L}_u L_x\mathbb{R}_{(u\backslash x)}$ for all $y\in Q$, which implies the identity $x\cdot u\backslash\{[y(u\backslash[u/(u\backslash x)])][u\backslash x]\} = \{x\cdot u\backslash[y(u\backslash x)]\}/(u\backslash x)$. Multiplying this identity by $(u\backslash x)$ to obtain $\{x\cdot u\backslash\{[y(u\backslash[u/(u\backslash x)])][u\backslash x]\}\}(u\backslash x) = x\cdot u\backslash[y(u\backslash x)]$, whence, for $z = u\backslash x$, i.e., $x = uz$ we deduce $\{(uz)\cdot u\backslash\{[y(u\backslash[u/z])]z\}\}z = (uz)\cdot u\backslash(yz)$. This proves $\mathrm{OSI}^{\rho}_{01.1}$.

If $R_{[u\backslash x]}R_{[x\backslash u]}\mathbb{R}_{[u\backslash x]}\gamma(x,u) = \gamma(x,u)\mathbb{R}_{(u\backslash x)}$, then $yR_{[u\backslash x]}R_{[x\backslash u]}\mathbb{R}_{[u\backslash x]}\gamma(x,u)$ $= y\gamma(x,u)\mathbb{R}_{(u\backslash x)}$ for all $y\in Q$. Hence $\{\{[y(u\backslash x)](x\backslash u)\}/[u\backslash x]\}\gamma(x,u) = [y\gamma(x,u)]/(u\backslash x)$, which after substitution of the value of $\gamma(x,u)$ and multiplication by $(u\backslash x)$ gives:

$$x\cdot u\backslash(\{\{[y(u\backslash x)](x\backslash u)\}/[u\backslash x]\}[u\backslash x]) = \{x\cdot u\backslash(y[u\backslash x])\}/(u\backslash x).$$

from this, for $z = u\backslash x$, i.e., $x = uz$, we obtain $\mathrm{OSI}^{\rho}_{01.2}$.

$\mathrm{OSI}^{\rho}_{01.1.1}$, $\mathrm{OSI}^{\rho}_{01.1.3}$ and $\mathrm{OSI}^{\rho}_{01.1.5}$ can be deduced from $\mathrm{OSI}^{\rho}_{01.1}$; $\mathrm{OSI}^{\rho}_{01.1.2}$ is a consequence of $\mathrm{OSI}^{\rho}_{01.1.1}$; $\mathrm{OSI}^{\rho}_{01.1.4}$ is deduced from $\mathrm{OSI}^{\rho}_{01.1.3}$.

Putting $y = z^\lambda$ in $\mathrm{OSI}^\rho_{01.1}$ we obtain $\mathrm{OSI}^\rho_{01.1.1}$. Substituting in $\mathrm{OSI}^\rho_{01.1.1}$ z by u we get $\mathrm{OSI}^\rho_{01.1.2}$. Putting $y = z$ in $\mathrm{OSI}^\rho_{01.1}$ we obtain $\mathrm{OSI}^\rho_{01.1.3}$. From this $z = u^\rho$ we conclude $\mathrm{OSI}^\rho_{01.1.4}$. Putting $y = z^\rho$ in $\mathrm{OSI}^\rho_{01.1}$ we obtain $\mathrm{OSI}^\rho_{01.1.5}$. The last for $u = z^\lambda$ gives $\mathrm{OSI}^\rho_{01.1.6}$. From this for $u = z$ we deduce $\mathrm{OSI}^\rho_{01.1.7}$. Similarly, $\mathrm{OSI}^\rho_{01.2}$ for $y = z^\lambda$ gives $\mathrm{OSI}^\rho_{01.2.1}$. This for $z = u$ implies $\mathrm{OSI}^\rho_{01.2.2}$. Putting $z = u^\lambda$ in $\mathrm{OSI}^\rho_{01.2.1}$ we obtain $\mathrm{OSI}^\rho_{01.2.3}$. $\mathrm{OSI}^\rho_{01.2}$ for $y = e$ gives $\mathrm{OSI}^\rho_{01.2.4}$, whence for $z = u^\lambda$ we obtain $\mathrm{OSI}^\rho_{01.2.5}$. Putting $y = z$ in $\mathrm{OSI}^\rho_{01.2}$ we get $\mathrm{OSI}^\rho_{01.2.6}$. $\mathrm{OSI}^\rho_{01.2}$ for $y = z^\rho$ gives $\mathrm{OSI}^\rho_{01.2.7}$. The last for $z = u$ implies $\mathrm{OSI}^\rho_{01.2.8}$, for $z = u^\rho$ – $\mathrm{OSI}^\rho_{01.2.9}$. Putting $z = u^\lambda$ in $\mathrm{OSI}^\rho_{01.2.7}$ we obtain $\mathrm{OSI}^\rho_{01.2.10}$. $\mathrm{OSI}^\rho_{01.2.4}$ for $z = u^\rho$ proves $u \cdot [u\backslash(u^\rho u)]u^\rho = u^\rho$. □

Corollary 3.28. *A right universal Osborn loop $(Q, \cdot, \backslash, /)$ is a RSIPL if and only if it satisfies the identity* $u^\lambda u^\rho \cdot u = u(uu)^\rho$.

Proof. By $\mathrm{OSI}^\rho_{01.1.2}$. □

Corollary 3.29. *A right universal Osborn loop $(Q, \cdot, \backslash, /)$ is a RSIPL if and only if it satisfies the identity* $u^\rho u^\rho = u[u\backslash(u^\rho u \cdot u^\rho) \cdot u^\rho]$.

Proof. By $\mathrm{OSI}^\rho_{01.1.4}$. □

Corollary 3.30. *A right universal Osborn loop $(Q, \cdot, \backslash, /)$ is a RSIPL if and only if it satisfies* $z^\lambda\backslash[z^\rho z^\lambda \cdot z] \cdot z = z^\lambda\backslash(z^\rho z)$.

Proof. By $\mathrm{OSI}^\rho_{01.1.6}$. □

Corollary 3.31. *A right universal Osborn loop $(Q, \cdot, \backslash, /)$ satisfies the identity* $zz \cdot z^\lambda = z$ *if and only if holds* $[zz \cdot z\backslash z^\rho]z = zz \cdot z\backslash(z^\rho z)$.

Proof. By $\mathrm{OSI}^\rho_{01.1.8}$. □

Corollary 3.32. *A right universal Osborn loop $(Q, \cdot, \backslash, /)$ satisfying the identity* $[zz \cdot z\backslash z^\rho]z = zz \cdot z\backslash(z^\rho z)$ *is a SFAIPL if and only if it is a SWIPL.*

Proof. By $\mathrm{OSI}^\rho_{01.1.8}$. □

Corollary 3.33. *A right universal Osborn loop $(Q, \cdot, \backslash, /)$ with the RSIP is a SFAIPL if and only if it satisfies the identity* $u \cdot u[u\backslash u^\rho \cdot u^\rho] = u^\rho$.

Proof. By Corollary 3.28, Corollary 3.29 and Corollary 3.20. □

Corollary 3.34. *A right universal Osborn loop $(Q, \cdot, \backslash, /)$ with the RSIP satisfies the identity* $u\backslash u^\rho = (uu)^\rho$.

Proof. By $\mathrm{OSI}^{\rho}_{01.2.2}$. □

Corollary 3.35. *A right universal Osborn loop $(Q, \cdot, \backslash, /)$ with the RSIP is a SFAIPL and $|J_\rho| = 6$.*

Proof. The first part follows from Corollary 3.34 and Corollary 3.20, the second can be deduced from the fact that SFAIPL implies $x^{\rho\rho\rho\rho\rho\rho} = x$ (sf. [21], p. 18). □

Corollary 3.36. *A right universal Osborn loop $(Q, \cdot, \backslash, /)$ satisfies the identity $uu^\lambda \cdot u^\rho = u^\lambda$ if and only if it satisfies the identity $u = (uu^\lambda) \cdot u(uu^\lambda)^\rho$.*

Proof. By $\mathrm{OSI}^{\rho}_{01.2.3}$. □

Corollary 3.37. *A right universal Osborn loop $(Q, \cdot, \backslash, /)$ satisfies the identity $u^\rho u = uu^\lambda$ if and only if it satisfies the identity $u \cdot u^\lambda u^\rho = u^\rho$.*

Proof. By using the identity $u \cdot [u\backslash(u^\rho u)]u^\rho = u^\rho$. □

Corollary 3.38. *A right universal Osborn loop $(Q, \cdot, \backslash, /)$ satisfies the identity $u^\rho u = uu^\lambda$ if and only if it satisfies the identity $\{(uu) \cdot u\backslash[(u^\rho u)[(uu)\backslash u]]\}u = uu \cdot u^\lambda$.*

Proof. By $\mathrm{OSI}^{\rho}_{01.2.8}$. □

Corollary 3.39. *A right universal Osborn loop $(Q, \cdot, \backslash, /)$ satisfying the identity $u^\rho u = uu^\lambda$ and the RSIP is a SWIPL.*

Proof. By Corollary 3.38. □

4. Concluding remarks and future studies

OSI_{01}, $\mathrm{OSI}_{01....}$; OSI^{ρ}_{01}, $\mathrm{OSI}^{\rho}_{01....}$ and $\mathrm{OSI}^{\lambda}_{01}$, $\mathrm{OSI}^{\lambda}_{01....}$ are all newly discovered identities that are true in universal, right universal and left universal Osborn loops respectively. So, all these identities are satisfied by any Moufang loop, extra loop, CC-loop, universal WIPL and VD-loop. This is a good news for CC-loop which has just received a tremendious growth increase by the works of Kinyon, Kunen, Drapal, Phillips e.t.c and especially for VD-loops which is yet to grow in study compared to CC-loops. We hope VD-loops will catch the attention of researchers with the newly found identities. A trilling observation in this study is the fact that identities $\mathrm{OSI}_{01}^{\lambda}$ and OSI_{01} are of the forms

$$[y(x^{-1}v)\cdot v^{-1}](xv) = [y(xv)\cdot v^{-1}](x^{-1}v) \qquad \text{and}$$

$$y\{u^{-1}([(uv)(v^{-1}x^{-1}\cdot u)]v)\} = \{(y[u^{-1}(xv)])v^{-1}\cdot x^{-1}(uv)\}[v^{-1}x^{-1}\cdot u]\cdot v,$$

respectively, in a Moufang loop or extra loop. If a Moufang or extra loop is of exponent 2 then, the first identity will be obviously true. Basarab [5] has shown that an Osborn loop of exponent 2 is an abelian group. So it is not wise to study identity $\mathrm{OSI}_{01}^{\lambda}$ for a loop of exponent 2 e.g. Steiner loops, but identity OSI_{01} can be studied for such a loop.

According to Phillips [31], a chain of five prominent varieties of CC-loops are: (1) groups, (2) extra loops, (3) WIP PACC-loops, (4) PACC-loops and (5) CC-loops. He was able to axiomatize the variety of WIP PACC-loops. With our new loop properties that are weaker forms of well known loop properties like inverse property, power associativity and diassociativity, we now have subvarieties of varieties of CC-loops mentioned above. It will be interesting to axiomatize some of them e.g. SWIP PACC-loops. These new algebraic properties give more insight into the algebraic properties of universal Osborn loops. Particularly, it can be used to fine tune some recent equations on CC-loop as shown in works of Kunen, Kinyon, Phillips and Drapal; [24, 22, 23], [13, 14], [26].

The continuation of this study will switch to the notations of Bryant and Schneider [9] for principal isotopes of quasigroups (loops) and use their results to deduce more algebraic equations for universal Osborn loops.

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