LGS–quasigroups

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Abstract. The concept of a LGS–quasigroup is defined and investigated in this paper. The geometric concepts of parallelograms and midpoints are introduced in a general LGS–quasigroup and the geometrical interpretation in the LGS–quasigroup $C(\frac{1}{2}(3+\sqrt{5}))$ is given. The theorem about the characterization of LGS–quasigroup by means of a commutative group is proved. RGS–quasigroup is introduced and the connection with LGS–qusigroup is investigated.

1. Definition and examples of LGS–quasigroups

We say that the quasigroup $(Q,\cdot)$ is the *left quasigroup of the golden section* or shorter *LGS–quasigroup*, if the identity of idempotency

$$aa = a, \tag{1}$$

and besides that the identity

$$ab \cdot c = cb \cdot a \tag{2}$$

is valid.

Example 1. Let $(G,+)$ be a commutative group in which there is an automorphism φ which satisfies the identity

$$\varphi^2(a) - 3\varphi(a) + a = 0. \tag{3}$$

If the binary operation $\cdot$ on the set G is defined by the identity

$$ab = a + \varphi(b-a), \tag{4}$$

then $(G,\cdot)$ is a LGS–quasigroup. We shall prove it here.

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For each $a, b \in G$ the equations $ax = b$, $ya = b$ are equivalent because of (4) with the equations

$$a + \varphi(x - a) = b,$$
$$y + \varphi(a) - \varphi(y) = b, \tag{5}$$

of which the first one has the unique solution $x = a + \varphi^{-1}(b - a)$, and the second one can be written in the form

$$\varphi^2(y) - \varphi(y) - \varphi^2(a) = -\varphi(b). \tag{6}$$

By addition of the equations (5) and (6) because of (3) we get the identity

$$\varphi(y) - \varphi^2(a) + \varphi(a) = -\varphi(b) + b$$

with the unique solution

$$y = \varphi^{-1}[\varphi^2(a) - \varphi(a) - \varphi(b) + b] = \varphi(a) - a - b + \varphi^{-1}(b),$$

which clearly shows that it satisfies the identity (5). The idempotency of the quasigroup $(G, \cdot)$ is obvious. Because of (4) and after making some arrangements we get

$$ab \cdot c = \varphi^2(a) - 2\varphi(a) + a + \varphi(b) - \varphi^2(b) + \varphi(c),$$

namely, because of (3) finally we get

$$ab \cdot c = \varphi(a) + \varphi(b) - \varphi^2(b) + \varphi(c). \tag{7}$$

The symmetry of the identity (7) with respect to a and c proves the identity (2).

Later we shall prove that the Example 1 is a characteristic example of LGS–quasigroup, namely we can get each LGS–quasigroup from a certain commutative group in the way given in Example 1.

Example 2. Let $(F, +, \cdot)$ be a field in which the equation

$$q^2 - 3q + 1 = 0 \tag{8}$$

has a solution q, and the operation $*$ on the set F is defined by the formula

$$a * b = (1 - q)a + qb. \tag{9}$$

Then by means of $\varphi(a) = qa$ an automorphism of the commutative group $(F, +)$ is obviously defined, and as the identity (8) is valid, for each $a \in F$ the identity (3) is valid. However, the equality (9) can be written in the form

$$a * b = a + \varphi(b - a),$$

so based on the Example 1 it follows that $(F, *)$ is a LGS–quasigroup.

Example 3. Let in the field $(C, +, \cdot)$ of complex numbers the operation $*$ be defined by the formula (9), where $q = \frac{1}{2}(3 + \sqrt{5})$ or $q = \frac{1}{2}(3 - \sqrt{5})$. Then the identity (8) is valid thus, based on Example 2, it follows that $(C, *)$ is LGS–quasigroup. This quasigroup has a nice geometric interpretation which can serve as a motivation for the study of LGS–quasigroups.

Let us consider the complex numbers as the points of the Euclidean plane. For any two different points a, b the formula (9) can also be written in the form

$$\frac{a * b - a}{b - a} = q$$

which means that the point $a * b$ divides the pair of points a, b in the ratio q. If $q = \frac{1}{2}(3 + \sqrt{5})$ namely $q = \frac{1}{2}(3 - \sqrt{5})$, then the point b divides the pair $a, a * b$ respectively the point $a * b$ divides the pair a, b in the ratio of golden section, which justifies the name (left) quasigroup of the golden section.

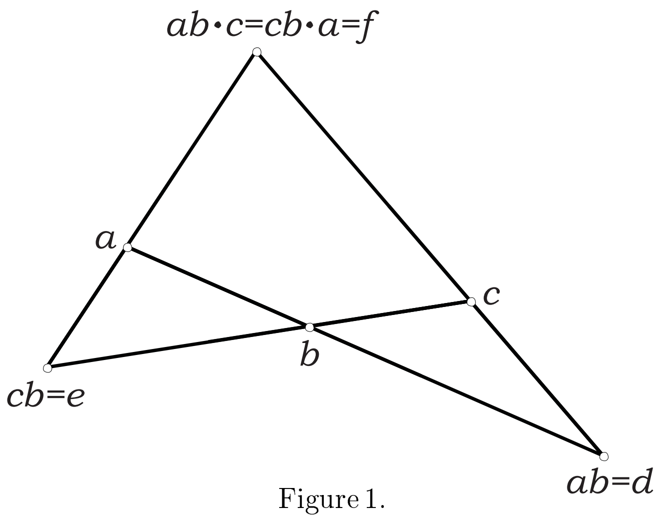

Figure 1.

Each identity in LGS–quasigroup $(C, *)$ interprets a certain geometric theorem, which could naturally also be proved directly, but the theory of LGS–quasigroup gives a better insight in the mutual relations of such theorems. So, for example on the Figure 1 the identity (2) is illustrated in the case of LGS–quasigroup $(C, *)$ with $q = \frac{1}{2}(3 + \sqrt{5})$, while instead of the symbol "$*$" the symbol $\cdot$ is used. (We are going to use this way of marking in the following figures.) All the figures will be presented in the mentioned quasigroup with $q = \frac{1}{2}(3 + \sqrt{5})$, but if in all the products of the form $x * y = z$ we exchange the roles of the elements y, z, on the same pictures, we shall get the illustration for the quasigroup $(C, *)$ with $q = \frac{1}{2}(3 - \sqrt{5})$.

2. The basic properties of LGS–quasigroups

Further, let $(Q, \cdot)$ be a LGS–quasigroup. The elements of the set Q are called the points. The relationship of two quasigroups from Example 3 suggests that the next theorem is valid.

Theorem 1. *If $(Q, \cdot)$ is a LGS–quasigroup and if the operation $\bullet$ on the set Q is defined by the equivalency*

$$a \bullet b = c \longleftrightarrow ac = b, \tag{10}$$

then $(Q, \bullet)$ is the LGS–quasigroup, too.

Proof. As the groupoid $(Q, \bullet)$ is conjugated (in the terminology of S. K. Stein [1]) to the quasigroup $(Q, \cdot)$, then $(Q, \bullet)$ is also a quasigroup. The idempotency of that quasigroup is obvious. We have to prove the identity

$$(a \bullet b) \bullet c = (c \bullet b) \bullet a,$$

i.e., the implication

$$a \bullet b = d,\ c \bullet b = e,\ d \bullet c = f \ \longrightarrow e \bullet a = f.$$

However, because of (10), that implication is equivalent to the implication

$$ad = b,\ ce = b,\ df = c \ \longrightarrow ef = a,$$

which should be proved. Because of (2) the assumptions of that implication imply successively

$$ef \cdot d = df \cdot e = ce = b = ad,$$

wherefrom $ef = a$ follows. □

In [3] the concept of GS–quasigroup is defined. A quasigroup $(Q,\cdot)$ is said to be GS–quasigroup if it is idempotent and if it satisfies the (mutually equivalent) identities

$$a(ab\cdot c)\cdot c=b, \qquad a\cdot(a\cdot bc)c=b.$$

Let us prove now

Theorem 2. *If the operations $\cdot$, $\bullet$ on the set Q are such that the equivalency*

$$ab=c\longleftrightarrow c\bullet b=a \tag{11}$$

is valid, then $(Q,\cdot)$ is a LGS–quasigroup if and only if $(Q,\bullet)$ is a GS–quasigroup.

Proof. It is obvious that $(Q,\cdot)$ is a quasigroup if and only if $(Q,\bullet)$ is a quasigroup, and the operation $\cdot$ is idempotent if and only if the operation $\bullet$ is idempotent too. The identity (2) can be written in the equivalent form as the implication

$$ab=d,\ cb=e,\ dc=f\ \longrightarrow ea=f,$$

i.e., because of (11) as the implication

$$d\bullet b=a,\ e\bullet b=c,\ f\bullet c=d\ \longrightarrow f\bullet a=e$$

or, after some eliminations, as the identity

$$f\bullet((f\bullet(e\bullet b))\bullet b)=e.$$

However, the last identity together with the idempotency characterizes GS–quasigroup $(Q,\bullet)$. □

Theorem 2 justifies the name LGS–quasigroup. Based on that theorem and on the properties of GS–quasigroup using (11) the properties of LGS–quasigroup can be deduced. Because it is not always simple, we will further deduce the properties of LGS–quasigroup $(Q,\cdot)$ independently from the theory of GS–quasigroup.

Theorem 3. *LGS–quasigroups are medial, i.e., the identity*

$$ab\cdot cd=ac\cdot bd \tag{12}$$

is valid.

Proof. Because of (2) we have successively
$$ab \cdot cd = (cd \cdot b)a = (bd \cdot c)a = ac \cdot bd. \qquad \square$$

Therefore $(Q, \cdot)$ is an IM–quasigroup, and it satisfies all the results from [2] and [4] with the geometric concepts of translation, paralellogram, midpoint and addition of points, defined in [2].

The identities of elasticity, right and left distributivity, as a consequence of the identities of idempotency and mediality, are valid in LGS–quasigroup, namely we have the following identities

$$ab \cdot a = a \cdot ba, \tag{13}$$

$$ab \cdot c = ac \cdot bc, \tag{14}$$

$$a \cdot bc = ab \cdot ac. \tag{15}$$

Besides that, out of (2) because of idempotency we get the identity

$$ab \cdot b = ba, \tag{16}$$

which is also valid in LGS–quasigroup $(Q, \cdot)$.

Theorem 4. *Any three of the four equalities*

$$ab = d, \tag{17}$$

$$cb = e, \tag{18}$$

$$dc = f, \tag{19}$$

$$ea = f, \tag{20}$$

imply the remaining equalities (*Figure 1*).

Proof. The substitutions $a \longleftrightarrow c$ and $d \longleftrightarrow e$ give the substitutions (17) $\longleftrightarrow$ (18) and (19) $\longleftrightarrow$ (20), so it is enough to prove the implications

$$(17) \;\&\; (18) \;\&\; (20) \;\longrightarrow\; (19)$$

$$(17) \;\&\; (19) \;\&\; (20) \;\longrightarrow\; (18),$$

i.e., that with the presumable equalities (17) and (20) the equalities (18) and (19) are mutually equivalent. We get successively

$$cb \cdot a \overset{(2)}{=} ab \cdot c \overset{(17)}{=} dc,$$

which together with (20) results in an equivalency of the equalities $cb \cdot a = ea$ and $dc = f$, i.e., the equalities (18) i (19). $\square$

Theorem 5. *Any two of the four equalities*

$$ab = d, \tag{21}$$

$$dc = a, \tag{22}$$

$$bc = d, \tag{23}$$

$$cb = a, \tag{24}$$

imply the remaining equality (Figure 2) .

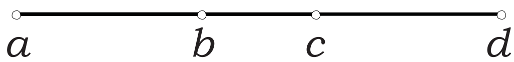

Figure 2.

Proof. The substitutions $a \longleftrightarrow d$, $b \longleftrightarrow c$ give the substitutions (21) $\longleftrightarrow$ (22) and (23) $\longleftrightarrow$ (24). Because of that, for the proof of the implications (21) & (22) $\longrightarrow$ (23) & (24) and (23) & (24) $\longrightarrow$ (21) & (22) it is sufficient to prove the implications

$$(21)\ \&\ (22) \longrightarrow (23) \tag{25}$$

$$(23)\ \&\ (24) \longrightarrow (22), \tag{26}$$

and from other implications

$$(21)\ \&\ (23) \longrightarrow (22)\ \&\ (24), \tag{27}$$

$$(21)\ \&\ (24) \longrightarrow (22)\ \&\ (23), \tag{28}$$

$$(22)\ \&\ (24) \longrightarrow (21)\ \&\ (23),$$

$$(22)\ \&\ (23) \longrightarrow (21)\ \&\ (24)$$

it is sufficient to prove the implications (27) and (28). We have successively

$$\begin{aligned} bc \cdot d &\overset{(2)}{=} dc \cdot b \overset{(22)}{=} ab \overset{(21)}{=} d \overset{(1)}{=} dd, \\ d \cdot dc &\overset{(23)}{=} bc \cdot dc \overset{(14)}{=} bd \cdot c \overset{(2)}{=} cd \cdot b \overset{(23)}{=} (c \cdot bc)b \\ &\overset{(13)}{=} (cb \cdot c)b \overset{(24)}{=} ac \cdot b \overset{(2)}{=} bc \cdot a \overset{(23)}{=} da, \end{aligned}$$

wherefrom the equalities (23) and (22) follow, which proves the implications (25) and (26). Further we get

$$\begin{aligned} bc &\overset{(16)}{=} cb \cdot b \overset{(24)}{=} ab \overset{(21)}{=} d, \\ cb \cdot b &\overset{(16)}{=} bc \overset{(23)}{=} d \overset{(21)}{=} ab, \end{aligned}$$

wherefrom the equalities (23) and (24) follow which proves the implications (21) & (24) $\longrightarrow$ (23) and (21) & (23) $\longrightarrow$ (24). The first one of the two implications together with (26) prove the implication (28), and the second one of these implications together with (26) prove the implication (27). $\square$

3. Parallelograms and midpoints in LGS–quasigroups

Equally, as in each medial quasigroup, based on the results from [2] we shall say that (a, b, c, d) is a *parallelogram* and we shall write $\mathrm{Par}(a, b, c, d)$ if there are two points p, q so that $pa = qb$ and $pd = qc$ are valid, respectively, as an equivalent to this, if there are two points u, v so that $au = bv$ and $du = cv$ are valid. From the two points p, q respectively u, v one can be chosen arbitrary, and then the second one is uniquely determined so that the mentioned equalities are valid. In [2] it was proved that (Q,Par) is a parallelogram space, i.e., the quaternary relation $\mathrm{Par} \subset Q^4$ has the following properties:

(P1) For any three points a, b, c there is one and only one point d such that $\mathrm{Par}(a, b, c, d)$,

(P2) If (e, f, g, h) is any cyclic permutation of (a, b, c, d) or of (d, c, b, a) then $\mathrm{Par}(a, b, c, d)$ implies $\mathrm{Par}(e, f, g, h)$,

(P3) From $\mathrm{Par}(a, b, c, d)$ and $\mathrm{Par}(c, d, e, f)$ it follows $\mathrm{Par}(a, b, f, e)$.

Let us prove:

Theorem 6. *In the assumptions of the Theorem 5 the statement Par*(a, b, d, c) *is valid.*

Proof. We have the equalities $ab = bc$, $cb = dc$. □

Theorem 7. *The statement Par*(a, d, e, f) *is valid, if and only if there are two points* b, c *so that* $d = ab$, $e = bc$, $f = ab \cdot c$ *(Figure 3).*

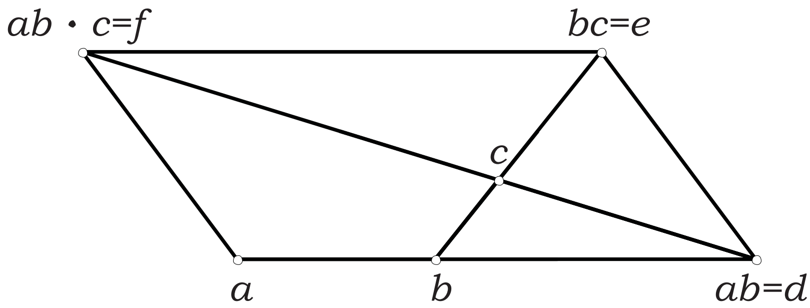

Figure 3.

Proof. Let a, b, c be any points and $d = ab$, $e = bc$, $f = ab \cdot c$. Then because of (1) respectively (2) we have $ab = ab \cdot ab$, $(ab \cdot c)b = bc \cdot ab$, so the statement $\mathrm{Par}(a, ab, bc, ab \cdot c)$, i.e., $\mathrm{Par}(a, d, e, f)$ is valid.
Conversely, let $\mathrm{Par}(a, d, e, f)$ be valid. There is the point b so that $ab = d$, and then the point c so that $bc = e$. Based on the first part of the theorem the statement $\mathrm{Par}(a, ab, bc, ab \cdot c)$, i.e., $\mathrm{Par}(a, d, e, ab \cdot c)$ is valid, which together with $\mathrm{Par}(a, d, e, f)$, according to (P1), gives the equality $f = ab \cdot c$. □

Theorem 8. *If* o, a, b *are any points,* $ab = c'$, *and the point* c *so that* $c'c = o$, *and* $bc = d'$, *and the point* d *so that* $d'd = o$, *then the statements Par*(o, a, c', d'), *Par*(a, b, d', d), *Par*(o, b, c', d) *are valid (Figure 4).*

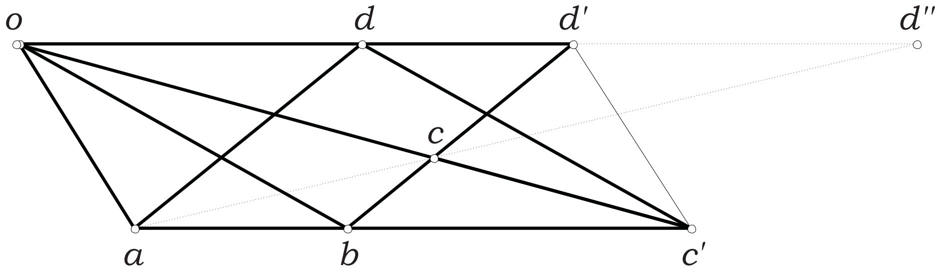

Figure 4.

Proof. We have $\mathrm{Par}(a, ab, bc, ab \cdot c)$, i.e., $\mathrm{Par}(a, c', d', o)$ or $\mathrm{Par}(o, a, c', d')$ according to Theorem 7. Let $d'' = ac$. Then we get equalities

$$d''d' = ac \cdot bc \overset{(14)}{=} ab \cdot c = c'c = o,$$

$d'd = o$, wherefrom according to Theorem 5 the equality $dd' = d''$ follows. From the equalities $ac = dd'$, $bc = d'd'$ the statement $\mathrm{Par}(a, d, d', b)$, i.e., $\mathrm{Par}(a, b, d', d)$ follows. Finally, from $\mathrm{Par}(0, a, c', d')$ and $\mathrm{Par}(a, d, d', b)$ because of Theorem 24 from [2] follows $\mathrm{Par}(d, c', b, o)$, i.e., $\mathrm{Par}(o, b, c', d)$ follows. □

We shall say that the point c is the *midpoint* of the pair of points a, b and we write $M(a,\ c,\ b)$ if the statement $\mathrm{Par}(a, c, b, c)$ is valid.

Besides the results listed and proved in [2] and [4] the following statement is valid

Theorem 9. *From $ab = c$ and $bc = d$ follows $M(a, c, d)$ (Figure 5).*

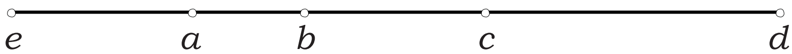

Figure 5.

Proof. There is the point e so that $ba = e$. From $ab = c$, $ba = e$, according to Theorem 5 follows $cb = e$, and from $bc = d$, $cb = e$, according to the same theorem, follows $dc = e$. So we have $ba = dc$, $bc = dd$, which proves $\mathrm{Par}(a, c, d, c)$, i.e., $M(a, c, d)$. □

4. The characterization of the LGS–quasigroup

Let 0 be a given point. For any two points a, b we define the sum $a + b$ by the equivalency

$$a + b = c \longleftrightarrow Par(0, a, c, b). \tag{29}$$

In [4] it is proved that $(Q, +)$ is a commutative group with the neutral element 0, and the identity

$$ab = a0 + 0b \tag{30}$$

is also proved, which can be proved directly. Really, because of (16) and (2) we get

$$0a = a0 \cdot 0,\ 0b \cdot a = ab \cdot 0,$$

so $\mathrm{Par}(0, a0, ab, 0b)$ is valid, which gives the identity (30), because of (29). In [2] it is proved that from (30) the identity

$$ab = a + \lambda_o(b - a) \tag{31}$$

follows, where $\lambda_o : x \to 0x$ is left translation of the quasigroup $(Q, \cdot)$ which is also the automorphism of the group $(Q, +)$. Let us prove now

Theorem 10. *For each point a the identity*

$$\lambda_o^2(a) - 3\lambda_o(a) + a = 0 \tag{32}$$

is valid.

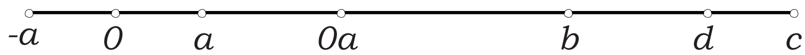

Figure 6.

Proof. Let first be $b = 2\lambda_o(a) = 0a + 0a$, i.e., because of (29) $\mathrm{Par}(0, 0a, b, 0a)$. As because of (13) $0 \cdot a0 = 0a \cdot 0$ is valid, so with $u = a0$, $v = 0$ from the definition of the parallelogram the identity

$$0a \cdot a0 = b0 \tag{33}$$

follows. If now $c = 3\lambda_o(a) = 0a+b$, then because of (29) we have $\mathrm{Par}(0, 0a, c, b)$, thus there is the point p so that $00 = 0a \cdot p$ and

$$b0 = cp \tag{34}$$

is valid.

The first of these two statements gets the form $pa \cdot 0 = 00$ because of (2), wherefrom

$$pa = 0 \tag{35}$$

follows.

From the equality $a + (-a) = 0$ follows $\mathrm{Par}(0, a, 0, -a)$, and as because of (16) $a0 = 0a \cdot a$ is valid, the identity

$$a(-a) = 0a \cdot 0 \tag{36}$$

follows. Let

$$d = 3\lambda_o(a) - a, \tag{37}$$

i.e., $d = -a + c$, so $\mathrm{Par}(0, -a, d, c)$ is valid, and as because of (36) we have $0a \cdot 0 = a(-a)$, so the identity

$$0a \cdot c = ad \tag{38}$$

is valid. Now, we have successively

$$\begin{aligned} ad &\overset{(38)}{=} 0a \cdot c \overset{(2)}{=} ca \cdot 0 \overset{(35)}{=} ca \cdot pa \overset{(14)}{=} cp \cdot a \overset{(34)}{=} b0 \cdot a \\ &\overset{(33)}{=} (0a \cdot a0)a \overset{(14)}{=} (0a \cdot a)(a0 \cdot a) \overset{(16),(13)}{=} a0 \cdot (a \cdot 0a) \overset{(15)}{=} a(0 \cdot 0a), \end{aligned}$$

wherefrom follows

$$d = 0 \cdot 0a = \lambda_o^2(a),$$

which together with (37) proves the identity (32). □

Comparing formulas (3) and (32), then the formulas (4) and (31) it follows that each LGS–quasigroup can be got in the way as in the Example 1, i.e., that it is valid

Theorem 11. *LGS–quasigroup $(Q, \cdot)$ exists if and only if there exists a commutative group $(Q, +)$ and its automorphism φ so that the identity (3) is valid. If the commutative group $(Q, +)$ is given and its automorphism φ, then the operation $\cdot$ is defined by the formula (4), and if the LGS–quasigroup $(Q, \cdot)$ is given and the element $0 \in Q$, then the operation $+$ is defined by the formula*

$$a + b = \rho_o^{-1}(a) \cdot \lambda_o^{-1}(b) \tag{39}$$

and 0 is the neutral element of the group $(Q, +)$, where $\varphi = \lambda_o$ and ρ_o are the left and the right translations of the quasigroup $(Q, \cdot)$ defined by the element 0.

In fact, the formula (39) follows from (30) by the substitution of the variables a, b with the variables $\rho_o^{-1}(a)$, $\lambda_o^{-1}(b)$.

5. RGS–quasigroups

We say that the quasigroup $(Q, \cdot)$ is the *right quasigroup of the golden section* or shorter *RGS–quasigroup*, if the identity of idempotency

$$aa = a, \tag{40}$$

and besides that the identity

$$a \cdot bc = c \cdot ba. \tag{41}$$

is valid.

Example 4. Let $(G,+)$ be a commutative group in which there is an automorphism φ which satisfies the identity

$$\varphi^2(a) + \varphi(a) - a = 0. \tag{42}$$

If the binary operation $\cdot$ on the set G is defined by the identity

$$ab = a + \varphi(b-a), \tag{43}$$

then it can be proved that $(G,\cdot)$ is a RGS–quasigroup.

Example 5. Let $(F,+,\cdot)$ be a field in which the equation

$$q^2 + q - 1 = 0 \tag{44}$$

has the solution q, and the operation $*$ on the set F is defined by the formula

$$a * b = (1-q)a + qb. \tag{45}$$

Then it can be proved that $(F,*)$ is a RGS–quasigroup.

Example 6. Let in the field $(C,+,\cdot)$ of complex numbers the operation $*$ be defined by the formula (45), where $q = \frac{1}{2}(-1+\sqrt{5})$ or $q = \frac{1}{2}(-1-\sqrt{5})$. Then the identity (44) is valid thus, based on Example 5, it follows that $(C,*)$ is RGS–quasigroup. This quasigroup has a nice geometric interpretation which can serve as a motivation for the study of RGS–quasigroups.

If we consider the complex numbers as the points of the Euclidean plane then for any two different points a, b the formula (45) can also be written in the form

$$\frac{a * b - a}{b - a} = q$$

which means that the point $a * b$ divides the pair of points a, b in the ratio q. If $q = \frac{1}{2}(-1-\sqrt{5})$ namely $q = \frac{1}{2}(-1+\sqrt{5})$, then the point a divides the pair $a * b, a$ respectively the point $a * b$ divides the pair b, a in the ratio of golden section, which justifies the name (right) quasigroup of the golden section.

We can introduce the same geometric concepts in any RGS–quasigroup analogously as in a LGS–quasigroup, and in fact in a RGS–quasigroup there exists the same (indeed dual) theory because the following theorem is valid

Theorem 12. *If the operations $\cdot$, $\bullet$ on the set Q are such that the equivalency*

$$ab = c \longleftrightarrow b \bullet a = c$$

is valid, then $(Q, \cdot)$ is a LGS–quasigroup if and only if $(Q, \bullet)$ is a RGS–quasigroup.

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